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Research Announcement:
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REVERSING HOMEOMORPHISMS OF
THE PLANE

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FIXED POINTS OF ORIENTATION REVERSING HOMEOMORPHISMS OF THE PLANE

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Let h be a homeomorphism of the plane \mathbb{R}^2 onto itself, and let X be a plane continuum invariant under h .

In 1951, M. L. Cartwright and J. C. Littlewood (see [5]) proved that if X does not separate the plane, then h has a fixed point in X . Simpler proofs of the Cartwright-Littlewood theorem were later provided by O. H. Hamilton in [6] and by Morton Brown in [4]. In [2], H. Bell proved the Cartwright-Littlewood theorem for an arbitrary homeomorphism of the plane (see also [1] and [3]).

In this note, we do not assume that the continuum X does not separate the plane. We assume that h is an orientation reversing homeomorphism of the plane onto itself.

Marcy Barge asked whether h has always a fixed point in X , and in some cases, for instance if X has exactly two complementary domains, whether h has two fixed points in X . The following results, whose proofs are based on Bell's theorem, answer Barge's question:

Theorem 1. If X has exactly two complementary domains, then h has at least two fixed points in X .

Denote by:

- (1) $[X, h]$ the union of X and the bounded complementary domains of X which contain no fixed points of h ,
- (2) $P(X, h)$ the set of fixed points of h in the bounded complementary domains of X ,
- (3) $LP(X, h)$ the set of the limit points of $P(X, h)$ in X ,
- (4) $Q(X, h)$ the set of the fixed points of h in X .

Lemma 1. If $LP(X, h) = \emptyset$, then $[X, h]$ has finitely many complementary domains.

Lemma 2. If $[X, h]$ does not separate \mathbb{R}^2 , then h has a fixed point in X .

Lemma 3. If $LP(X, h)$ consists of exactly one point, then there is an orientation reversing homeomorphism f of \mathbb{R}^2 onto itself, and there exists a continuum Y invariant under f such that 1) Y has finitely many complementary domains, and 2) if $Q(X, h)$ contains n (finitely many) points, then $Q(Y, f)$ contains at most $n - 1$ points.

Lemma 4. Let $k > 2$ be an integer. If $[X, h]$ has k complementary domains, then there exist an orientation reversing homeomorphism f of \mathbb{R}^2 onto itself, a continuum Y invariant under f , and an integer j , $2 \leq j \leq k - 1$, such that 1) Y has j invariant complementary domains, and 2) the cardinality of $Q(Y, f)$ does not exceed the cardinality of $Q(X, h)$.

Theorem 2. If at least one of the bounded complementary domains of X is invariant under h , then h has at least two fixed points in X . Otherwise h has at least one fixed point in X .

Proof. By Lemma 2, if there are no invariant bounded complementary domains, then h has a fixed point in X .

If $LP(X,h)$ contains more than one point, then clearly h has at least two fixed points in X .

By Lemma 3, if $LP(X,h)$ contains exactly one point which is the only point of $Q(X,h)$, then there exists an orientation reversing homeomorphism f of the plane and a continuum Y invariant under f , such that f has no fixed points in Y , and Y has finitely many complementary domains. If $[Y,f]$ does not separate the plane, then f has a fixed point in Y . If $[Y,f]$ separates the plane, then by Lemma 4 applied inductively, f has a fixed point in Y . Hence, $Q(X,h)$ contains at least two points.

Assume now that X has at least one invariant bounded complementary domain U . If $LP(X,h) = \emptyset$, then $[X,h]$ has finitely many complementary domains. The continuum $[X,h] \cup U$ separates the plane. Either $[X,h] \cup U$ has exactly two complementary domains, or by Lemma 4, applied inductively, we obtain an orientation reversing homeomorphism g of the plane having an invariant continuum Z with exactly two complementary domains. By Theorem 1, g has at least two fixed points in Z . Therefore, h has at least two fixed points in X .

The proofs of the above results are given in [7].
The following generalizations of Theorem 2 are contained in [8]:

Theorem 3. If X has at least 2^k , $k \geq 0$, bounded complementary domains which are invariant under h , then h has at least $k + 2$ fixed points in X .

Theorem 4. If X has infinitely many complementary domains which are invariant under h , then h has infinitely many fixed points in X .

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