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by

Beverly Diamond

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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All spaces in this paper will be completely regular and Hausdorff. A space X is *rimcompact* if X has a base of open sets with compact boundaries, *almost rimcompact* if X has a compactification KX in which each point of KX\X has a base in KX of open sets whose boundaries lie in X, and a *0-space* if X has a compactification with zero-dimensional remainder.

As mentioned in [Ku], Misic' has pointed out that the property of rimcompactness is preserved under mappings that are simultaneously open and closed, denoted open-closed. The argument is straightforward--if f: X → Y is open-closed and U is open in X with compact boundary, then f[U] is an open subset of Y having compact boundary. According to [Ku], this statement also holds for monotone open maps, so that monotone open maps preserve the property of rimcompactness. This last result is extended in [Di_3] to the case in which the space X is almost rimcompact or a 0-space. In this paper we indicate that monotone open maps and open-closed maps possess a more general property sufficient for this extension.

An open set U of βX is CI *in* βX (denoting clopen at infinity) if U ∩ (βX\X) is clopen in βX\X. An open set U of X is *π-open* in X if $bd_X U$ is compact. For a map f: X → Y, f^β will denote the extension map from βX into βY.

Definition 1. An open map f: $X \to Y$ is CI *preserving* if whenever U is a CI open subset of βX, $f^{\beta}[U]$ is a CI open subset of βY.

As in [Di_3], for a space X and $p \in \beta X$, K_p will denote $\cap \{\beta X \setminus U$: U is CI open in βX, $p \notin U\}$. According to 2.2 and 2.3 of [Di_3], for $p \in \beta X$, K_p is a connected compact subset of βX. If $K_p \subseteq \beta X \setminus X$, then K_p is the quasicomponent of p in $\beta X \setminus X$ and has a base of CI open sets in βX. If X is a 0-space, for $p \in \beta X \setminus X$, $K_p \subseteq \beta X \setminus X$.

Theorem 1. *If* f: $X \to Y$ *is* CI *preserving and* X *is a* 0-*space,* Y *is a* 0-*space.*

Proof. We first show that if f: $X \to Y$ is CI preserving and X is a 0-space, then for $p \in \beta Y \setminus Y$, $K_p \subseteq \beta Y \setminus Y$. Choose $p \in \beta Y \setminus Y$, $y \in Y$ and $x \in f^{\leftarrow}(y) \cap X$. For each $z \in f^{\beta\leftarrow}(p)$, $z \in \beta X \setminus X$ so that $K_z \subseteq \beta X \setminus X$ and $x \notin K_z$. There is a CI open set V_z of βX such that $x \in V_z$ while $z \notin V_z$. For each z, $(\beta X \setminus V_z) \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$. Then $f^{\beta\leftarrow}(p) \subseteq \cup \{(\beta X \setminus V_z) \cap (\beta X \setminus X): z \in f^{\beta\leftarrow}(p)\}$, thus there is a finite subset $\{z_i$: i = 1 to n$\}$ of $f^{\beta\leftarrow}(p)$ such that $f^{\beta\leftarrow}(p) \subseteq \cup^n_{i=1} (\beta X \setminus V_{z_i}) \cap (\beta X \setminus X)$. Let $V = \cap^n_{i=1} V_{z_i}$. Then $x \in V$, $f^{\beta\leftarrow}(p) \cap V = \emptyset$ and V is CI open in βX. It follows that $y = f(x) \in f^{\beta}[V]$ which is CI and open in βY, while $p \notin f^{\beta}[V]$. Thus $K_p \subseteq \beta Y \setminus Y$.

The theorem now follows from the proof of 2.6 of [Di_3].

Theorem 2.6 of $[Di_3]$ is stated for $f: X \to Y$ monotone open. However, a careful reading of the proof will indicate that the property of the function f actually used is that such a function is CI preserving.

To work with rimcompact and almost rimcompact spaces we need to look at images of π-open sets. In the following, if U is open in βX, $Ex_{\beta X}U$ will denote $\beta X \setminus cl_{\beta X}(X \setminus U)$, the largest open subset of βX whose intersection with X is U.

Lemma 1. *If* $f: X \to Y$ *is* CI *preserving and* U *is* π*-open in* X, *then* $f[U]$ *is* π*-open in* Y *and* $cl_Y f[U] = f[cl_X U]$.

Proof. The set $f[U]$ is clearly open in Y. For any continuous $f: X \to Y$, f^β is closed, so that $cl_{\beta Y} f[U] = f^\beta[cl_{\beta X}U]$. Now U is π-open in X, hence $cl_{\beta X}U \setminus Ex_{\beta X}U = bd_{\beta X}Ex_{\beta X}U = cl_{\beta X}bd_X U = bd_X U \subseteq X$ (see [Sk] or $[Di_3]$).

Since f is CI preserving, $bd_{\beta Y} f^\beta[cl_{\beta X}U] \subseteq f^\beta[cl_{\beta X}U] \setminus f^\beta[Ex_{\beta X}U] \subseteq f^\beta[cl_{\beta X}U \setminus Ex_{\beta X}U] = f^\beta[bd_X U] \subseteq Y$. Finally, $bd_Y f[U] \subseteq bd_{\beta Y} f^\beta[cl_{\beta X}U] \cap Y \subseteq f[bd_X U]$, so that $cl_Y f[U] \subseteq f[U] \cup f[bd_X U] = f[cl_X U]$ and $bd_Y f[U]$ is compact.

Example 1 will indicate that a CI preserving map need not be closed on all closed sets, even if the space X is rimcompact, so that the π-open sets form a base.

Corollary 1. *Suppose that* $f: X \to Y$ *is* CI *preserving, and that* X *is almost rimcompact (rimcompact). Then* Y *is almost rimcompact (rimcompact).*

Proof. Suppose first that X is almost rimcompact. It follows from Theorem 1 that Y is a 0-space. According to 2.7 of [Di_1], Y is almost rimcompact if and only if each $y \in Y$ has the property (*): there is a compact set K_y of Y such that if F is closed in Y and $F \cap K_y = \emptyset$, there is a π-open subset V of Y with $y \in V$ and $cl_Y V \cap F = \emptyset$. Since X is almost rimcompact, it also follows from 2.7 of [Di_1] that each $x \in X$ has property (*).

Suppose that $y \in Y$; choose $x \in f^{\leftarrow}(y)$ and K_x witnessing the fact that x has property (*). Let $K_y = f[K_x]$, and suppose that F is closed in Y with $F \cap K_y = \emptyset$. Then $f^{\leftarrow}[F]$ is a closed subset of X with $f^{\leftarrow}[F] \cap K_x = \emptyset$. Choose V to be π-open in X with $x \in V$ and $cl_X V \cap f^{\leftarrow}[F] = \emptyset$. Then $y \in f[V]$ which is π-open in Y with $F \cap cl_Y f[V] = F \cap f[cl_X V] = \emptyset$.

If X is rimcompact, we can choose $K_x = \{x\}$ in the above argument. Then $K_y = \{y\}$, indicating that Y has a base of π-open sets.

In the above argument it is not necessary that each $x \in X$ have property (*) in either the rimcompact or almost rimcompact case. In [Di_3], pointwise definitions of rimcompactness and almost rimcompactness are made. (That $x \in X$ have property (*) is one of two conditions defining almost rimcompactness of X at x.) The use of Theorem 1 of this paper is no longer valid under the hypotheses of the next result, but arguments similar to those in Corollary 4 above and 2.8 and 2.9 of [Di_3] yield the following:

Theorem 2. *Suppose that* $f: X \to Y$ *is* CI *preserving, and that for* $y \in Y$, $f^{\leftarrow}(y)$ *contains a point at which* X *is almost rimcompact (rimcompact). Then* Y *is almost rimcompact (rimcompact).*

Finally, we indicate the existence of nontrivial CI preserving maps. The next result generalizes 2.1 of [Di_3] by removing the hypothesis that X be a 0-space.

Theorem 3. *If* $f: X \to Y$ *is monotone and open, then* f *is* CI *preserving.*

Proof. Suppose that U is open and CI in βX, and $p \in (\beta Y \setminus Y) \cap f^{\beta}[U]$. Then $f^{\beta\leftarrow}(p) \subseteq \beta X \setminus X$ and $f^{\beta\leftarrow}(p) \cap U \neq \emptyset$. According to 4.7 of [$Di_4$], f^{β} is monotone; since $U \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$, $f^{\beta\leftarrow}(p) \subseteq U$. Then $f^{\beta\leftarrow}[f^{\beta}[U] \cap (\beta Y \setminus Y)] = U \cap f^{\beta\leftarrow}[\beta Y \setminus Y]$, thus $f^{\beta}[U] \cap (\beta Y \setminus Y)$ is clopen in $\beta Y \setminus Y$. Since f^{β} is closed, $p \in \text{int}_{\beta Y} f^{\beta}[U]$.

It remains to show that $f^{\beta}[U] \cap Y \subseteq \text{int}_{\beta Y} f^{\beta}[U]$. We first show that $f^{\beta}[U] \cap Y = f[U \cap X]$. If $p \in [f^{\beta}[U] \cap Y] \setminus f[U \cap X]$, then $f^{\leftarrow}(p) \subseteq X \setminus U$, so that $\text{cl}_{\beta X} f^{\leftarrow}(p) \cap U = \emptyset$. It follows that $f^{\beta\leftarrow}(p) \cap U = f^{\beta\leftarrow}(p) \cap U \cap (\beta X \setminus X)$ and is open in $f^{\beta\leftarrow}(p)$. Since $(\beta X \setminus X) \setminus U$ is clopen in $\beta X \setminus X$, there is an open set W of βX such that $W \cap (\beta X \setminus X) = (\beta X \setminus X) \setminus U$, thus $W \cap U \subseteq X$. Then $f^{\beta\leftarrow}(p) \cap U = f^{\beta\leftarrow}(p) \setminus W$ and so is closed in $f^{\beta\leftarrow}(p)$. That is, $f^{\beta\leftarrow}(p) \cap U$ is clopen in the connected set $f^{\beta\leftarrow}(p)$, a contradiction.

Choose $x \in f^{\leftarrow}(p) \cap U$ and W open in βX with $x \in W \subseteq \text{cl}_{\beta X} W \subseteq U$. Since $f[W \cap X]$ is open in Y,

$p \in \mathrm{int}_{\beta Y}\mathrm{cl}_{\beta Y} f[W \cap X] \subseteq \mathrm{cl}_{\beta Y} f[W \cap X] \subseteq f^{\beta}[\mathrm{cl}_{\beta X} W] \subseteq f^{\beta}[U]$. That is, $p \in \mathrm{int}_{\beta Y} f^{\beta}[U]$.

In 3.4 of [Di_2], a rimcompact space X, nonrimcompact space Y and monotone closed map $f: X \to Y$ are constructed, indicating that monotone closed maps are not CI preserving.

Theorem 4. *Suppose that* f *and* f^{β} *are open. Then* f *is* CI *preserving.*

Proof. If U is open and CI in βX, then $f^{\beta}[U]$ is an open subset of βY. Also, $U \cap f^{\beta\leftarrow}[\beta Y \setminus Y]$ is clopen in $f^{\beta\leftarrow}[\beta Y \setminus Y]$. Since f^{β} restricted to $f^{\beta\leftarrow}[\beta Y \setminus Y]$ is a closed map, $f^{\beta}[U] \cap (\beta Y \setminus Y)$ is clopen in $\beta Y \setminus Y$.

A map $f: X \to Y$ is a WZ map if $\mathrm{cl}_{\beta X} f^{\leftarrow}(y) = f^{\beta\leftarrow}(y)$ for each $y \in Y$. As pointed out in 1.1 of [Is], a closed map is a WZ map. Theorem 4.4 of the same paper states that if $f: X \to Y$ is a WZ map, then f^{β} is open if and only if f is open. Thus we have the following:

Corollary 2. *If* $f: X \to Y$ *is open and* WZ, *then* f *is* CI *preserving.*

Corollary 3. *If* $f: X \to Y$ *is open-closed, then* f *is* CI *preserving.*

The next example indicates that, as mentioned earlier, a CI preserving map on a space X need not be closed, even if the π-open sets form a base for X. It also indicates that an open WZ map need not be closed.

Example 1. Let X be the space of countable ordinals, and Y the space X in addition to the first uncountable ordinal. As discussed in 3.10.16 of [En], the projection map $\pi_Y: X \times Y \to Y$ is not closed. Since $X \times Y$ is pseudocompact, $\beta(X \times Y) = \beta X \times Y$ (Theorems 1 and 4 of [Gl]). The extension of π_Y over $\beta(X \times Y)$ is clearly the projection map $\pi_{\beta X}: \beta X \times Y \to Y$. Then $cl_{\beta X \times Y}\pi_Y^{\leftarrow}(y) = \beta X \times \{y\} = \pi_{\beta X}^{\leftarrow}(y)$, so that π_Y is an open WZ map.

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College of Charleston

Charleston, South Carolina 29424