TOPOLOGY PROCEEDINGS

Volume 14, 1989

Pages 265–277

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

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In 1979 David Bellamy [1] showed that there exist tree-like continua which admit fixed point free mappings. There has been interest since that time in determining conditions under which a tree-like continuum will have the fixed point property. A few results of this nature can be found in [2], [3], [4], [7], [8], and [9]. However, it is still unknown if a simple triod-like continuum must have the fixed point property. This paper establishes several fixed point related theorems for T-like continua, where T is a fixed tree. Corollary 3 gives a necessary condition for a T-like continuum to admit a fixed point free mapping, and Theorem 2 generalizes the fixed point theorem in [7].

A *continuum* is a nondegenerate compact connected metric space. A continuous function will be referred to as a *map* or *mapping*. A continuum X has the *fixed point property* provided that whenever f is a mapping of X into itself, there is a point x in X such that $f(x) = x$. A *tree* is a finite connected, simply connected graph. If ε is a positive number, the mapping $f: X \to Y$ is an ε-*mapping* if $\mathrm{diam}(f^{-1}(y)) < \varepsilon$ for each $y \in Y$. If H is a family of continua, we say that the continuum X is *H-like* provided that, for each positive number ε, there is an ε-mapping of X onto a member of $\mathcal{H}$. For example, if $\mathcal{H}$ is the family of all trees, we simply say that X is *tree-like*; or if $\mathcal{H}$ is a set whose only member is the continuum T, we say that X is *T-like*.

Let T be a tree. The point $v \in T$ is a *branchpoint (an endpoint)* of T if $T - \{v\}$ has at least three components (only one component). If v is either a branchpoint or an endpoint of T, we say that v is a *vertex* of T. If v and w are points of T, let $[v,w]$ denote the arc in T with endpoints v and w, and let $T(v,w]$ denote the component of $T - \{v\}$ that contains w.

Lemma. Let F *be a function from the vertex set of the tree* T *into the set of all subsets of* T. *If for each vertex* v *of* T, F(v) *is a subset of the closure of some component of* $T - \{v\}$, *then there exist neighboring (adjacent) vertices* v *and* w *in* T *such that* $F(v) \subseteq \overline{T(v,w]}$ *and* $F(w) \subseteq \overline{T(w,v]}$.

Proof. Let v_1 be any branchpoint of T and let C_1 be the component of $T - \{v_1\}$ such that $F(v_1)$ is a subset of $\overline{C_1}$. Let v_2 be the vertex of C_1 that is adjacent to v_1. So, $C_1 = T(v_1,v_2]$. If $F(v_2) \subseteq \overline{T(v_2,v_1]}$, then v_1 and v_2 have the desired properties. Otherwise, v_2 must be a branchpoint of T and there is a component C_2 of $T - \{v_2\}$ such that $C_2 \neq T(v_2,v_1]$ and $F(v_2) \subseteq \overline{C_2}$. Now, $C_2 \subseteq C_1$ and C_2 contains fewer branchpoints than C_1. Since C_1 has finitely many branchpoints, a repetition of the process above must yield adjacent vertices with the desired properties.

We introduce the following terminology. Given a sequence $\{F_n\}_{n=1}^{\infty}$, to say that

$\{F_n\}_{n=1}^{\infty}$ *frequently* has some property means that for each positive integer N, there is an integer $n \geq N$ such that F_n has the property,

and to say that

$\{F_n\}_{n=1}^{\infty}$ *eventually* has some property means that there is a positive integer N such that if $n \geq N$, then F_n has the property.

We are now ready for our main theorems.

Theorem 1. *Suppose that* T *is a tree,* X *is* T-*like, and for each* $n \geq 1$, $g_n\colon X \to T$ *is a* δ_n-*mapping onto* T, *where* $\{\delta_n\}_{n=1}^{\infty}$ *converges to zero. If* $f\colon X \to X$ *is a mapping,* $\{n_i\}_{i=1}^{\infty}$ *is an increasing sequence of positive integers, and there are adjacent vertices* v *and* w *of* T *such that* $\{g_{n_i} f g_{n_i}^{-1}(v)\}_{i=1}^{\infty}$ *is eventually a subset of* $\overline{T(v,w]}$ *and* $\{g_{n_i} f g_{n_i}^{-1}(w)\}_{i=1}^{\infty}$ *is eventually a subset of* $\overline{T(w,v]}$, *then* f *has a fixed point.*

Proof. Suppose that f is fixed point free. Let d denote the metric on X. Assume that each edge of T has length one and let p denote the "arc length" metric on T. Let ε be a positive number such that $d(x,f(x)) \geq \varepsilon$ for each $x \in X$.

Fix n large enough so that g_n is an $\frac{\varepsilon}{2}$-mapping, $g_n f g_n^{-1}(v) \subseteq \overline{T(v,w]}$, and $g_n f g_n^{-1}(w) \subseteq \overline{T(w,v]}$. Since g_n is an $\frac{\varepsilon}{2}$-mapping, it follows that $t \notin g_n f g_n^{-1}(t)$ for any $t \in T$. So, we have that $g_n f g_n^{-1}(v) \subseteq T(v,w]$ and $g_n f g_n^{-1}(w) \subseteq T(w,v]$.

Let $0 < \delta < 1$ such that if $d(x,y) \geq \varepsilon$, then $p(g_n(x),g_n(y)) \geq \delta$. That such a δ exists is easily seen (argument by contradiction).

Let V be an open set in X such that $g_n^{-1}(v) \subseteq V$, $\text{diam}\overline{V} < \varepsilon$, and if $x \in V$, then $g_n f(x) \in T(v,w]$ and $p(g_n(x),v) < \frac{\delta}{2}$. Similarly, let W be an open set in X such that $g_n^{-1}(w) \subseteq W$, $\text{diam}\overline{W} < \varepsilon$, and if $x \in W$, then $g_n f(x) \in T(w,v]$ and $p(g_n(x),w) < \frac{\delta}{2}$.

Pick any point q in $g_n^{-1}(w)$ and let L be the component of X - V that contains q. Now, L must intersect the boundary of V at some point y. We point out that $g_n(L) \subseteq T(v,w]$. For if not, there is a point $x \in L$ such that $g_n(x) \in T(w,v] - (v,w]$. Also, $q \in L$ and $g_n(q) = w$. Since L is connected and g_n is continuous, it follows that there is a point of L that is also in $g_n^{-1}(v) \subseteq V$, a contradiction.

Let K be the component of L - W that contains y. Let z be a point of the boundary of W that is also in K. As above, $g_n(K) \subseteq T(w,v]$. For if not, there is a point $x \in K$ such that $g_n(x) \in T(v,w] - [v,w)$. Since $y \in \overline{V}$, $g_n(y) \in \overline{T(w,v]}$. Now, y is also in K; hence, there is a point of K that is also in $g_n^{-1}(w) \subseteq W$, a contradiction.

Since $K \subseteq L$, we get that $g_n(K) \subseteq (v,w)$. Let

$$R = \{x \in K \mid g_n(x) \text{ separates } g_n f(x) \text{ from } v \text{ in } T\}$$

and

$$S = \{x \in K \mid g_n(x) \text{ separates } g_nf(x) \text{ from } w \text{ in } T\}.$$

Clearly, $R \cup S = K$, and R and S are disjoint open sets in K. We will show that $y \in R$ and $z \in S$.

Suppose that $y \notin R$. Then $y \in S$ and $g_n(y)$ must separate $g_nf(y)$ from w in T. Since $y \in \overline{V}$, $\rho(g_n(y),v) \leq \frac{\delta}{2}$ and $g_nf(y) \in \overline{T(v,w]}$. Hence, we must have that $g_nf(y) \in [v,w]$ and that $\rho(g_nf(y),g_n(y)) \leq \frac{\delta}{2} < \delta$. But, by choice of δ, $d(y,f(y)) \geq \varepsilon$ implies that $\rho(g_n(y),g_nf(y)) \geq \delta$, a contradiction.

A symmetric argument gives us that $z \in S$. But then K is not connected, which is a contradiction.

Since an arc is a tree with exactly two vertices, namely its endpoints, we get Hamilton's [5] fixed point theorem as an immediate corollary.

Corollary 1. *If* X *is an arc-like continuum, then* X *has the fixed point property.*

Corollary 2. *Suppose that* T *is a simple k-od with branchpoint* v, X *is* T-*like, and for each* $n \geq 1$, $g_n: X \to T$ *is a* δ_n-*mapping onto* T, *where* $\{\delta_n\}_{n=1}^{\infty}$ *converges to zero. If* $f: X \to X$ *is a fixed point free mapping, then* $\{g_nfg_n^{-1}(v)\}_{n=1}^{\infty}$ *eventually intersects two components of* $T - \{v\}$.

Proof. Suppose that $\{g_nfg_n^{-1}(v)\}_{n=1}^{\infty}$ does not eventually intersect two components of $T - \{v\}$. Then there is a component L of $T - \{v\}$ such that $\{g_nfg_n^{-1}(v)\}_{n=1}^{\infty}$ is frequently a subset of L. Let e be the endpoint of T that belongs to L. Then v and e are adjacent vertices of T. Also, $\{g_n f g_n^{-1}(e)\}_{n=1}^{\infty}$ is a subset of $\overline{T(e,v]}$ for all $n \geq 1$ since $\overline{T(e,v]} = \overline{T - \{e\}} = T$. It follows from Theorem 1 that f has a fixed point, which is a contradiction.

Corollary 3. *Suppose that* T *is a tree,* X *is* T-*like, and for each* $n \geq 1$, g_n: X → T *is a* δ_n-*mapping onto* T, *where* $\{\delta_n\}_{n=1}^{\infty}$ *converges to zero. If* f: X → X *is a fixed point free mapping, then there is a branchpoint* v *of* T *such that* $\{g_n f g_n^{-1}(v)\}_{n=1}^{\infty}$ *frequently intersects two components of* T - {v}.

Proof. By way of contradiction, we assume that for each branchpoint v of T, there is a positive integer N_v such that if $n \geq N_v$, then $g_n f g_n^{-1}(v)$ is a subset of the closure of some component of T - {v}.

Let $N = \max\{N_v \mid v \text{ is a branchpoint of } T\}$ and fix $n \geq N$. We recall that if e is an endpoint of T and v is the vertex of T adjacent to e, then $g_n f g_n^{-1}(e) \subseteq \overline{T(e,v]}$. Hence, by the lemma, there exist adjacent vertices v and w in T such that $g_n f g_n^{-1}(v) \subseteq \overline{T(v,w]}$ and $g_n f g_n^{-1}(w) \subseteq \overline{T(w,v]}$. So, if $n \geq N$, we may associate with n a pair of adjacent vertices in T that have the properties above. Since there are only finitely many pairs of adjacent vertices in T, it follows that there is an increasing sequence $\{n_i\}_{i=1}^{\infty}$, each term of which is associated with the same pair of adjacent vertices. By Theorem 1, f has a fixed point, which is a contradiction.

Our next theorem generalizes, in the case of finite fans, the fixed point result in [7].

Theorem 2. *Let* T *be a tree, and for each branchpoint* v *of* T, *let* $\{L_i(v)\}_{i=1}^{k_v}$ *be a labeling of the components of* T - {v}. *If* $X = \varprojlim\{T, g_n^{n+1}\}$, *where for each* $n \geq 1$ *and each branchpoint* v *of* T, $g_n^{n+1}(L_i(v)) = L_i(v)$ *for* $2 \leq i \leq k_v$, *then* X *has the fixed point property.*

Proof. Let d denote the metric on X and, for each $n \geq 1$, let g_n be the projection mapping of X onto T. Now, X is T-like and for $\varepsilon > 0$, n can be chosen so that g_n is an ε-mapping (see [6]).

By way of contradiction, we assume that f is a fixed point free mapping on X and that ε is a positive number such that $d(x,f(x)) \geq \varepsilon$ for each $x \in X$.

Let v be any branchpoint of T. We notice that $g_n^{n+1}(v) = v$ for each $n \geq 1$. So, let p_v be the point of X such that $g_n(p_v) = v$ for each $n \geq 1$. Also, let $M_v = \cup_{i=2}^{k_v}\overline{L_i(v)}$. We further observe that

(*) if $x \in X$ and there is an integer N such that $g_N(x)$ is not in M_v, then for $n \geq N$, $g_n(x) \notin M_v$.

Suppose that (*) is not the case. Then there is a point $x \in X$ and positive integers N and n with $n \geq N$ such that $g_N(x) \notin M_v$ but $g_n(x) \in M_v$. However, this implies that $g_N(x) = g_N^n g_n(x) \in M_v$, which is a contradiction.

Hence, by (*) and the fact that $g_n^{n+1}(M_v) \subseteq M_v$ for each $n \geq 1$, we may choose a positive integer m such that g_m is an ε-mapping and so that either

i) $g_n(f(p_v)) \in L_1(v)$ for $n \geq m$ or

ii) $g_n(f(p_v)) \in M_v$ for $n \geq m$.

Note that $g_n(f(p_v)) \neq v$, for $n \geq m$, since g_m is an ε-mapping and $v = g_n(p_v)$. Since $g_n^{n+1}(L_i(v)) = L_i(v)$ for $n \geq 1$ and $2 \leq i \leq k_v$, it follows that if $g_n(f(p_v)) \in M_v$ for $n \geq m$, then there is an integer $2 \leq j \leq k_v$ such that $g_n(f(p_v)) \in L_j(v)$ for $n \geq m$. So, in fact, we have that there is an integer $1 \leq i \leq k_v$ such that $g_n(f(p_v)) \in L_i(v)$ for $n \geq m$.

Let δ be a positive number such that if $x \in X$ and $d(x,p_v) < \delta$, then $g_m f(x) \in L_i(v)$. Let $n \geq m$ and large enough so that g_n is a δ-mapping. Since $p_v \in g_n^{-1}(v)$ and $\operatorname{diam}(g_n^{-1}(v)) < \delta$, it follows that if $x \in g_n^{-1}(v)$, then $d(x,p_v) < \delta$ and $g_m f(x) \in L_i(v)$. Thus, $g_m f g_n^{-1}(v) \subseteq L_i(v)$. Now, if $i = 1$, then by (*), $g_n f g_n^{-1}(v) \subseteq L_1(v)$. If $i \neq 1$, we get that $g_n f g_n^{-1}(v) \subseteq L_1(v) \cup L_i(v)$.

We have shown that for each branchpoint v of T, there is a positive integer m_v and an integer $1 \leq i_v \leq k_v$ such that for $n \geq m_v$,

(1) $g_n(f(p_v)) \in L_{i_v}(v)$, and

(2) $g_n f g_n^{-1}(v) \subseteq L_1(v) \cup L_{i_v}(v)$.

Let $N = \max\{m_v | v$ is a branchpoint of $T\}$. For $n \geq N$, and v a branchpoint of T, let

$$F_n(v) = \begin{cases} g_n f g_n^{-1}(v) & \text{if } g_n f g_n^{-1}(v) \text{ intersects only one of } L_1(v) \text{ and } L_{i_v}(v), \\ g_n f g_n^{-1}(v) \cap L_{i_v}(v) & \text{otherwise.} \end{cases}$$

For $n \geq N$ and e an endpoint of T, let $F_n(e) = g_n f g_n^{-1}(e)$. By our lemma, for each $n \geq N$, there are adjacent vertices v and w of T such that $F_n(v) \subseteq \overline{T(v,w]}$ and $F_n(w) \subseteq \overline{T(w,v]}$. By the finiteness of the set of all pairs of adjacent vertices in T, we can pick an increasing number sequence $\{n_i\}_{i=1}^{\infty}$ and a pair of adjacent vertices v and w such that for each $i \geq 1$, $F_{n_i}(v) \subseteq \overline{T(v,w]}$ and $F_{n_i}(w) \subseteq \overline{T(w,v]}$. Let $\leq$ be a partial order on T that is consistent with the metric on T and such that v is the least element of $\overline{T(v,w]}$ and w is the maximum element of $\overline{T(w,v]}$.

The remainder of the proof involves three cases.

Case 1. $\{g_{n_i} f g_{n_i}^{-1}(v)\}_{i=1}^{\infty}$ eventually intersects only one of $L_1(v)$ and $L_{i_v}(v)$, and $\{g_{n_i} f g_{n_i}^{-1}(w)\}_{i=1}^{\infty}$ eventually intersects only one of $L_1(w)$ and $L_{i_w}(w)$.

In this case, by definition, $F_{n_i}(v) = g_{n_i} f g_{n_i}^{-1}(v)$ and $F_{n_i}(w) = g_{n_i} f g_{n_i}^{-1}(w)$ for all i beyond some integer. It follows from Theorem 1 that f has a fixed point, which is a contradiction.

Case 2. $\{g_{n_i} f g_{n_i}^{-1}(v)\}_{i=1}^{\infty}$ frequently intersects both of $L_1(v)$ and $L_{i_v}(v)$ and $\{g_{n_i} f g_{n_i}^{-1}(w)\}_{i=1}^{\infty}$ frequently intersects both of $L_1(w)$ and $L_{i_w}(w)$.

We observe that if $i_v \neq 1$ and $g_r f g_r^{-1}(v)$ intersects $L_{i_v}(v)$ for any integer r, then $g_k f g_k^{-1}(v)$ intersects $L_{i_v}(v)$ for each integer $k \leq r$. To see this, let $k \leq r$ and first notice that $g_r^{-1}(v) \subseteq g_k^{-1}(v)$ since v is fixed by all bonding mappings. Thus, $g_r f g_r^{-1}(v) \subseteq g_r f g_k^{-1}(v)$. So, there is a point x in $L_{i_v}(v) \cap g_r f g_k^{-1}(v)$. Since $i_v \neq 1$, $g_k^r(x) \in L_{i_v}(v)$. Hence, $g_k^r(g_r f g_k^{-1}(v)) = g_k f g_k^{-1}(v)$ intersects $L_{i_v}(v)$.

By our assumption in this case, $i_v \neq 1$ and $i_w \neq 1$. Hence, since $\{g_{n_i} f g_{n_i}^{-1}(u)\}_{i=1}^{\infty}$ frequently intersects $L_{i_u}(u)$ for $u \in \{v,w\}$, it follows from our observation in the preceding paragraph that $\{g_n f g_n^{-1}(u)\}_{i=1}^{\infty}$ intersects $L_{i_u}(u)$ for all $n \geq 1$. So, by definition, $F_n(u) \subseteq L_{i_u}(u)$ for all $n \geq 1$ and $u \in \{v,w\}$. It follows that $L_{i_v}(v) = T(v,w]$ and $L_{i_w}(w) = T(w,v]$. Hence, for each $n \geq 1$, $g_n^{n+1}(T(v,w]) = T(v,w]$ and $g_n^{n+1}(T(w,v]) = T(w,v]$. It follows that for $n \geq 1$, $g_n^{n+1}([v,w]) = [v,w]$.

Let $C = \varprojlim\{[v,w], g_n^{n+1}|_{[v,w]}\}$. Now, C is an arc-like continuum containing the points p_v and p_w. Recall that for each $n \geq m_v$, $g_n f(p_v) \in L_{i_v}(v) = T(v,w]$ and for $n \geq m_w$, $g_n f(p_w) \in L_{i_w}(w) = T(w,v]$. Let n be large enough so that $n \geq \max\{m_v, m_w\}$ and g_n is an ε-mapping. Let

$$R = \{x \in C \mid g_n(x) < g_n f(x)\}$$

and

$$S = \{x \in C \mid g_n(x) > g_n f(x)\}.$$

Clearly, $R \cup S = C$, R and S are open disjoint sets in C, $p_v \in R$, and $p_w \in S$. But then C is not connected, which is a contradiction.

Case 3. $\{g_{n_i} f g_{n_i}^{-1}(v)\}_{i=1}^{\infty}$ eventually intersects only one of $L_1(v)$ and $L_{i_v}(v)$, and $\{g_{n_i} f g_{n_i}^{-1}(w)\}_{i=1}^{\infty}$ frequently intersects both of $L_1(w)$ and $L_{i_w}(w)$.

As in Case 2, it follows that $i_w \neq 1$, $L_{i_w}(w) = T(w,v]$, and $F_n(w) \subseteq T(w,v]$ for all $n \geq 1$.

Now, if $i_v \neq 1$ and $\{g_{n_i} f g_{n_i}^{-1}(v)\}_{i-1}^{\infty}$ is frequently a subset of $L_{i_v}(v)$, then the argument beginning with the second paragraph in Case 2 applies and we are done. So, we may assume that $\{g_{n_i} f g_{n_i}^{-1}(v)\}_{i=1}^{\infty}$ is eventually a subset of $L_1(v)$. Thus, for all i beyond some integer, $F_{n_i}(v) = g_{n_i} f g_{n_i}^{-1}(v)$, and it follows that $L_1(v) = T(v,w]$. We may choose an integer n large enough so that $n \geq m_w$, $g_n f g_n^{-1}(v) \subseteq \overline{T(v,w]}$, $g_n f g_n^{-1}(w) \cap T(w,v] \neq \emptyset$, and g_n is an $\frac{\varepsilon}{2}$-mapping. Let δ be a positive number such that $d(x,y) \geq \varepsilon$ in X implies that $p(g_n(x), g_n(y)) \geq \delta$ in T.

Let V be an open set in X such that $g_n^{-1}(v) \subseteq V$, $\text{diam}\overline{V} < \varepsilon$, and if $x \in V$, then $g_n f(x) \in T(v,w]$ and

$p(g_n(x),v) < \frac{\delta}{2}$. Let $M = \underleftarrow{\lim}\{\overline{T(w,v]}, g_i^{i+1}|\overline{T(w,v]}\}$, and let C be the component of $M - V$ that contains p_w. Recall that $g_n f(p_w) \in L_{i_w}(w) = T(w,v]$ since $n \geq m_w$. Now, C must intersect the boundary of V at some point y. We point out that $g_n(C) \subseteq \overline{T(v,w]}$. For if not, there is a point $x \in C$ such that $g_n(x) \in T(w,v] - [v,w]$. Also, $p_w \in C$ and $g_n(p_w) = w$. Since C is connected and g_n is continuous, it follows that there is a point of C that is also in $g_n^{-1}(v) \subseteq V$, a contradiction.

Furthermore, $g_n(C) \subseteq \overline{T(w,v]}$ simply because $C \subseteq M$. It follows that $g_n(C) \subseteq [v,w]$. Let

$$R = \{x \in C \mid g_n(x) < g_n f(x)\}$$

and

$$S = \{x \in C \mid g_n(x) > g_n f(x)\}.$$

Clearly, $R \cup S = C$, and R and S are disjoint open sets in C. We will show that $y \in R$ and $p_w \in S$.

Now, $p_w \in S$ since $g_n(p_w) = w$ and $g_n f(p_w) \in T(w,v]$.

Suppose $y \notin R$. Then $y \in S$ and $g_n(y) > g_n f(y)$. Since $y \in \overline{V}$, $p(g_n(y),v) \leq \frac{\delta}{2}$, and $g_n f(y) \in \overline{T(v,w]}$. Hence, we must have that $g_n f(y) \in [v,w]$ and that $p(g_n f(y), g_n(y)) \leq \frac{\delta}{2} < \delta$. But by choice of δ, $d(y,f(y)) \geq \varepsilon$ implies that $p(g_n(y), g_n f(y)) \geq \delta$, a contradiction.

But now we have that C is not connected, which is a contradiction.

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