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OPEN MAPS ON RIMCOMPACT SPACES

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All spaces in this paper will be completely regular and Hausdorff. A space X is *rimcompact* if X has a base of open sets with compact boundaries, *almost rimcompact* if X has a compactification KX in which each point of $KX \setminus X$ has a base in KX of open sets whose boundaries lie in X , and a *0-space* if X has a compactification with zero-dimensional remainder.

As mentioned in [Ku], Misić' has pointed out that the property of rimcompactness is preserved under mappings that are simultaneously open and closed, denoted open-closed. The argument is straightforward--if $f: X \rightarrow Y$ is open-closed and U is open in X with compact boundary, then $f[U]$ is an open subset of Y having compact boundary. According to [Ku], this statement also holds for monotone open maps, so that monotone open maps preserve the property of rimcompactness. This last result is extended in [Di₃] to the case in which the space X is almost rimcompact or a 0-space. In this paper we indicate that monotone open maps and open-closed maps possess a more general property sufficient for this extension.

An open set U of βX is *CI in βX* (denoting clopen at infinity) if $U \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$. An open set U of X is *π -open* in X if $\text{bd}_X U$ is compact. For a map $f: X \rightarrow Y$, f^β will denote the extension map from βX into βY .

Definition 1. An open map $f: X \rightarrow Y$ is CI preserving if whenever U is a CI open subset of βX , $f^\beta[U]$ is a CI open subset of βY .

As in $[Di_3]$, for a space X and $p \in \beta X$, K_p will denote $\cap \{\beta X \setminus U: U \text{ is CI open in } \beta X, p \notin U\}$. According to 2.2 and 2.3 of $[Di_3]$, for $p \in \beta X$, K_p is a connected compact subset of βX . If $K_p \subseteq \beta X \setminus X$, then K_p is the quasicomponent of p in $\beta X \setminus X$ and has a base of CI open sets in βX . If X is a 0-space, for $p \in \beta X \setminus X$, $K_p \subseteq \beta X \setminus X$.

Theorem 1. If $f: X \rightarrow Y$ is CI preserving and X is a 0-space, Y is a 0-space.

Proof. We first show that if $f: X \rightarrow Y$ is CI preserving and X is a 0-space, then for $p \in \beta Y \setminus Y$, $K_p \subseteq \beta Y \setminus Y$. Choose $p \in \beta Y \setminus Y$, $y \in Y$ and $x \in f^+(y) \cap X$. For each $z \in f^{\beta+}(p)$, $z \in \beta X \setminus X$ so that $K_z \subseteq \beta X \setminus X$ and $x \notin K_z$. There is a CI open set V_z of βX such that $x \in V_z$ while $z \notin V_z$. For each z , $(\beta X \setminus V_z) \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$. Then $f^{\beta+}(p) \subseteq \cup \{(\beta X \setminus V_z) \cap (\beta X \setminus X): z \in f^{\beta+}(p)\}$, thus there is a finite subset $\{z_i: i = 1 \text{ to } n\}$ of $f^{\beta+}(p)$ such that $f^{\beta+}(p) \subseteq \cup_{i=1}^n (\beta X \setminus V_{z_i}) \cap (\beta X \setminus X)$. Let $V = \cap_{i=1}^n V_{z_i}$. Then $x \in V$, $f^{\beta+}(p) \cap V = \emptyset$ and V is CI open in βX . It follows that $y = f(x) \in f^\beta[V]$ which is CI and open in βY , while $p \notin f^\beta[V]$. Thus $K_p \subseteq \beta Y \setminus Y$.

The theorem now follows from the proof of 2.6 of $[Di_3]$.

Theorem 2.6 of [Di₃] is stated for $f: X \rightarrow Y$ monotone open. However, a careful reading of the proof will indicate that the property of the function f actually used is that such a function is CI preserving.

To work with rimcompact and almost rimcompact spaces we need to look at images of π -open sets. In the following, if U is open in βX , $Ex_{\beta X} U$ will denote $\beta X \setminus cl_{\beta X}(X \setminus U)$, the largest open subset of βX whose intersection with X is U .

Lemma 1. *If $f: X \rightarrow Y$ is CI preserving and U is π -open in X , then $f[U]$ is π -open in Y and $cl_Y f[U] = f[cl_X U]$.*

Proof. The set $f[U]$ is clearly open in Y . For any continuous $f: X \rightarrow Y$, f^β is closed, so that $cl_{\beta Y} f[U] = f^\beta[cl_{\beta X} U]$. Now U is π -open in X , hence $cl_{\beta X} U \setminus Ex_{\beta X} U = bd_{\beta X} Ex_{\beta X} U = cl_{\beta X} bd_X U = bd_X U \subseteq X$ (see [Sk] or [Di₃]).

Since f is CI preserving, $bd_{\beta Y} f^\beta[cl_{\beta X} U] \subseteq f^\beta[cl_{\beta X} U] \setminus f^\beta[Ex_{\beta X} U] \subseteq f^\beta[cl_{\beta X} U \setminus Ex_{\beta X} U] = f^\beta[bd_X U] \subseteq Y$. Finally, $bd_Y f[U] \subseteq bd_{\beta Y} f^\beta[cl_{\beta X} U] \cap Y \subseteq f[bd_X U]$, so that $cl_Y f[U] \subseteq f[U] \cup f[bd_X U] = f[cl_X U]$ and $bd_Y f[U]$ is compact.

Example 1 will indicate that a CI preserving map need not be closed on all closed sets, even if the space X is rimcompact, so that the π -open sets form a base.

Corollary 1. *Suppose that $f: X \rightarrow Y$ is CI preserving, and that X is almost rimcompact (rimcompact). Then Y is almost rimcompact (rimcompact).*

Proof. Suppose first that X is almost rimcompact. It follows from Theorem 1 that Y is a 0-space. According to 2.7 of [Di₁], Y is almost rimcompact if and only if each $y \in Y$ has the property (*): there is a compact set K_Y of Y such that if F is closed in Y and $F \cap K_Y = \emptyset$, there is a π -open subset V of Y with $y \in V$ and $\text{cl}_Y V \cap F = \emptyset$. Since X is almost rimcompact, it also follows from 2.7 of [Di₁] that each $x \in X$ has property (*).

Suppose that $y \in Y$; choose $x \in f^+(y)$ and K_X witnessing the fact that x has property (*). Let $K_Y = f[K_X]$, and suppose that F is closed in Y with $F \cap K_Y = \emptyset$. Then $f^+[F]$ is a closed subset of X with $f^+[F] \cap K_X = \emptyset$. Choose V to be π -open in X with $x \in V$ and $\text{cl}_X V \cap f^+[F] = \emptyset$. Then $y \in f[V]$ which is π -open in Y with $F \cap \text{cl}_Y f[V] = F \cap f[\text{cl}_X V] = \emptyset$.

If X is rimcompact, we can choose $K_X = \{x\}$ in the above argument. Then $K_Y = \{y\}$, indicating that Y has a base of π -open sets.

In the above argument it is not necessary that each $x \in X$ have property (*) in either the rimcompact or almost rimcompact case. In [Di₃], pointwise definitions of rimcompactness and almost rimcompactness are made. (That $x \in X$ have property (*) is one of two conditions defining almost rimcompactness of X at x .) The use of Theorem 1 of this paper is no longer valid under the hypotheses of the next result, but arguments similar to those in Corollary 4 above and 2.8 and 2.9 of [Di₃] yield the following:

Theorem 2. Suppose that $f: X \rightarrow Y$ is CI preserving, and that for $y \in Y$, $f^+(y)$ contains a point at which X is almost rimcompact (rimcompact). Then Y is almost rimcompact (rimcompact).

Finally, we indicate the existence of nontrivial CI preserving maps. The next result generalizes 2.1 of [Di₃] by removing the hypothesis that X be a 0-space.

Theorem 3. If $f: X \rightarrow Y$ is monotone and open, then f is CI preserving.

Proof. Suppose that U is open and CI in βX , and $p \in (\beta Y \setminus Y) \cap f^\beta[U]$. Then $f^{\beta+}(p) \subseteq \beta X \setminus X$ and $f^{\beta+}(p) \cap U \neq \emptyset$. According to 4.7 of [Di₄], f^β is monotone; since $U \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$, $f^{\beta+}(p) \subseteq U$. Then $f^{\beta+}[f^\beta[U] \cap (\beta Y \setminus Y)] = U \cap f^{\beta+}[\beta Y \setminus Y]$, thus $f^\beta[U] \cap (\beta Y \setminus Y)$ is clopen in $\beta Y \setminus Y$. Since f^β is closed, $p \in \text{int}_{\beta Y} f^\beta[U]$.

It remains to show that $f^\beta[U] \cap Y \subseteq \text{int}_{\beta Y} f^\beta[U]$. We first show that $f^\beta[U] \cap Y = f[U \cap X]$. If $p \in [f^\beta[U] \cap Y] \setminus f[U \cap X]$, then $f^+(p) \subseteq X \setminus U$, so that $\text{cl}_{\beta X} f^+(p) \cap U = \emptyset$. It follows that $f^{\beta+}(p) \cap U = f^{\beta+}(p) \cap U \cap (\beta X \setminus X)$ and is open in $f^{\beta+}(p)$. Since $(\beta X \setminus X) \setminus U$ is clopen in $\beta X \setminus X$, there is an open set W of βX such that $W \cap (\beta X \setminus X) = (\beta X \setminus X) \setminus U$, thus $W \cap U \subseteq X$. Then $f^{\beta+}(p) \cap U = f^{\beta+}(p) \setminus W$ and so is closed in $f^{\beta+}(p)$. That is, $f^{\beta+}(p) \cap U$ is clopen in the connected set $f^{\beta+}(p)$, a contradiction.

Choose $x \in f^+(p) \cap U$ and W open in βX with $x \in W \subseteq \text{cl}_{\beta X} W \subseteq U$. Since $f[W \cap X]$ is open in Y ,

$p \in \text{int}_{\beta Y} \text{cl}_{\beta Y} f[W \cap X] \subseteq \text{cl}_{\beta Y} f[W \cap X] \subseteq f^\beta[\text{cl}_{\beta X} W] \subseteq f^\beta[U]$.
That is, $p \in \text{int}_{\beta Y} f^\beta[U]$.

In 3.4 of [Di₂], a rimcompact space X , nonrimcompact space Y and monotone closed map $f: X \rightarrow Y$ are constructed, indicating that monotone closed maps are not CI preserving.

Theorem 4. Suppose that f and f^β are open. Then f is CI preserving.

Proof. If U is open and CI in βX , then $f^\beta[U]$ is an open subset of βY . Also, $U \cap f^{\beta+}[\beta Y \setminus Y]$ is clopen in $f^{\beta+}[\beta Y \setminus Y]$. Since f^β restricted to $f^{\beta+}[\beta Y \setminus Y]$ is a closed map, $f^\beta[U] \cap (\beta Y \setminus Y)$ is clopen in $\beta Y \setminus Y$.

A map $f: X \rightarrow Y$ is a WZ map if $\text{cl}_{\beta X} f^+(y) = f^{\beta+}(y)$ for each $y \in Y$. As pointed out in 1.1 of [Is], a closed map is a WZ map. Theorem 4.4 of the same paper states that if $f: X \rightarrow Y$ is a WZ map, then f^β is open if and only if f is open. Thus we have the following:

Corollary 2. If $f: X \rightarrow Y$ is open and WZ, then f is CI preserving.

Corollary 3. If $f: X \rightarrow Y$ is open-closed, then f is CI preserving.

The next example indicates that, as mentioned earlier, a CI preserving map on a space X need not be closed, even if the π -open sets form a base for X . It also indicates that an open WZ map need not be closed.

Example 1. Let X be the space of countable ordinals, and Y the space X in addition to the first uncountable ordinal. As discussed in 3.10.16 of [En], the projection map $\pi_Y: X \times Y \rightarrow Y$ is not closed. Since $X \times Y$ is pseudo-compact, $\beta(X \times Y) = \beta X \times Y$ (Theorems 1 and 4 of [Gl]). The extension of π_Y over $\beta(X \times Y)$ is clearly the projection map $\pi_{\beta X}: \beta X \times Y \rightarrow Y$. Then $\text{cl}_{\beta X \times Y} \pi_Y^{-1}(y) = \beta X \times \{y\} = \pi_{\beta X}^{-1}(y)$, so that π_Y is an open WZ map.

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