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NECESSARY AND SUFFICIENT CONDITIONS FOR PRODUCTS OF k -SPACES

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Introduction

As is well-known, the product of a k -space with a separable metric space need not be a k -space ([8, 15]). In [27, 30, and 31] etc., the author obtained some necessary and sufficient conditions for products of various kinds of k -spaces to be k -spaces.

In this paper, we give some characterizations for the product $X \times Y$ or X^{ω} to be a k -space, if X and Y are more general types of k -spaces, as well as X and Y are dominated by these types of k -spaces. Also, we shall pose some questions concerning products of k -spaces.

We assume that all spaces are regular T_1 , and all maps are continuous and surjections.

1. Definitions

Definition 1.1. Let X be a space and \mathcal{C} be a cover of X . Then X is determined by \mathcal{C} [11] (or X has the weak topology with respect to \mathcal{C}), if $F \subset X$ is closed in X whenever $F \cap C$ is relatively closed in C for every $C \in \mathcal{C}$. Here we can replace "closed" by "open." Obviously, every space is determined by an open cover.

We recall that a space is a k -space; sequential space; c -space (= space of countable tightness), if it is

determined by the cover of all compact subsets; compact metric subsets; countable subsets, respectively. First countable spaces are sequential, and sequential spaces are k -spaces. We note that a space is a c -space if and only if whenever $x \in \bar{A}$, then $x \in \bar{C}$ for some countable $C \subset A$, and sequential spaces and hereditarily separable spaces are c -spaces; see [17; p. 123], for example.

For an infinite cardinal number α , a space is a k_α -space [36], if it is determined by a cover \mathcal{C} of compact subsets with the cardinality of $\mathcal{C} \leq \alpha$. k_ω -spaces (= spaces belonging to class \mathcal{G}' in the sense of K. Morita [21]) are defined by E. Michael [16]. A space X is locally $< k_\alpha$ [36], if each point $x \in X$ has a neighborhood whose closure is a $k_{\beta(x)}$ -space, where $\beta(x) < \alpha$. We shall say that a space is locally k_ω if it is locally $< k_{\omega_1}$.

Let X be a space, and \mathcal{C} be a closed cover of X . Then X is dominated by \mathcal{C} [13], if the union of any subcollection \mathcal{C}' of \mathcal{C} is closed in X , and the union is determined by \mathcal{C}' . Every space is dominated by a hereditarily closure-preserving closed cover. If X is dominated by \mathcal{C} , then it is determined by \mathcal{C} . But the converse does not hold. We note that if X is determined by an increasing countable, closed cover \mathcal{C} , then X is dominated by \mathcal{C} . As is well-known, every CW-complex, more generally, chunk-complex in the sense of J. G. Ceder [3] is dominated by a cover of compact metric subsets.

Definition 1.2. We recall that a space X is *Fréchet* (= Fréchet-Urysohn), if whenever $x \in \bar{A}$, then there exists a sequence in A converging to the point x .

A space X is *strongly Fréchet* [25] (= countably bi-sequential in the sense of E. Michael [17]), if whenever (A_n) is a decreasing sequence accumulating at x in X (i.e., $\overline{A_n - \{x\}} \ni x$ for any $n \in \mathbb{N}$), then there exist $x_n \in A_n$ such that $\{x_n; n \in \mathbb{N}\}$ converges to the point x . First countable spaces are strongly Fréchet, strongly Fréchet spaces are Fréchet, and Fréchet spaces are sequential.

Definition 1.3. E. Michael [17] introduced the notion of bi- k -spaces (resp. countably bi- k -space), and he showed that every bi- k -space (resp. countably bi- k -space) is characterized as the bi-quotient (resp. countably bi-quotient) image of a paracompact M -space. Here, a space is a paracompact M -space if it admits a perfect map onto a metric space. A space X is a *bi- k -space* if, whenever a filter base F accumulating at x in X (i.e., $\bar{F} \ni x$ for any $F \in F$), then there exists a k -sequence (A_n) in X such that $x \in \overline{F \cap A_n}$ and $x \in A_n$ for all $n \in \mathbb{N}$ and all $F \in F$. Here (A_n) is a k -sequence if $K = \bigcap \{A_n; n \in \mathbb{N}\}$ is compact, and each nbd of K contains some A_n . Paracompact M -spaces and first countable spaces, more generally spaces of pointwise countable type [1] are bi- k -spaces. A space X is a *countably bi- k -space* [17] if, whenever (A_n) is a

decreasing sequence accumulating at x in X , then there exists a k -sequence (B_n) in X such that $x \in \overline{A_n \cap B_n}$ for any $n \in \mathbb{N}$. Strongly Fréchet spaces and bi- k -spaces are countably bi- k -spaces.

Definition 1.4. According to E. Michael [18], a space X is an *inner-one A-space*, if whenever (A_n) is a decreasing sequence accumulating at x in X , then there exists a non-closed subset $\{x_n; n \in \mathbb{N}\}$ of X with $x_n \in A_n$. Some properties of inner-one A-spaces and related spaces are investigated in [19]. Countably bi- k -spaces are inner-one A-spaces, but the converse does not hold.

We conclude this section by recording some elementary facts which will be often used later on. These are well-known, or easily proved. (4)(ii) is due to [13] or [20].

Proposition 1.5. (1) Let X be determined by $\{X_\alpha; \alpha \in A\}$. For each $\alpha \in A$, let $X_\alpha \subset Y_\alpha \subset X$. Then X is determined by $\{Y_\alpha; \alpha \in A\}$.

(2) Let X be determined by $\{X_\alpha; \alpha \in A\}$. If each X_α is determined by $\{X_{\alpha\beta}; \beta \in B\}$, then X is determined by $\{X_{\alpha\beta}; \alpha \in A, \beta \in B\}$.

(3) Let $f: X \rightarrow Y$ be a quotient map. If X be determined by C , then Y is determined by $\{f(C); C \in C\}$.

(4) (i) If X is determined by k -spaces; sequential spaces; c -spaces, then so is X respectively.

(ii) If X is dominated by paracompact spaces, then so is X .

2. k-ness of $X \times Y$

The following theorem is essentially proved in [29], but let us give a direct proof.

Theorem 2.1. *Let X be a c-space, and let Y be a bi-k-space. If $X \times Y$ is determined by $\{C \times Y; C \text{ is countably compact in } X\}$ (in particular, $X \times Y$ is a k-space), then X is an inner-one A-space, or Y is locally countably compact.*

Proof. Suppose that X is not inner-one A. Then there exist a point $p \in X$ and a decreasing sequence (A_n) with $p \in \overline{A_n - \{p\}}$ satisfying (*) below.

(*) For any $x_n \in A_n$, $\{x_n; n \in \mathbb{N}\}$ is closed in X .

Let $A_n' = A_n - \{p\}$, and let $A = \bigcap \{A_n'; n \in \mathbb{N}\}$.

Suppose A is not closed in X . Since X is a c-space, there exists a countable, non-closed subsets of A . Then the sequence (A_n) does not satisfy (*). This is a contradiction. Hence A is closed in X . Let $B_n = A_n' - A$. Then the sequence (B_n) satisfies (*) with $p \in \overline{B_n}$ and

$\bigcap \{B_n; n \in \mathbb{N}\} = \emptyset$. Since X is a c-space, there exists a countable $C_n \subset B_n$ with $p \in \overline{C_n}$ for each $n \in \mathbb{N}$. Moreover, suppose that Y is not locally countably compact. Then

for some point y in Y , $F = \{Y - K; K \text{ is countably compact in } Y\}$ is a filter base accumulating at y . Since Y is bi-k, there exists a k-sequence (E_n) such that $y \in \overline{E_n \cap F}$

for any $n \in \mathbb{N}$ and $F \in F$. Thus E_n is not countably compact.

Then there exists a sequence $\{D_n; n \in \mathbb{N}\}$ of pairwise

disjoint, closed discrete, and infinite countable subsets in Y such that $D_n \subset E_n$, $D_n \cap L = \emptyset$, where $L = \bigcap \{E_n; n \in \mathbb{N}\}$. Let $Z = (\bigcup \{D_n; n \in \mathbb{N}\} \cup L)$, and let $Z^* = Z/L$. Since Z is closed in Y , $X \times Z$ is determined by $\{C \times Z; C \text{ is countably compact in } X\}$. But Z^* is the perfect image of Z , so $X \times Z^*$ is the perfect image of $X \times Z$. Thus $X \times Z^*$ is determined by $\{C \times Z^*; C \text{ is countably compact in } X\}$ by Proposition 1.5(3).

Now, let $p = [L]$ in Z^* . Then any point except p is isolated in Z^* , and any nbd of p contains all D_n except finitely many D_n . Since the C_n are infinite countable sets, we can assume that for each $n \in \mathbb{N}$, $D_n = C_n$ (as a set) in Z^* . Let $S = \{(x, x); x \in C_n \text{ for some } n \in \mathbb{N}\}$. Then $(p, p) \in \bar{S} - S$ in $X \times Z^*$. Thus S is not closed in $X \times Z^*$. But S is closed in $X \times Z^*$. Indeed, let C be a countably compact subset of X . Then, by (*) it follows that there exists $n \in \mathbb{N}$ such that $\bigcup \{C_m; m \geq n\} \cap C$ is finite. Also, $\{C_n; n \in \mathbb{N}\}$ is point-finite. Thus we see that $S \cap (C \times Z^*)$ is closed in $C \times Z^*$. This implies that S is closed in $X \times Z^*$. This is a contradiction. Thus X is inner-one A , or Y is locally countably compact. (In particular, if $X \times Y$ is a k -space, then the cover of compact subsets of $X \times Y$ is a refinement of $\mathcal{C} = \{C \times Y; C \text{ is countably compact in } X\}$. Thus, by proposition 1.5(1), $X \times Y$ is determined by \mathcal{C} . Hence X is inner-one A , or Y is locally countably compact.)

In the previous theorem, the property "X is a c-space" is essential under (CH); see Remark 2.16(1).

Concerning necessary and sufficient conditions for the product $X \times Y$ to be a k-space, the following question is posed in view of Theorem 2.1.

Question 2.2. Let X be a k-space and c-space (in particular, let X be a sequential space). Let Y be a paracompact, bi-k-space.

(1) If X is an inner-one A-space, then $X \times Y$ is a k-space?

(2) If $X \times Y$ is a k-space, then X is a countably bi-k-space, or Y is locally compact?

If (1); or (2) is affirmative, then a characterization for the product $X \times Y$ to be a k-space is respectively "X is inner one A, or Y is locally compact;" or "X is countably bi-k, or Y is locally compact" by Theorem 2.1 or [28: Proposition 4.6].

We will give some affirmative answers to Question 2.2 for fairly general types of k-spaces. We recall that every k-space is precisely the quotient image of a paracompact, locally compact space ([6]). Here we can replace "locally compact space" by "bi-k-space" in view of [17]. In terms of this, first, let us consider the following conditions (C) and (Q).

(C) Closed image of a paracompact bi-k-space.

As modifications of (C), we shall also consider (C_0) stronger than (C), and (C_1) weaker than (C).

(C_0) Closed image of a locally compact paracompact space.

(C_1) Closed image of a countably bi-k-space.

(Q) Quotient Lindelöf image of a paracompact bi-k-space. Here a Lindelöf image denotes the image under a map with the pre-image of each point Lindelöf.

Every paracompact M-space X , as well as every closed image of X satisfies (C). In particular, every Lašnev space (= closed image of a metric space) satisfies (C). A characterization of closed images of paracompact M-spaces (resp. metric spaces) is given in [22] (resp. [7]). Every Fréchet space dominated by paracompact bi-k-spaces (resp. paracompact locally compact spaces) satisfies (C) (resp. (C_0)); see [35].

Because k - and k_0 -spaces are precisely the quotient images of separable metric spaces [14], they satisfy (Q). Because k_ω -spaces (resp. spaces determined by a point-countable cover of compact subsets) are precisely the quotient (resp. quotient Lindelöf) images of locally compact Lindelöf (resp. locally compact paracompact) spaces [21], they satisfy (Q).

Lemma 2.3. The following are equivalent.

- (1) X satisfies (C_0) .
- (2) X has a hereditarily closure-preserving cover of compact subsets.

(3) X satisfies (C) and the condition (*): Any closed subset of X , which is a paracompact M -space, is locally compact.

Proof. The equivalence (1) \Leftrightarrow (2) is known, or easy. Indeed, (1) \Rightarrow (2) is easy. For (2) \Rightarrow (1), let L be the topological sum of the compact subsets. Then X is the closed image of a paracompact locally compact space L , hence X satisfies (C_0) . The equivalence (1) \Leftrightarrow (3) is due to [33; Theorem 1.1].

We recall two canonical quotient spaces S_ω and S_2 . For $\alpha \geq \omega$, let S_α be the quotient space obtained from the topological sum of α convergent sequences by identifying all the limit points with a single point $\{\infty\}$ (in particular, S_ω is called the sequential fan). Let $S_2 = (N \times N) \cup N \cup \{0\}$ with each point of $N \times N$ isolated. A local base of $n \in N$ consists of all sets of the form $\{n\} \cup \{(m,n); m \geq m_0\}$, and U is a neighborhood of 0 if and only if $0 \in U$ and U is a neighborhood of all but finitely many $n \in N$ (S_2 is called the Arens' space).

Lemma 2.4. Let X be an inner-one A -space. Then each of the following conditions implies that X is a countably bi- k -space. Indeed, X is a metric space for (a); strongly Fréchet space for (b), (c), and (d); countably bi- k -space for (e); and bi- k -space (resp. locally compact space) for (f) and (g).

- (a) Lašnev space.
- (b) Fréchet space.
- (c) k -space in which every point is a G_δ -set.
- (d) Hereditarily normal, sequential space.
- (e) Space satisfying (C_1) .
- (f) Space satisfying (C) (resp. (C_0)).
- (g) c -space satisfying (Q) .

Proof. For (a), X is a metric space by [17; Corollary 9.10]. For (b), X is a strongly Fréchet space by [24; Theorem 5.1]. For (c), X is sequential by [17; Theorem 7.3]. While, any inner-one A -space contains no closed copy of S_ω and no S_2 . Then, for (c) and (d), a sequential space X is strongly Fréchet by [34; Theorem 3.1]. For (e), X is a countably bi- k -space by Theorem 6.3 and Proposition 2.4 in [19], and [17; p. 114]. For (f) and (g), X is a bi- k -space (resp. locally compact space) by Theorems 9.5 and 9.9, and Proposition 3.E.4 in [17].

Let $\{X_\alpha; \alpha < \gamma\}$ be a cover of X . For each $\alpha < \gamma$, let $L_0 = X_0$, $L_\alpha = X_\alpha - \cup\{X_\beta; \beta < \alpha\}$, and $X_\alpha^* = \text{cl } L_\alpha$. We will use these notations.

Lemma 2.5. Let X be dominated by $\{X_\alpha; \alpha < \gamma\}$. Then the following hold.

- (1) X is determined by $\{X_\alpha^*; \alpha < \gamma\}$.
- (2) Let A be a subset of X . For each $\alpha < \gamma$, let B_α be a subset of L_α such that $A \cup B_\alpha$ is closed in X , and let $B = \cup\{B_\alpha; \alpha < \gamma\}$. Then $S = A \cup B$ is closed in X .

(3) If $F_\alpha \subset L_\alpha$ is finite for each $\alpha < \gamma$, then $D = \cup\{F_\alpha; \alpha < \gamma\}$ is closed and discrete in X .

Proof. (1) For $F \subset X$, let $F \cap X_\alpha^*$ be closed in X_α^* for each $\alpha < \gamma$. Then $F \cap X_0$ is closed in X_0 . Suppose that $F \cap X_\alpha$ is closed in X_α for each $\alpha < \delta$. Let $F_\delta = (F \cap X_\delta) \cup \{X_\alpha; \alpha < \delta\}$. Then $F_\delta \subset \cup\{X_\alpha; \alpha < \delta\}$, and $F_\delta \cap X_\alpha = (F \cap X_\alpha) \cap X_\delta$ is closed in X_α for each $\alpha < \delta$. Then F_δ is closed in X . While $F \cap X_\delta = F_\delta \cup (F \cap X_\delta^*)$. Thus, since $F \cap X_\delta^*$ is closed in X_δ , $F \cap X_\delta$ is closed in X_δ . Hence, by induction, $F \cap X_\alpha$ is closed in X_α for each $\alpha < \gamma$. Then F is closed in X .

(2) $S \cap X_0 = (A \cap X_0) \cup (B \cap X_0) = ((A \cup B_0) \cap X_0)$. Then $S \cap X_0$ is closed in X_0 . Suppose that $S \cap X_\alpha$ is closed in X_α for each $\alpha < \delta$. Let $E_\delta = (S \cap X_\delta) \cap \cup\{X_\alpha; \alpha < \delta\}$. Then as is seen, E_δ is closed in X . While $S \cap X_\delta = (A \cap X_\delta) \cup (B \cap L_\delta) \cup E_\delta = ((A \cup B_\delta) \cap X_\delta) \cup E_\delta$. Thus $S \cap X_\delta$ is closed in X_δ . Hence, by induction, $S \cap X_\alpha$ is closed in X_α for each $\alpha < \gamma$. Then S is closed in X .

(3) In (2), putting $A = \emptyset$ and $B_\alpha = F_\alpha$, we see that any subset of D is closed in X . Hence D is closed and discrete in X .

Lemma 2.6. Let X be a c -space dominated by a cover (or determined by a point-countable cover). If X is an inner-one A -space, then each point of X has a neighborhood which is contained in a finite union of elements of the cover.

Proof. First, we show the parenthetic part. Suppose that the assertion does not hold. Let X be

determined by a point-countable cover \mathcal{C} . For each countable subset A of X , let $\{P_n(A); n \in \mathbb{N}\} = \{C \in \mathcal{C}; C \cap A \neq \emptyset\}$. Then, since X is a c -space, for some $x \in X$ there exists a sequence $\{B_n; n \in \mathbb{N}\}$ of countable subsets of X such that $\{x\} = B_1$, $x \in \overline{B_n}$, and $B_n \cap P_i(B_j) = \emptyset$ if $i, j < n$. We note that any element of \mathcal{C} meets only finitely many B_n . Let $D_n = \cup\{B_i; i \geq n\}$ for each $n \in \mathbb{N}$. Then (D_n) is a decreasing sequence accumulating at x . Since X is inner-one A , there exists a non-closed subset $D = \{x_n; n \in \mathbb{N}\}$ with $x_n \in D_n$. Thus there exists $C_0 \in \mathcal{C}$ such that $C_0 \cap D$ is not closed in \mathcal{C} . Hence C_0 meets infinitely many B_n . This is a contradiction. The parenthetic part holds.

Now, let X be dominated by $\{X_\alpha; \alpha < \gamma\}$. Suppose that $\{X_\alpha^*; \alpha < \gamma\}$ is not point-finite. Then, for some $x \in X$, there exists an infinite sequence $\{X_{\alpha_n}^*; n \in \mathbb{N}\}$ with $X_{\alpha_n}^* \ni x$. Let $B_n = \cup\{L_{\alpha_n}; n \in \mathbb{N}\} - \cup\{L_{\alpha_i}; i \leq n\}$. Then (B_n) is a decreasing sequence accumulating at x . Since X is inner-one A , there exists a non-closed subset $B = \{x_n; n \in \mathbb{N}\}$ with $x_n \in B_n$. But $B \cap L_\alpha$ is finite for each $\alpha < \gamma$. Then B is closed discrete in X by Lemma 2.5(3). This is a contradiction. Thus $\{X_\alpha^*; \alpha < \gamma\}$ is point-finite. But, by Lemma 2.5(1) X is determined by $\{X_\alpha^*; \alpha < \gamma\}$. Thus the lemma holds by the parenthetic part.

Lemma 2.7. Each of the following conditions implies that $X \times Y$ is a k -space.

- (a) *X is a k -space, and Y is locally compact ([6]).*
- (b) *X is a sequential space, and Y is a locally countably compact, sequential space ([2]).*
- (c) *X is a countably bi- k -space, and Y is a bi- k -space ([28]).*
- (d) *X and Y are locally k_ω -spaces (cf. [16]).*
- (e) (MA). *X is locally $< k_c$, where $c = 2^\omega$, and Y is a locally k_ω -space each of whose compact set is metric (cf. [35]).*

Now we give some partial answers to Question 2.2.

Theorem 2.8. Let X be dominated by spaces satisfying one of the following conditions (in particular, let X be a Lašnev space, CW-complex, or the quotient s -image of a metric space). Let Y be a paracompact bi- k -space (resp. let Y be a sequential, bi- k -space).

- (a) *Fréchet space.*
- (b) *k -space in which every point is a G_δ -set.*
- (c) *Hereditarily normal, sequential space.*
- (d) *c -space (resp. sequential space) satisfying (C_1) or (Q) .*

Then the following are equivalent.

- (1) *$X \times Y$ is a k -space.*
- (2) *X is a countably bi- k -space, or Y is locally compact (resp. locally countably compact).*

(3) X is an inner-one A -space, or Y is locally compact (resp. locally countably compact).

Proof. We note that each of (a) \sim (d) (resp. parenthetic part of (d)) implies that X is a c -space (resp. sequential space) by Proposition 1.5(4)(i). (1) \Rightarrow (3) follows from Theorem 2.1. (3) \Rightarrow (2) follows from Lemmas 2.4 & 2.6, and the fact every locally countably bi- k -space is countably bi- k . (2) \Rightarrow (1) follows from Lemma 2.7.

Lemma 2.9. (1) Let X be determined by $\{X_\alpha; \alpha \in A\}$. Let γ be a regular cardinal with $\gamma \geq \omega_1$. For any $\alpha \in A$, suppose that $\{\beta \in A; X_\alpha \cap X_\beta \neq \emptyset\}$ has cardinality $< \gamma$. Then X is the topological sum of $\{T_d; d \in D\}$ such that each T_d is determined by $\{X_\alpha; \alpha \in A_d\}$, where $A_d \subset A$ has cardinality $< \gamma$.

(2) A space X is a paracompact locally k_ω -space if and only if it is determined by countably many locally compact paracompact, closed subsets (equivalently, X is the topological sum of k_ω -spaces).

Proof. (1) This can be proved by the same method as in the proof of (a) \Rightarrow (c) and (d) in [32; Theorem 1].

(2) For the "if" part, let X be determined by locally compact paracompact, closed subsets X_n ($n \in \mathbb{N}$). Since each X_n is paracompact, it is determined by a locally finite cover $\{X_{n\alpha}; \alpha \in A\}$ of compact subsets. Then by Proposition 1.5(2), X is determined by a cover $\mathcal{C} = \{X_{n\alpha}; n \in \mathbb{N}, \alpha \in A\}$ of compact subsets. But any element of \mathcal{C} meets only countably many members. Thus X is the

topological sum of k_ω -spaces by (1). Then X is a paracompact locally k_ω -space. For the "only if" part, X has a locally finite closed cover $\{K_\alpha; \alpha \in A\}$ of k_ω -spaces. Each K_α can be dominated by an increasing countable cover $\{K_{\alpha n}; n \in \mathbb{N}\}$ of compact subsets. Let $L_n = \cup\{K_{\alpha i}; \alpha \in A, i \leq n\}$ for each $n \in \mathbb{N}$. Then each compact subset of X is contained in some L_n by means of Lemma 2.5(3). But X is a k -space, for it is a locally k -space. Thus, by Proposition 1.5(1), X is determined by a closed cover $\{L_n; n \in \mathbb{N}\}$ of locally compact paracompact, closed subsets.

We obtained a characterization for $X \times Y$ to be a k -space if X is various kinds of k -spaces and Y is a bi- k -space (Theorem 2.8). Next, let us consider the k -ness of $X \times Y$ if Y is not necessarily a bi- k -space.

First, when X and Y are dominated by countably many certain bi- k spaces, we have the following theorem.

Theorem 2.10. Let X and Y be dominated by countably many paracompact bi- k , and c -spaces (in particular, let X and Y be dominated by countably many metric spaces). Then $X \times Y$ is a k -space if and only if one of the following properties holds.

- (a) X or Y is locally compact.
- (b) X and Y are bi- k -spaces.
- (c) X and Y are locally k_ω -spaces.

Proof. The "if" part holds by Lemma 2.7. We prove the "only if" part. Note that X and Y are c -spaces by

Proposition 1.5(4), and that every locally bi-k-space is bi-k. Thus, if $X \times Y$ is a k-space, by Theorem 2.1 and Lemma 2.6, (a) and (b) hold, otherwise X and Y are dominated by countably many locally compact paracompact spaces. The last property implies (c) holds by Lemma 2.9(2).

Second, when X and Y are dominated by (not necessarily countably many) certain Fréchet spaces, the assertion of the previous theorem is equivalent to a certain set-theoretic axiom weaker than (CH). We will show this.

Lemma 2.11. *Let X be dominated by Fréchet spaces satisfying (C). If X contains no closed copy of S_ω and no S_2 , then X is a bi-k-space.*

Proof. Let X be dominated by $\{X_\alpha; \alpha \in A\}$, where each X_α is Fréchet. Since any X_α contains no closed copy of S_2 , X is Fréchet in view of the proof of [34; Theorem 2.1(a)]. But X contains no closed copy of S_ω . Then, by [26; p. 31] X is strongly Fréchet, and so is each X_α . Then X is a bi-k-space by Lemmas 2.4 and 2.6, and the fact that every locally bi-k-space is bi-k.

Lemma 2.12. *Let X be dominated by c-spaces satisfying (C). Let Y be a space satisfying (C). If $X \times Y$ is a k-space, then X is a bi-k-space, or Y has a hereditarily closure-preserving cover of compact subsets.*

Proof. Suppose X is not a bi-k-space. Then X is not an inner-one A-space by Lemmas 2.4 and 2.6. Then, by

Theorem 2.1, Y satisfies $(*)$ in Lemma 2.3(3). Then, by Lemma 2.3, Y has a hereditarily closure-preserving cover of compact subsets.

Let F be the set of all functions from N to N . For $f, g \in F$, we define $f \geq g$ if $\{n \in N; f(n) < g(n)\}$ is finite. Let $b = \min \{\gamma; \text{there exists an unbounded family } A \subset F \text{ with cardinality } \gamma\}$. By $BF(\alpha)$, we mean " $b \geq \alpha$ ". It is well-known that (MA) implies " $b = c$ ".

The following lemma is due to G. Gruenhagen [10].

Lemma 2.13. (1) $S_{\omega_1} \times S_{\omega_1}$ is not a k -space.
 (2) $S_\alpha \times S_\omega$ is a k -space if and only if $BF(\alpha+)$ holds, where $\alpha+$ means the least cardinal greater than α .

Now, we show that the assertion of Theorem 2.10 is equivalent to " $BF(\omega_2)$ is false" if X and Y are dominated by certain Fréchet spaces.

Theorem 2.14. Let X and Y be dominated by Fréchet spaces satisfying (C) . Then the following (1) and (2) are equivalent. When $X = Y$, the assertion (2) holds; that is, X^2 is a k -space if and only if X is a bi- k -space, or a locally k_ω -space.

(1) $BF(\omega_2)$ is false.
 (2) $X \times Y$ is a k -space if and only if one of the properties (a), (b) and (c) in Theorem 2.10 holds.

Proof. (2) \Rightarrow (1). Any S_α is dominated by compact metric spaces. But S_ω is neither bi- k nor locally

compact, and S_{ω_1} is not locally k_ω . Then $S_{\omega_1} \times S_\omega$ is not a k -space by the "only if" part of (2). Thus $BF(\omega_2)$ is false by Lemma 2.13(2).

(1) \Rightarrow (2). The "if" part holds by Lemma 2.7. We prove the "only if" part. Let X be dominated by $\{X_\alpha; \alpha < \gamma\}$, where each X_α is a Fréchet space satisfying (C). First, suppose that one of X and Y is bi- k , but another is not bi- k . We can assume that X is bi- k , but Y is not bi- k . Since each $X_\alpha \times Y$ is closed in a k -space $X \times Y$, $X_\alpha \times Y$ is a k -space. Then, by Lemma 2.12, each X_α is dominated by a cover of compact subsets. Since each X_α is bi- k , by Lemma 2.6 X_α is locally compact. Then, a bi- k -space X is locally compact by Lemma 2.6.

Next suppose that neither X nor Y is bi- k . Since each $X_\alpha^* \times Y$ is closed in a k -space $X \times Y$, $X_\alpha^* \times Y$ is a k -space. But each X_α^* is a space satisfying (C). Then, by Lemma 2.12, X_α^* has a hereditarily closure-preserving cover $\{C_{\alpha\beta}; \beta < \delta_\alpha\}$ of compact subsets. Hence X_α^* is dominated by this cover. Let $C_\alpha = \{C_{\alpha\beta}^*; \beta < \delta_\alpha\}$ for each $\alpha < \gamma$, and $C = \cup\{C_\alpha; \alpha < \gamma\}$. Suppose that for some $C \in \mathcal{C}$, $\Lambda = \{\alpha; C \cap X_\alpha^* \neq \emptyset\}$ has cardinality $\zeta \geq \omega_1$. Let $x_\alpha \in C \cap X_\alpha^*$ for each $\alpha \in \Lambda$. Note that if $p_\alpha \in L_\alpha$ for $\alpha \in \Lambda$, $\{p_\alpha; \alpha \in \Lambda\}$ is closed, discrete in X by Lemma 2.5(3). Then, since C is compact, we can assume that $L_\alpha \cap C = \emptyset$ for any $\alpha \in \Lambda$. For each $\alpha \in \Lambda$, since $x_\alpha \in X_\alpha^*$ and X_α is Fréchet, there exists an infinite sequence A_α in L_α converging to $x_\alpha \in C$ with $A_\alpha \cap C = \emptyset$. Let

$S = C \cup \cup\{A_\alpha; \alpha \in \Lambda\}$, and let T be the quotient space obtained from S by identifying all the points of C . Then S is closed in X , and T is a copy of S_ζ by Lemma 2.5(2) and (3). Now, since Y is not bi- k , Y contains a closed copy S' of S_ω or S_2 by Lemma 2.11. Since $S \times S'$ is a closed subset of a k -space $X \times Y$, it is a k -space. But S_ζ is the perfect image of S , and similarly S_ω is the perfect image of S_2 . Then $S_\zeta \times S_\omega$ is the perfect (hence quotient) image of a k -space $S \times S'$. Thus $S_\zeta \times S_\omega$ is a k -space, for every quotient image of a k -space is a k -space. But $BF(\omega_2)$ is false. Then $\zeta < \omega_1$ by Lemma 2.13(2). This is a contradiction. Then any $C \in \mathcal{C}$ meets only countably many X_α^* . Besides, for any $C \in \mathcal{C}'$ and for any $\alpha < \gamma$, a compact subset $C \cap X_\alpha^*$ of X meets only countably many elements of C_α in view of the above arguments. Then any $C \in \mathcal{C}$ meets only countably many elements of \mathcal{C} . While, by Proposition 1.5(2) and Lemma 2.5(1), it follows that X is determined by the cover \mathcal{C} of compact subsets. Thus X is locally k_ω by Lemma 2.9(2). Similarly, Y is also locally k_ω . Therefore the "only if" part holds. When $X = Y$, using Lemma 2.13(1), the assertion (2) holds by the same way as in the above.

Remark 2.15. In the following, (2) is a generalization of Theorem 1.6 in [35], where $X = S_C$ and Y is dominated by metric spaces.

(1) Let X and Y be dominated by Fréchet spaces satisfying (C). If $X \times Y$ is a k -space, then X or Y is a

locally k_ω -space (equivalently, X or Y is the topological sum of k_ω -spaces), otherwise X and Y are bi- k -spaces.

(2) Let X be a Fréchet space satisfying (C), but is not locally $< k_c$. Let Y be dominated by Fréchet space satisfying (C). Then $X \times Y$ is a k -space if and only if Y is locally compact, otherwise X and Y are bi- k -spaces.

Indeed, note that neither $S_{\omega_1} \times S_{\omega_1}$ nor $S_c \times S_\omega$ is a k -space by Lemma 2.13. Then (1) holds by the same way as in the proof of (1) \Rightarrow (2) of Theorem 2.14. For (2), suppose that Y is not bi- k . Then, in view of the proof of (1) \Rightarrow (2) of Theorem 2.14, X has a hereditarily closure-preserving cover \mathcal{C} by compact subsets such that for each $x \in X$, $\{C \in \mathcal{C}; x \in C\}$ has cardinality $< c$. Then X is locally $< k_c$. This is a contradiction. Thus Y is a bi- k -space. Hence, (2) holds in view of the proof of Theorem 2.14.

Remark 2.16. (1) In Theorem 2.14, the property "each piece is Fréchet" is essential. In Theorem 2.1 (resp. Theorems 2.8 and 2.10), the property " X is a c -space" (resp. "each piece is a c -space") is essential under (CH).

Indeed, quite recently Chen Huaipeng [5] showed that, under (CH), there exists a k_ω -space X satisfying (C_0) such that X^ω is a k -space, but X is not locally compact. Since X is not countably compact, X contains a closed copy of N . Let Y be N^ω , and let Z be the topological sum $X + Y$ of X

and Y . Then Z is dominated by countably many paracompact M -spaces. Since $X \times Y$ is a closed subset of X^ω , it is a k -space. Then $Z \times Z$ is a k -space. X satisfies (C), but it is not locally compact. Then X is not an inner-one A -space by Lemma 2.4, hence not a bi- k -space. Y is a metric space, but not locally compact. Then Y is not locally k_ω by Lemma 2.6. Then Z is neither bi- k nor locally k_ω . Then (1) (indeed, (CH)) \Rightarrow (2) in Theorem 2.14 does not hold if we omit the Fréchet-ness of each piece. The space Z is also the desired space for the latter part.

(2) In Theorem 2.14, the condition (C) of each Fréchet piece is essential.

Indeed, G. Gruenhage [9] showed that, under (MA), there exists a countable, Fréchet space X such that any finite product X^n is Fréchet, but the countable product X^ω is not Fréchet. Thus X^ω is not even a k -space by Theorem 3.1(2) in the next Section. Then X is not a bi- k -space. Because any countable product of bi- k -spaces is a bi- k -space by [17; Proposition 3.E.4], hence a k -space. Now, X^2 is Fréchet. Then X is strongly Fréchet in view of the proof of Proposition 4.E.4 in [17]. But X is not locally compact. Then X is not locally k_ω by Lemma 2.6. Hence, (1) (indeed, (CH)) \Rightarrow (2) in Theorem 2.14 does not hold if we omit the condition (C).

In view of Theorem 2.14 and Remark 2.16, the author has the following question.

Question 2.17. The equivalence (1) \Leftrightarrow (2) in Theorem 2.14 holds under case (a) or (b) below?

(a) X and Y are dominated by compact (or compact sequential) spaces.

(b) X and Y are determined by point-countable covers of compact (or compact metric) spaces.

As for case (b), we note that if X and Y are k' -spaces (in the sense of [1]) determined by point-countable covers of compact spaces, then $X \times Y$ is a k -space in view of [4]. But the property " X and Y are k' -spaces" is essential even if $X = Y$ and X is a paracompact space determined by a point-finite cover of compact metric spaces; see [32; Example 3].

In the following corollary, (A) is a generalization for cases " X and Y are Lašnev spaces ([10])" and " X and Y are CW-complexes ([31]), more generally they are closed images of CW-complexes ([36])". Note that every closed image of a CW-complex is dominated by compact metric spaces, but not every Lašnev space is dominated by metric spaces [37]. The latter half of (A) is also a generalization of Theorem 1.7 (1) in [35], where $X = Y$ and X is a Fréchet space dominated by metric spaces.

Corollary 2.18. Let X and Y be dominated by Lašnev spaces.

(A) The following (1) and (2) are equivalent. When $X = Y$, the assertion (2) holds; that is, X^2 is a k -space if and only if X is metric, or locally k_ω .

(1) $BF(\omega_2)$ is false.

(2) $X \times Y$ is a k -space if and only if one of the following holds.

(a) X or Y is locally compact metric.

(b) X and Y are metric.

(c) X and Y are locally k_ω .

(B) (MA). $X \times Y$ is a k -space if and only if one of the following holds. When $X = Y$, we can omit (MA).

(a) X or Y is locally compact metric.

(b) X and Y are metric.

(c) One of X and Y is locally k_ω , and another is locally $< k_c$.

Proof. (A) Since X and Y are dominated by paracompact spaces, they are paracompact by Proposition 1.5(4). But, by Lemmas 2.4 and 2.6, every bi- k -space dominated by Lašnev spaces is locally metric. While every paracompact, locally metric space is metric. Hence if X is a bi- k -space, it is metric. Thus (1) \Leftrightarrow (2) holds by Theorem 2.14.

(B) From the above, every compact subset of X and Y is metric. Then the "if" part holds by Lemma 2.7. For the "only if" part, suppose that neither (a) nor (b) holds. Then neither X nor Y is a bi- k -space in view of the proof of (1) \Rightarrow (2) of Theorem 2.14. We note that neither $S_{\omega_1} \times S_{\omega_1}$ nor $S_c \times S_\omega$ is a k -space by Lemma 2.13, and the cardinal c is regular under (MA). Then using Lemma 2.9(1), X or Y is locally k_ω , and both X and Y are locally $< k_c$

by the same way as in the proof of (1) \Rightarrow (2) of Theorem 2.14. Then (c) holds.

We shall say that a space X is *locally α -compact* if each point of X has a neighborhood whose closure is α -compact. Here a space is α -compact if any subset of with cardinality α has an accumulation point.

Lemma 2.19. *Let X be dominated by a cover \mathcal{C} of compact subsets. Then for an infinite regular cardinal γ , X is locally γ -compact if and only if it is locally $< k_\gamma$.*

Proof. The "if" part is easily proved. For the "only if" part, for $x \in X$, let $V(x)$ be a neighborhood of x such that $\overline{V(x)}$ is γ -compact. Then by Lemma 2.5(3), $\overline{V(x)} \subset \cup \mathcal{C}'$ for some $\mathcal{C}' \subset \mathcal{C}$ with cardinality $< \gamma$. Then $\overline{V(x)}$ is determined by a cover $\{\overline{V(x)} \cap C; C \in \mathcal{C}'\}$ of compact subsets, with cardinality $< \gamma$. This implies that X is locally $< k_\gamma$.

We have the following by Corollary 2.18(B) and Lemma 2.19.

Corollary 2.20. (MA). *Let X and Y be dominated by compact metric spaces. Then the following are equivalent. When $X = Y$, we can omit (MA).*

- (1) $X \times Y$ is a k -space.
- (2) X or Y is locally compact metric, otherwise one of X and Y is locally k_ω and another is locally $< k_c$.

(3) X or Y is locally compact metric, otherwise one of X and Y is locally ω_1 -compact and another is locally c -compact.

Lemma 2.21. Let X be dominated by a cover C of compact c -spaces, and let Y have the same property. Then $X \times Y$ is a k -space if and only if it is a c -space.

Proof. "If": Since $X \times Y$ is a c -space, it is determined by countable subsets of $X \times Y$. Then, by Proposition 1.5(1), $X \times Y$ is determined by $G = \{\bar{D} \times \bar{E}; D \text{ and } E \text{ are countable}\}$. Let D be a countable subset of X . Since X is dominated by C , $\bar{D} \subset \cup C'$ for some countable $C' \subset C$. Then \bar{D} is determined by a countable cover $\{\bar{D} \cap C; C \in C'\}$ of compact subsets. Hence \bar{D} is a k_ω -space. Similarly, any separable closed subset of Y is a k_ω -space. Then any element of G is a k -space by Lemma 2.7. Thus $X \times Y$ is a k -space by Proposition 1.5(4).

"Only if": Since $X \times Y$ is a k -space, it is determined by $H = \{K \times L; K \text{ and } L \text{ are compact}\}$. But by Proposition 1.5(4), X and Y are c -spaces, then so is any compact subset of X and Y . Thus, by [12; Theorem 4], any element of H is a c -space. Then $X \times Y$ is a c -space.

Lemma 2.22. Let X be dominated by $C = \{X_\alpha; \alpha \in A\}$, and let Y be dominated by $\mathcal{D} = \{Y_n; n \in \mathbb{N}\}$ with $Y_n \subset Y_{n+1}$ ($n \in \mathbb{N}$). If $X \times Y$ is a k -space, then it is dominated by $F = \{X_\alpha \times Y_n; \alpha \in A, n \in \mathbb{N}\}$.

Proof. Since C and \mathcal{D} are closure-preserving closed covers, so is F in $X \times Y$. Let F' be any subcollection of

F , and let $S = \cup F'$. Since S is closed in a k -space $X \times Y$, S is also a k -space. Besides, any element of F' is closed in $X \times Y$. Then, to show S is determined by F' , it suffices to show that each compact subset of S is contained in a finite union of elements of F' . For each $n \in \mathbb{N}$, let $A_n = \cup\{X_\alpha; X_\alpha \times Y_n \in F'\}$ (if there are no $X_\alpha \times Y_n \in F'$, let $A_n = \emptyset$), $B_n = \cup\{A_i; i \geq n\}$, and let $L_n = Y_n - Y_{n-1}$, $Y_0 = \emptyset$. Then $S = \cup\{A_n \times Y_n; n \in \mathbb{N}\} = \cup\{B_n \times L_n; n \in \mathbb{N}\}$. Let C be a compact subset of S . Then there exist compact subsets K in X and L in Y such that $C \subset K \times L$. Note that the compact set L meets only finitely many L_n by Lemma 2.5(3). Let $m = \text{Max}\{n \in \mathbb{N}; L \cap L_n \neq \emptyset\}$. Then $C \subset (K \times L) \cap S \subset \cup\{B_n \times Y_n; n \leq m\}$. Now, each B_n is dominated by $G_n = \{X_\alpha; X_\alpha \times Y_i \in F', i \geq n\}$. Then each compact subset of B_n is contained in a finite union of elements of G_n by Lemma 2.5(3). Then each compact subset of $B_n \times Y_n$ is contained in a finite union of elements of $H_n = \{X_\alpha \times Y_n; X_\alpha \in G_n\}$, in particular, so is a compact subset $(K \times L) \cap (B_n \times Y_n)$ of $B_n \times Y_n$. Hence $(K \times L) \cap S$ is contained in a finite union of elements of $\cup\{H_n; n \leq m\}$, hence so is C . But $Y_n \subset Y_{n+1}$ for each $n \in \mathbb{N}$. Then C is contained in a finite union of elements of F' . Hence each compact subset of S is contained in a finite union of elements of F' .

For spaces X and Y dominated by compact metric spaces, let us give equivalent properties to the k -ness of $X \times Y$.

We note that $S_\omega \times Q$, Q is the rationals, is a c -space, but is not a k -space. Thus the compactness of the metric piece is essential in (2) \Rightarrow (1) of the following theorem.

Theorem 2.23. *Let X and Y be dominated by compact metric spaces. Then the following are equivalent.*

- (1) $X \times Y$ is a k -space.
- (2) $X \times Y$ is a c -space.
- (3) $X \times Y$ is dominated by compact metric spaces.

Proof. (1) \Leftrightarrow (2) follows from Lemma 2.21. (3) \Rightarrow (1) follows from Proposition 1.5(4). We prove (1) \Rightarrow (3). We note that if X is bi- k , by Lemma 2.6, X is locally compact, hence is locally k_ω . Then, in view of Remark 2.15(1), X or Y is the topological sum of k_ω -spaces. Then, using Lemma 2.22, we show that $X \times Y$ is dominated by compact spaces. But any compact subset of X and Y is metric. Then (3) holds.

3. k -ness of X^ω

Theorem 3.1. (1) *Let X^ω be a k -space with X a c -space. Then X^ω is an inner-one A -space.*

(2) *Let X be dominated by $\{X_\alpha; \alpha \in A\}$, where every point of X_α is a G_δ -set in X_α . Then X^ω is a k -space if and only if it is strongly Fréchet.*

(3) *Let X be dominated by c -spaces satisfying (C) or (Q). Then X^ω is a k -space if and only if it is a bi- k -space (equivalently, X is a bi- k -space).*

Proof. (1) Let $Y = X^\omega$. Since Y is a k -space with X a c -space, using Remark 3 in [12], Y is a c -space by the same way as in the proof of the "only if" part of Lemma 2.19. Every countably compact space is inner-one A. Then, to show that Y is inner-one A, let Y be not countably compact. Then Y contains a closed copy of N . Then X^ω , which is homeomorphic to $Y \times Y^\omega$, contains a closed copy of $Y \times N^\omega$. But $Y \times N^\omega$ is a k -space with Y a c -space. Then Y is an inner-one A-space by Theorem 2.1.

(2) Since every strongly Fréchet space is a k -space, it suffices to prove the "only if" part. Using Lemma 2.5(3), any compact or separable subset S of X is covered by countably many X_α . Then every point of S is a G_δ -set in X . Hence every point of a compact or separable subset of $Y = X^\omega$ is a G_δ -set in Y . Then any compact subset of Y is first countable. Thus a k -space Y is sequential hence a c -space. Then, to show Y is strongly Fréchet, by [17; Proposition 8.7] it suffices to show that any countable subset A of Y is strongly Fréchet. Let $B = \bar{A}$. Since B is a separable closed subset of Y , B is a sequential space in which every point is a G_δ -set. But B is an inner-one A-space by means of (1). Thus B is strongly Fréchet by Lemma 2.4. Hence A is strongly Fréchet.

(3) For the "if" part, we note that every product of countably many bi- k -spaces is a bi- k -space [17], hence a k -space. For the "only if" part, since X is a c -space, X is an inner-one A-space by (1). Thus the "only if" part follows from Lemmas 2.4 and 2.6.

Remark 3.2. (1) In Theorem 3.1(1) (resp. Theorem 3.1(3)), the property "X is a c-space" (resp. "each piece is a c-space") is essential under (CH).

Indeed, let us consider the space $Z = X + N^\omega$ in Remark 2.16(1). $X^\omega \times N^\omega$ is closed in $X^\omega \times X^\omega$, and $X^\omega \times X^\omega$ is homeomorphic to a k-space X^ω . Then $X^\omega \times N^\omega$ is a k-space. Thus Z^ω is a k-space. Then Z is the desired space in view of Remark 2.16(1).

(2) The condition (C) or (Q) in Theorem 3.1(3) is essential under (CH).

Indeed, T. Nogura [23] showed that, under (CH), there exists a countable Fréchet space X such that X^ω is Fréchet (hence strongly Fréchet), but X is not a bi-sequential space (in the sense of [17; 3D]). Then X is not a bi-k-space, because any b-k-space in which every point is a G_δ -set is bi-sequential by [17; Theorem 7.3].

In view of Theorem 3.1 and Remark 3.2, the author has the following question.

Question 3.3. Let X^ω be a k-space with X a c-space. Then X is a countably bi-k-space (equivalently, X^ω is countably bi-k)?

In the following corollary, (1) is a generalization of Theorem 1.7 in [35], where X is a Fréchet space dominated by metric spaces.

Corollary 3.4. Let X be dominated by Lašnev spaces. Then the following hold.

(1) For $n \geq 2$, X^n is a k -space if and only if X is metric or locally k_ω .

(2) X^ω is a k -space if and only if X is metric.

Proof. For (1), the "only if" part follows from Corollary 2.17(A). For the "if" part, note that every finite product of locally k_ω -spaces is locally k_ω by means of [16; (7.5)]. Then each X^n is a locally k -space, hence is a k -space. For (2), note that if X is a bi- k -space, then it is metric as in the proof of Corollary 2.17(A). Thus (2) follows from Theorem 3.1(2) or (3).

References

- [1] A. V. Arhangel'skii, *Biocompact sets and the topology of spaces*, Trans. Moscow Math. Soc., 13(1965), 1-65.
- [2] T. K. Boehme, *Linear s-spaces*, Proc. Symp. Convergence Structures, Univ. of Oklahoma, 1965.
- [3] J. G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math., 4(1961), 105-126.
- [4] Chen Huaipeng, *Products of k' -spaces with point-countable k -systems*, Kexue Tongbao (= Science Bulletin), 31(1986).
- [5] _____, *A conjecture on the countable products of k -spaces*, to appear in Proc. Amer. Math. Soc.
- [6] C. H. Cohen, *Spaces with weak topology*, Quart. J. Math. Oxford Ser., (2) 5(1954), 77-80.
- [7] L. Foged, *A characterization of closed images of metric spaces*, Proc. Amer. Math. Soc., 95(1985), 487-490.
- [8] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math., 57(1965), 107-115.
- [9] G. Gruenhage, *A note on the product of Fréchet spaces*, Topology Proc., 3(1978), 109-115.

- [10] _____, *k*-spaces and products of closed images of metric spaces, Proc. Amer. Math. Soc., 80(1980), 478-482.
- [11] G. Gruenhage, E. Michael and Y. Tanaka, Spaces determined by point-countable covers, Pacific J. Math., 113(1984), 303-332.
- [12] V. I. Malyhin, On tightness and Suslin number in $\exp X$ and in a product of spaces, Soviet Math. Dokl., 13(1972); 496-499.
- [13] E. Michael, Continuous selections I, Ann. Math., 63(1956), 361-382.
- [14] _____, \aleph_0 -spaces, J. Math. Mech., 15(1966), 983-1002.
- [15] _____, Local compactness and cartesian products of quotient maps and *k*-spaces, Ann. Inst. Fourier (Grenoble), 18(1968), 281-286.
- [16] _____, Bi-quotient maps and cartesian products of quotient maps, *ibid*, 287-302.
- [17] _____, A quintuple quotient quest, General Topology and Appl., 2(1972), 91-138.
- [18] _____, Countably bi-quotient maps and A-spaces, Topology Conf. Virginia Polytech. Inst. and State Univ., 1973, 183-189.
- [19] E. Michael, R. C. Olson and F. Siwiec, A-spaces and countably bi-quotient maps, Dissertations Mathematicae (Warszawa), 133(1976), 4-43.
- [20] K. Morita, On spaces having the weak topology with respect to closed coverings II, Proc. Japan Acad., 30(1954), 711-717.
- [21] _____, On decomposition spaces of locally compact spaces, *ibid*, 32(1956), 544-548.
- [22] K. Morita and T. Rishel, Results related to closed images of M-spaces, I, II, Proc. Japan Acad., Supplements to 47(1971), 1004-1007, 1008-1011.

- [23] T. Nogura, *Products of Fréchet spaces*, General Topology and its Relations to Modern Analysis and Algebra VI, Proc. Sixth Prague Topological Symposium (1986), Z. Frolik (ed.), 371-378.
- [24] C. Olson, *Bi-quotient maps, countably bi-sequential spaces, and related topics*, General Topology and Appl., 4(1974), 1-28.
- [25] F. Siwiec, *Sequence-covering and countably bi-quotient mappings*, General Topology and Appl., 8(1971), 143-154.
- [26] _____, *Generalizations of first axiom of countability*, Rocky Mountain J. Math., 5(1975), 1-60.
- [27] Y. Tanaka, *Products of sequential spaces*, Proc. Amer. Math. Soc., 54(1976), 371-375.
- [28] _____, *A characterization for the products of k -and- \aleph_0 -spaces and related spaces*, *ibid.*, 59(1976), 149-155.
- [29] _____, *Some necessary conditions for products of k -spaces*, Bull. of Tokyo Gakugei Univ. Ser. IV, 30(1978), 1-16.
- [30] _____, *A characterization for the product of closed images of metric spaces to be a k -space*, Proc. Amer. Math. Soc., 74(1979), 166-170.
- [31] _____, *Products of CW-complexes*, *ibid.*, 86(1982), 503-507.
- [32] _____, *Point-countable k -systems and products of k -spaces*, Pacific J. of Math., 101(1982), 199-208.
- [33] _____, *Closed images of locally compact spaces and Fréchet spaces*, Topology Proc., 7(1982), 279-291.
- [34] _____, *Metrizability of certain quotient spaces*, Fund. Math., 119(1983), 157-168.
- [35] Y. Tanaka and Zhou Hao-xuan, *Spaces dominated by metric subsets*, Topology Proc., 9(1984), 149-163.
- [36] _____, *Products of closed images of CW-complexes and k -spaces*, Proc. Amer. Math. Soc., 92(1984), 465-469.
- [37] _____, *Spaces determined by metric subsets, and their character*, Questions and Answers in General Topology, 3(1985/86), 145-160.

- [38] Zhou Hao-xuan, *Weak topology and J. Whitehead's problem* (preprint).

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