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All spaces below are assumed to be Hausdorff continuum means a compact and connected space.

1. Classical results

About 1880 C. Jordan defined a continuous curve as a continuous image of the closed interval of real numbers. However, in 1890 G. Peano found an example of a continuous map of [0,1] onto the square $[0,1]^2$ (see [48]). This motivated the subsequent efforts in order to find a satisfactory definition of a dimension of topological spaces.

About 1914 Peano's result was generalized by H. Hahn and S. Mazurkiewicz. They proved independently the following famous Hahn-Mazurkiewicz theorem (see [13], [33]):

Theorem 1. *Each locally connected metrizable continuum is a continuous image of* [0,1].

It is rather easy to see that the converse of Theorem 1 holds. Thus we have the following characterization result:

Theorem 2. *A space is a continuous image of* [0,1] *if and only if it is a metrizable locally connected continuum.*

Let us also state the following Moore theorem, [35], which can be proved as a corollary to the Hahn-Mazurkiewicz theorem:

Theorem 3. *Each metrizable locally connected continuum is arcwise connected (and locally arcwise connected).*

Another result related to the Hahn-Mazurkiewicz theorem is again a characterization theorem. It was proved by P. S. Alexandrov and F. Hausdorff about 1925 (see [2] - the result was announced in [1]; [15]).

Theorem 4. *A space is a continuous image of the Cantor set* $\{0,1\}^{\omega}$ *if and only if it is compact and metrizable.*

The Alexandrov-Hausdorff theorem has several simple proofs. Together with the Moore theorem it can be applied to produce a rather easy proof of the Hahn-Mazurkiewicz theorem (see e.g. [17, p. 129]).

2. The Hahn-Mazurkiewicz problem: A history

The Alexandrov-Hausdorff theorem motivated the following definition: a space is said to be *dyadic* if it is a continuous image of $\{0,1\}^{\alpha}$ for some cardinal number α. Simple arguments show that not every compact space is a dyadic space.

However, there is another way to produce large classes of compact spaces in a manner related to the Alexandrov-Hausdorff, Hahn-Mazurkiewicz and Moore theorems.

This way makes a use of the orderability properties of $[0,1]$ and $\{0,1\}^\omega$.

Let us start with recalling some definitions and simple facts.

A triple $(X,\mathcal{T},\leq)$, where X is a set, $\mathcal{T}$ a topology on X, and $\leq$ a linear ordering on X, is said to be a *linearly ordered topological space* provided the family of all intervals $(\leftarrow,a) = \{x \in X: x < a\}$, $(a,b) = \{x \in X: a < x < b\}$ and $(b,\rightarrow) = \{x \in X: b < x\}$, $a,b \in X$, $a < b$, is a basis for the space $(X,\mathcal{T})$. Moreover, a space $(X,\mathcal{T})$ is said to be *orderable* if there exists a linear ordering $\leq$ on X such that $(X,\mathcal{T},\leq)$ is a linearly ordered topological space. Such an ordering $\leq$ is said to be a *natural ordering* of the space $(X,\mathcal{T})$.

It is well-known that a linearly ordered topological space $(X,\mathcal{T},\leq)$ is compact if and only if $(X,\leq)$ is order complete, i.e., each nonempty subset of X has its supremum and infimum in $(X,\leq)$. Moreover, $(X,\mathcal{T},\leq)$ is a continuum if and only if $(X,\leq)$ is order complete and has no jump, i.e., $(a,b) \neq \emptyset$ for any $a,b \in X$, $a < b$.

A point x of a continuum X is a *cut point* of X provided $X - \{x\}$ is not connected. If $x \in X$ is not a cut point of X, then it is called a *non-cut point* of X. Recall that, by another well-known Moore theorem (see e.g. [17, p. 49]) each continuum has at least two non-cut points.

We will say that a continuum is an arc provided it is a nondegenerate orderable space. It is well known (see e.g. [17, pp. 50-56]) that a continuum is an arc if and only if it has exactly two non-cut points. It is easy to see that each arc has exactly two natural orderings and that its weight coincides with its density. Hence there exists exactly one metrizable arc (this was proved in 1905 by O. Veblen, [65]). Finally, recall that each compact orderable space X can be embedded into an arc Y in a manner such that each component of Y - X is homeomorphic to [0,1]. Therefore the class of arcs is proper (i.e., it is not a set).

We will say that a space is *arcwise connected* if any two distinct points of the space can be joined by an arc.

Now, let us give three false statements which served for years as "folk conjectures" (see [25, p. 929] and [61, p. 96]):

Statement 1 -- a "generalization" of the Alexandrov-Hausdorff theorem. Each compact space is a continuous image of a compact orderable space.

Statement 2 -- a "generalization" of the Hahn-Mazurkiewicz theorem. Each locally connected continuum is a continuous image of an arc.

Statement 3 -- a "generalization" of the Moore theorem. Each locally connected continuum is arcwise connected.

In 1960 S. Mardešić produced an example of a locally connected continuum which is not arcwise connected ([25], a simpler example can be found in [8]). This proved that Statement 3 is false. Moreover, Statement 2 is false because a continuous image of an arc is arcwise connected (the latter fact was proved only in [14] and [66]). A few years later Mardešić gave another surprising example: he constructed a locally connected continuum no proper subcontinuum of which is locally connected, [29]. That construction was done under the additional assumption of Continuum Hypothesis. Recently, G. Gruenhage constructed in ZFC only another example with the same properties, [12].

Again, in 1960, in [31] which is subsequent to [25], S. Mardešić and P. Papić proved that if a product of locally connected continua is a continuous image of an arc, then only countably many factor spaces are nondegenerate and all of them are metrizable. Hence each nonmetric Tichonov cube is a locally connected and arcwise connected continuum which is a continuous image of no arc. The result of Mardešić and Papić was then generalized. In particular, in 1962, G. S. Young proved that the Tichonov plank $(\omega+1) \times (\omega_1+1)$ is a continuous image of no compact orderable space, [72]. A very simple proof of the latter fact is attributed to A. J. Ward (see [61, p. 96]): each continuous image of a compact orderable space is hereditarily normal (see e.g. [16]) and the Tichonov plank is not hereditarily normal (see e.g. [10]). By Young's result, Statement 1 is false as well. Theorems of Mardešić and Papić, and Young were generalized in 1964 by L. B. Treybig ([57]; for simple proofs see [16] and [5]):

Theorem 5. *If a product space* $X_1 \times X_2$ *is a continuous image of a compact orderable space and* X_1, X_2 *are infinite, then both* X_1 *and* X_2 *are metrizable.*

The above-mentioned counterexamples caused a lot of natural questions, some of which were asked explicitly in a 1963 paper of S. Mardešić and P. Papić, [32]. The most important of them was the so-called (nonseparable) Hahn-Mazurkiewicz problem, [32, Problem 1]: find a topological characterization of continuous images of arcs. Up to this time over 60 papers have been published which deal with continuous images of arcs and compact orderable spaces. Those papers are due to S. Mardešić, S. Mardešić and P. Papić, A. J. Ward, L. B. Treybig, J. L. Cornette, B. J. Pearson, J. N. Simone, L. E. Ward, the author, and other topologists. There are even two survey articles: the 1966 paper [27] of S. Mardešić and the beautiful 1981 paper [61] of L. B. Treybig and L. E. Ward; see also [21, pp. 287-289] and [71, p. 315]. The deep results, which were obtained within 25 years, finally led to a solution of the Hahn-Mazurkiewicz problem which was given in the author's paper [37]; see Theorem 14, below. Let us follow some steps on that way.

3. The Hahn-Mazurkiewicz problem: A solution

We will say, [37, p. 92], that a subset A of a locally connected continuum X is a *Treybig set* (abbreviated as T-*set*) in X provided A is closed and the boundary of each component of X - A consists of exactly two points.

It is rather easy to show the following Theorem 6 which relates continuous images of compact orderable spaces to continuous images of arcs:

Theorem 6. (see [28, Lemma 8], [40, Theorem 2]). *If* X *is a continuous image of a compact orderable space, then there exists a continuum* Y *such that* Y *is a continuous image of an arc,* $X \subset Y$, X *is a* T-*set in* Y, *and each component of* Y - X *is homeomorphic to*]0,1[.

In 1965 L. B. Treybig proved the following very deep Theorem 7.

Theorem 7. [58]. *If* X *is a continuum which is a continuous image of a compact orderable space, then either* X *is metrizable or there is a subset* C *of* X *such that* $|C| < 3$ *and* X - C *is not connected.*

In 1984 L. B. Treybig and then R. Maehara noticed that the original proof of Theorem 7 can produce another very strong result:

Theorem 8. ([59, Theorem 4] and [23, Lemma]). *If* X *is a locally connected continuum which has no cut point* *and is a continuous image of a compact orderable space, and* A *is a countable subset of* X, *then there exists a separable* T-*set* B *in* X *such that* $A \subset B$.

Theorem 8 has an immediate corollary which was never stated explicitly:

Corollary 1. *Let* X *be as in* Theorem 8. *If* M *is a closed and metrizable subset of* X, *then there exists a separable* T-*set* B *in* X *such that* $M \subset B$.

Corollary 1 was strengthened by the author:

Theorem 9. (see [37, Theorem 4.9 and Corollary 4.10]). *Let* X *be as in* Theorem 8 *and* Corollary 1. *If* M *is a closed and metrizable subset of* X, *then there exists a metrizable* T-*set* B *in* X *such that* $M \subset B$.

Theorem 9 is strong enough to solve the Hahn-Mazurkiewicz problem. Before describing a solution we need some definitions and facts of the classical cyclic element theory (see e.g. [19], [69]; see also [70] and [7]), the theory of approximation by finite dendrons, and the theory of T-sets.

Let X be a locally connected continuum. A subset Y of X is said to be a *cyclic element* of X if Y is connected and maximal with respect to the property: no point separates Y (i.e., $Y - \{y\}$ is connected for every $y \in Y$). It follows that each cyclic element Z of X is a locally connected subcontinuum of X and there exists a (unique) monotone retraction $r_Z: X \to Z$. Note that X is the unique cyclic element of itself if and only if it has no cut point. Moreover, if Z_1, Z_2 are distinct cyclic elements of X then $|Z_1 \cap Z_2| \leq 1$.

Suppose that $x,y \in X$ and let $E(x,y) = \{z \in X: x$ and y are in distinct components of $X - \{z\}\}$. It is well-known that $E(x,y) \cup \{x,y\}$ is closed in X. Recall that X is said to be a *cyclic chain* from x to y provided $X = E(x,y) \cup \{x,y\} \cup \bigcup H$, where H is the family of all cyclic elements Z of X such that $|Z \cap (E(x,y) \cup \{x,y\})| = 2$.

In 1974 J. L. Cornette noticed that the cyclic element theory is a useful tool in some considerations involving continuous images of arcs. He proved the following:

Theorem 10. [7]. *Let* X *be a locally connected continuum. Then* X *is a continuous image of an arc if and only if each cyclic element of* X *is a continuous image of an arc.*

Another fruitful idea is due to L. E. Ward. First recall that a locally connected continuum is said to be a *dendron* (other names: tree, compact tree-like space, compact dendritic space, dendrite) if each of its cyclic elements consists of a single point. Saying equivalently, a continuum X is a dendron if and only if, for any $x,y \in X$, $x \neq y$, there exists $z \in X$ such that x,y are in distinct components of $X - \{z\}$. Metrizable dendrons are called *dendrites*. A point x of a dendron X is an *end-point* of X if it is a non-cut point of X. A dendron is said to be a *finite dendron* if it has finitely many end-points. Obviously, each arc is a finite dendron.

Let X be a continuum and J a family of finite dendrons which are subcontinua of X. We will say, [68], that J approximates X provided:

(i) J is directed by inclusion;

(ii) $\bigcup J$ is dense in X; and

(iii) if $\mathcal{U}$ is an open covering of X, then there exists $T_{\mathcal{U}} \in J$ such that for each $T \in J$ and each component S of $T - T_{\mathcal{U}}$ there exists $V \in \mathcal{U}$ such that $S \subset V$.

If, moreover,

(ii') $\bigcup J = X$,

then we will say that J *strongly approximates* X, [37].

In 1976 L. E. Ward proved the following two theorems:

Theorem 11. [68, Theorem 1]. *If* X *is a continuum which can be approximated by a family of finite dendrons, then* X *is a continuous image of an arc.*

Theorem 12. (see [68, Theorem 2]). *If* X *is a metrizable and locally connected continuum, then there exists a countable family of finite dendrites which approximates* X.

The method of T-sets was introduced by L. B. Treybig in his 1986 paper [59] in order to study continuous images of arcs. Treybig's method is very useful in such considerations. It was further developed in the author's papers [37] and [38].

Let X,Y be locally connected continua, A a T-set in X and $f: X \to Y$ a function. We will say, [38], that f is a *T-map* with respect to A if the following conditions are satisfied:

(i) f is a continuous surjection;

(ii) $B = f(A)$ is a T-set in Y and $f|_A$ is a homeomorphism from A onto B;

(iii) each component of $Y - B$ is homeomorphic to $]0,1[$; and

(iv) for each component Q of $Y - B$ there is the unique component P_Q of $X - A$ such that $f(P_Q) \subset cl(Q)$, and each component of $X - A$ is a P_Q for some component Q of $Y - B$.

Theorem 13. (see [59, Theorem 6], see also [38, Lemma 2.2]). *If* X *is a locally connected continuum and* A *is a* T-*set in* X, *then there are a locally connected continuum* X_A *and a function* $f: X \to X_A$ *such that* f *is a* T-*map with respect to* A. *Moreover,* X_A *is determined uniquely up to a homeomorphism.*

Roughly speaking, each component of $X - A$ is replaced by a copy of $]0,1[$ in X_A.

Let Y be a locally connected continuum which is the unique cyclic element of itself (i.e., Y has no cut point) and let $\mathcal{A} = (A_1, A_2, \ldots)$ be a sequence of T-sets in Y. We will say, [38], that $\mathcal{A}$ *T-approximates* Y provided

(i) $A_1 \subset A_2 \subset \ldots$;

(ii) if P is a component of $Y - A_n$, $n = 1,2,\ldots$, then the set of all cut points of cl(P) is contained in A_{n+1};

(iii) if P is a component of $Y - A_n$, $n = 1,2,\ldots$, and C is a nondegenerate cyclic element of cl(P), then the set $C \cap A_{n+1}$ is metrizable and consists of at least 3 points; and

(iv) A_1 is metrizable.

It turns out that if a sequence A of T-subsets of Y fulfills the conditions (i) and (ii) of the definition of T-approximation and has the following property (iii'):

(iii') if P is a component of $Y - A_n$, $n = 1,2,\ldots$, and C is a nondegenerate cyclic element of cl(P), then the set $C \cap A_{n+1}$ contains at least 3 points,

then $A = \bigcup_{n=1}^{\infty} A_n$ is a dense subset of Y (see [37, Lemma 3.4]).

Note also that if Y is a locally connected continuum which has no cut point, A is a T-set in Y, and P is a component of Y - A, $bd(P) = \{p,q\}$, then cl(P) is a cyclic chain from p to q.

Now, we are ready to state the main result of [37].

Theorem 14. (see [37, Theorem 1.1, p. 92]). *If* X *is a continuum, then the following conditions are equivalent:*

(i) X *is a continuous image of an arc;*

(ii) X *is a continuous image of a compact orderable space and* X *is locally connected;*

(iii) X *can be strongly approximated by a family of finite dendrons;*

(iv) X *can be approximated by a family of finite dendrons;*

(v) X *is locally connected and for each cyclic element* Y *of* X *the following conditions hold:*

(a) *if* p,q,r $\in$ Y, *then there is a separable* T-*set* E *in* Y *such that* p,q,r $\in$ E,

(b) *if* E $\subset$ E' $\subset$ Y *and* E' *is separable, then* E *is also separable, and*

(c) *if* E' *is an image of* Y *under a continuous map and* E *is a separable continuum in* E', *then* E *is metrizable;*

(vi) X *is locally connected, and if* Y *is a cyclic element of* X *and* p,q,r $\in$ Y, *then there is a metrizable* T-*set* A *in* Y *such that* p,q,r $\in$ A;

(vii) X *is locally connected and each cyclic element* Y *of* X *can be* T-*approximated by a sequence of* T-*subsets of* Y.

The proof of Theorem 14 is very hard. It makes a use of many results and methods of L. B. Treybig and L. E. Ward.

Theorem 14 is a solution of the Hahn-Mazurkiewicz problem. Moreover, the implication (ii) $\Rightarrow$ (i) answers affirmatively another question of Mardešić and Papić, [32, Problem 18], and the implication (i) $\Rightarrow$ (iv) shows that a conjecture of L. E. Ward is true ([68, p. 371 ; see also

[61, Problem 2, p. 100]). Recall that the implication (ii) ⇒ (i) was proved independently by L. B. Treybig, [60, Theorem 3]. Recall also that the proof of (iv) ⇒ (i) is due to L. E. Ward (Theorem 11, above) and the proof of (ii) ⇒ (v) can be found in Treybig's papers [57], [58] and [59].

In [37], the condition (v) (c) of Theorem 14 was stated for continuous and monotone maps. Unfortunately, in general, this form is too restrictive to give the desired equivalence.

Another solution of the Hahn-Mazurkiewicz problem was given in 1984 by W. Bula and M. Turzański, [6]. Unfortunately, no application has been found to their theorems yet.

The condition (vii) of Theorem 14 can be used to prove many properties of continuous images of arcs (see e.g. Theorems 17 and 19, below). It was a little bit modified in [39]:

Theorem 15. (see [39, Theorem 4]). *A locally connected continuum* X *is a continuous image of an arc if and only if for each cyclic element* Y *of* X *there exists a sequence* $(A_1^Y, A_2^Y, \ldots)$ *of* T-*subsets of* Y *such that* $(A_1^Y, A_2^Y, \ldots)$ T-*approximates* Y*, and for each positive integer* n *and each component* Z *of* $Y - A_n^Y$ *the set of all cut points of* cl(Z) *is not metrizable.*

4. Further properties of continuous images of arcs

Let Y be a continuous image of an arc and suppose that Y has no cut point. We will say, [38], that the *rank* of Y is less than or equal to a positive integer m, denoted $r(Y) \leq m$, provided there is a sequence $(A_1, A_2, \ldots)$ of T-subsets of Y such that $(A_1, A_2, \ldots)$ T-approximates Y and $A_m = Y$.

Recall that a continuum is said to be *rim-finite* (another name: regular) if it has a basis of open sets with finite boundaries. It is well-known that each dendron is a rim-finite continuum. Moreover, each rim-finite continuum is locally connected. Actually, a much stronger result holds: each rim-finite continuum is a continuous image of an arc (see [50], [67], [64] and [42]).

In [38, Example 4.1] a rim-finite continuum Y is constructed such that Y has no cut point and $r(Y) = \infty$, i.e., there is no positive integer m such that $r(Y) \leq m$. Thus continuous images of arcs of infinite rank do exist. They are rather complicated spaces. However, inverse limit techniques together with Treybig's method of T-maps can be applied to describe them as inverse limits of simpler spaces.

Theorem 16. [38, Theorem 3.2]. *Let* Y *be a continuum without cut points which is a continuous image of an arc. If* $(A_1, A_2, \ldots)$ *is a sequence of* T-*subsets of* Y *which* T-*approximates* Y, *then*

(i) $r(Y_{A_n}) \leq n$ *for* $n = 1,2,\ldots$ (the meaning of Y_{A_n} is explained in Theorem 13, above);

(ii) *there are functions* $f_n\colon Y \to Y_{A_n}$ *and* $g_n\colon Y_{A_{n+1}} \to Y_{A_n}$, $n = 1,2,\ldots$, *such that*

(a) *each* f_n *is a* T-*map with respect to* A_n,

(b) *each* g_n *is a* T-*map with respect to* $f_{n+1}(A_n)$,

(c) $g_n \circ f_{n+1} = f_n$ *for* $n = 1,2,\ldots$, *and*

(d) Y *is homeomorphic to* $\lim \operatorname{inv}(Y_{A_{n+1}}, g_n)$.

Recently the author has constructed a rim-finite continuum Y such that, in particular, Y has no cut point and no one of functions f_n, g_n of Theorem 16 can be monotone (see [46]). Y is constructed by a method somewhat similar to that of [38, Example 4.1]. The example was produced in order to show that there are many false results in some papers which were sent to the author.

The following Theorem 17 was proved as an application of the condition (vii) of Theorem 14:

Theorem 17. [39, Theorem 6]. *If* X *is a continuous image of an arc and* $x_0 \in X$, *then there exists a partial ordering* $\leq$ *on* X *such that*

(i) $\leq$ *is a closed subset of* $X \times X$;

(ii) *for each* $x \in X$ *the set* $\{y \in X\colon y \leq x\}$ *is connected; and*

(iii) x_0 *is the least element of* $(X,\leq)$.

The result of Theorem 17 is well-known in the case of X being a metrizable locally connected continuum, [18]. It does not hold for arbitrary locally connected continua (see e.g. [39, pp. 361-362]). Note that Theorem 17 shows that a conjecture of L. B. Treybig and L. E. Ward (see [61, p. 102]) is true.

5. The Hahn-Mazurkiewicz theorem for some classes of continua

Let X be a continuum. We will say (see [20] and [64]) that X is *finitely Suslinian* provided, for any open covering $\mathcal{U}$ of X and any family A of pairwise disjoint subcontinua of X, the family $\{Y \in A: Y \text{ is not contained in any } V \in \mathcal{U}\}$ is finite. Recall that X is said to be *hereditarily locally connected* if each of its subcontinua is locally connected, and X is said to be *rim-countable* (another name: rational) if it has a basis B such that bd(U) is countable for each $U \in B$.

Each rim-finite continuum is finitely Suslinian and each finitely Suslinian continuum is hereditarily locally connected (see e.g. [64, Lemma 2]). Moreover, it is well-known that each metrizable hereditarily locally connected continuum is rim-countable (see e.g. [69, Theorem 3.3 of Chapter V]). About 1974 A. E. Brouwer, J. L. Cornette and B. J. Pearson proved that each dendron is a continuous image of an arc; [3], [7] and [49]. Then B.J. Pearson and L.E. Ward showed that each rim-finite continuum is a continuous image of an arc; [50] and [67]. A year later, E.D. Tymchatyn proved that each finitely Suslinian continuum is a continuous image of an arc, [64]. And in 1988 the author proved the strongest result in this direction:

Theorem 18. [42, Theorem 3.4]. *Each hereditarily locally connected continuum is a continuous image of an arc.*

The proof of Theorem 18 which was given in [42] is an application of some inverse limit methods. It depends on general facts developed by the author in [44]. In [45] another proof of Theorem 18 can be found. That alternative proof applies a transfinite induction on weights of hereditarily locally connected continua under the consideration. It uses Lemma 3.3 of [42], Theorem 3 of [24] and some results of [26].

Hereditarily locally connected continua were studied by J. N. Simone (see e.g. [54], [55] and [56]) and B. J. Pearson (see e.g. [51]). However, their internal structure had been almost unknown before Theorem 18 was proved. In particular, in [55, Theorem 9] it was proved that if X is a hereditarily locally connected continuum which is a continuous image of a compact orderable space, then the small inductive dimension of X does not exceed 3, ind $X \leq 3$. The latter fact can be strengthened.

Theorem 19. [42, Theorem 4.1]. *Each hereditarily locally connected continuum is rim-countable. Hence its small inductive dimension is equal to* 1.

Theorem 19 is a nontrivial generalization of the analogous fact for metrizable hereditarily locally connected continua. Its proof applied Theorem 18 and then the powerful condition (vii) of Theorem 14.

6. Some open problems

Let us start with the following Problem 1 which is related to the preceding considerations:

Problem 1. Suppose that X is a locally connected and rim-countable continuum. Is X a continuous image of an arc?

Problem 1 was posed by E. D. Tymchatyn during the Sacramento State Topology Conference in 1987, and then it was repeated in [42, Remark 4.2]. Concerning Problem 1 it is worth to note that in [11] a locally connected continuum X can be found such that X is a continuous image of no arc and X admits a basis **B** such that bd(U) is metrizable and zero-dimensional for each U ∈ **B** (that example is constructed with the use of Continuum Hypothesis).

We will say that a space X is *rim-metrizable* if it admits a basis **B** such that bd(U) is metrizable for each U ∈ **B**.

In 1967 S. Mardešić proved that each continuous image of a compact orderable space is rim-metrizable, [28]. Moreover, he gave in [28] a simple example of a locally connected rim-metrizable continuum which is a continuous image of no arc. In 1987 E. D. Tymchatyn proposed to investigate rim-metrizable continua. In particular he posed the following:

Problem 2. Suppose that $f: X \to Y$ is a continuous map of a locally connected rim-metrizable continuum X onto a space Y. Does it follow that Y is rim-metrizable?

Some results concerning Problem 2 can be found in [42, Remark 4.2], [62] and [63].

Problem 3. (see [38, Problem 4.4], [40, Problem 2]). Suppose that $T = (X_n, f_n)$ is an inverse sequence such that all the spaces X_n are continuous images of arcs and all the bonding maps $f_n: X_n \to X_{n-1}$ are monotone surjections. Does it follow that lim inv T is a continuous image of an arc?

If all the spaces X_n are, moreover, hereditarily locally connected continua then Corollary 3.5 of [42] gives the positive answer to Problem 3.

Note that some questions of [32] are still open. Probably Theorem 14 can be applied to solve them. In particular, it seems to the author that the condition (vii) of Theorem 14 can be used to give the positive answer to the following:

Problem 4. [32, Problem 14]. Do dim X, ind X, and Ind X coincide for each space X which is a continuous image of an arc? a continuous image of a compact orderable space?

There exist some generalizations of dyadic spaces (see e.g. [36], [9]). Recall that a space is said to be m-adic (resp. ξ-adic), [36], provided it is a continuous image of some power of the space A_m (resp. $W(\xi)$), where A_m denotes the one-point compactification of the discrete space of cardinality m, and $W(\xi)$ denotes the set of all ordinal numbers not exceeding the ordinal number ξ with its usual order topology. Therefore the following Problem 5 might be of some interest.

Problem 5. Investigate the class of all spaces which are continuous images of arbitrary products of compact orderable spaces.

Note that even the case of continuous images of products of finitely many compact orderable spaces is rather interesting. In fact, the following old question of S. Mardešić is still open:

Problem 6. (see [30, Conjecture on p. 163]; see also [40, Problem 3]). Let m,n be positive integers, $X_1,\ldots,X_{m+n}$ be infinite compact spaces, $K_1,\ldots,K_m$ be compact orderable spaces, and $f\colon K_1 \times \ldots \times K_m \to X_1 \times \ldots \times X_{m+n}$ be a continuous surjection. Does it follow that there are at least n+1 metrizable factors X_i?

Recall that a space is said to be *suborderable* if it can be embedded into an orderable one. It is well-known that there exist suborderable spaces which are not orderable (for example the Sorgenfrey line is such a space, see e.g. [21, pp. 269-270]).

Theorem 20. (see [4, Corollary 2]). *If a compact space is an image of a suborderable space under a closed continuous map, then it is a continuous image of a compact orderable space.*

Theorem 20 shows that the following general Problem 7 is related to our considerations of continuous images of compact orderable spaces.

Problem 7. Investigate the class of all spaces which are images of suborderable spaces under continuous and closed surjections.

We will say, [16], that a space X is *monotonically normal* provided for each point x in an open subset U of X there is an open set H(x,U) such that

(i) $x \in H(x,U) \subset U$,

(ii) if $x \in U \subset V$ and U,V are open then $H(x,U) \subset H(x,V)$, and

(iii) if $x,y \in X$ and $x \neq y$ then $H(x,X-\{y\}) \cap H(y,X-\{x\}) = \emptyset$.

Recall that monotone normality is a strong separation property. Indeed, each monotonically normal space is collectionwise normal, [16]. Moreover, monotone normality is a hereditary property.

Theorem 21. (see [16]). *Each space which is a continuous image of a compact orderable space is monotonically normal.*

Problem 8. ([40, Problem 6]; see also [47, p. 38]). Is every compact and monotonically normal space a continuous image of a compact orderable space?

Recall the following two results which relate Problem 8 to subsequent Problems 9 and 10.

Theorem 22. (see [41, Theorem 2.1]). *Each zero-dimensional space which is a continuous image of a compact orderable space can be embedded into a dendron.*

Theorem 23. (see [41, Theorem 3.1] and [47, Theorem 3]). *Each separable and zero-dimensional space which is a continuous image of a compact orderable space is orderable.*

Problem 9. [40, Problem 5]. Can a compact, monotonically normal and zero-dimensional space be embedded into a dendron?

Problems 8 and 9 were solved by P. Nyikos and S. Purisch in 1987 under the additional very strong assumption that the considered spaces are scattered. Namely:

Theorem 24. [73, Theorem 2]. *A space is monotonically normal, compact and scattered if and only if it is a continuous image of a compact well-ordered space.*

Now, let us recall an old question of S. Purisch:

Problem 10. (see [53, p. 63], [22, Question 1], [52] and [40, Problem 4]). Let X be a compact, monotonically normal, separable and zero-dimensional space. Is then X orderable?

Let X be a set and **A** be a family of its subsets. We will say that **A** is *binary* provided, for every subfamily **B** of **A**, if $\cap\mathbf{B} = \emptyset$, then there are $C,D \in \mathbf{B}$ such that $C \cap D = \emptyset$. We will write $\wedge.\mathbf{A}$ to denote the family of all finite intersections of members of **A**.

Recall that a space is said to be *supercompact* if it admits a subbase **S** for closed sets such that **S** is a binary family (by the well-known Alexander's Lemma, each supercompact space is compact, however the converse does not hold--see e.g. [34]). Moreover, a space is said to be *regular supercompact* if it admits a subbase **T** for closed sets such that **T** is a binary family and $\wedge.\mathbf{T}$ consists of closed domains (i.e., $cl(int(A)) = A$ for each $A \in \wedge.T$).

Recall the following author's result which was proved in order to solve a question of J. van Mill, [34]:

Theorem 25. [43, Corollary 6.9]. *A compact subset of a dendron is regular supercompact.*

By Theorem 22, we have the following immediate corollary:

Corollary 2. *Each zero-dimensional space which is a continuous image of a compact orderable space is regular supercompact.*

Now, recall two problems on supercompactness and continuous images of compact orderable spaces:

Problem 11. [34, Question 1.5.24]. Are rim-finite continua supercompact?

Problem 12. (see [40, Problem 7]). Let X be a continuous image of a compact orderable space. Does it follow that X is supercompact? regular compact?

Finally, let us pose the following problem of a different nature than those above:

Problem 13. Find a simpler proof of Theorem 9.

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ADDED IN PROOF: 1. Problem 4 has been recently solved in the paper: J. Nikiel, H. M. Tuncali and E. D. Tymchatyn, *On the rim-structure of continuous images of ordered compacta*, Pacific J. Math., to appear; where the following fact is proved: If X is a continuous image of a compact orderable space and X is not zero-dimensional, then dim X = ind X = Ind X = max {1, sup {dim Z: Z is a closed and metrizable subset of X}}.