NECESSARY AND SUFFICIENT CONDITIONS FOR PRODUCTS OF k-SPACES

by

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Introduction

As is well-known, the product of a k-space with a separable metric space need not be a k-space ([8, 15]). In [27, 30, and 31] etc., the author obtained some necessary and sufficient conditions for products of various kinds of k-spaces to be k-spaces.

In this paper, we give some characterizations for the product $X \times Y$ or X^ω to be a k-space, if X and Y are more general types of k-spaces, as well as X and Y are dominated by these types of k-spaces. Also, we shall pose some questions concerning products of k-spaces.

We assume that all spaces are regular T_1, and all maps are continuous and surjections.

1. Definitions

Definition 1.1. Let X be a space and C be a cover of X. Then X *is determined by* C [11] (or X has the weak topology with respect to C), if $F \subset X$ is closed in X whenever $F \cap C$ is relatively closed in C for every $C \in C$. Here we can replace "closed" by "open." Obviously, every space is determined by an open cover.

We recall that a space is a *k-space; sequential space; c-space* (= space of countable tightness), if it is

determined by the cover of all compact subsets; compact
metric subsets; countable subsets, respectively. First
countable spaces are sequential, and sequential spaces are
k-spaces. We note that a space is a c-space if an only if
whenever $x \in \overline{A}$, then $x \in \overline{C}$ for some countable $C \subset A$, and
sequential spaces and hereditarily separable spaces are
c-spaces; see [17; p. 123], for example.

   For an infinite cardinal number α, a space is a
k_α-*space* [36], if it is determined by a cover C of compact
subsets with the cardinality of $C \leqq \alpha$. k_ω-spaces (=
spaces belonging to class $\mathfrak{G}'$ in the sense of K. Morita
[21]) are defined by E. Michael [16]. A space X is
locally $< k_\alpha$ [36], if each point $x \in X$ has a neighborhood
whose closure is a $k_{\beta(x)}$-space, where $\beta(x) < \alpha$. We shall
say that a space is *locally* k_ω if it is locally $< k_{\omega_1}$.

   Let X be a space, and C be a closed cover of X. Then
X *is dominated by* C [13], if the union of any subcollec-
tion C' of C is closed in X, and the union is determined
by C'. Every space is dominated by a hereditarily
closure-preserving closed cover. If X is dominated by C,
then it is determined by C. But the converse does not
hold. We note that if X is determined by an increasing
countable, closed cover C, then X is dominated by C. As
is well-known, every CW-complex, more generally, chunk-
complex in the sense of J. G. Ceder [3] is dominated by
a cover of compact metric subsets.

Definition 1.2. We recall that a space X is *Fréchet*
(= Fréchet-Urysohn), if whenever $x \in \overline{A}$, then there exists
a sequence in A converging to the point x.

A space X is *strongly Fréchet* [25] (= countably bi-
sequential in the sense of E. Michael [17]), if whenever
(A_n) is a decreasing sequence accumulating at x in X
(i.e., $\overline{A_n - \{x\}} \ni x$ for any $n \in N$), then there exist
$x_n \in A_n$ such that $\{x_n; n \in N\}$ converges to the point x.
First countable spaces are strongly Fréchet, strongly
Fréchet spaces are Fréchet, and Fréchet spaces are se-
quential.

Definition 1.3. E. Michael [17] introduced the notion
of bi-k-spaces (resp. countably bi-k-space), and he
showed that every bi-k-space (resp. countably bi-k-space)
is characterized as the bi-quotient (resp. countably bi-
quotient) image of a paracompact M-space. Here, a space
is a paracompact M-space if it admits a perfect map onto
a metric space. A space X is a *bi-k-space* if, whenever
a filter base F accumulating at x in X (i.e., $\overline{F} \ni x$ for
any $F \in F$), then there exists a k-sequence (A_n) in X such
that $x \in \overline{F \cap A_n}$ and $x \in A_n$ for all $n \in N$ and all $F \in F$.
Here (A_n) is a k-sequence if $K = \cap\{A_n; n \in N\}$ is compact,
and each nbd of K contains some A_n. Paracompact M-spaces
and first countable spaces, more generally spaces of
pointwise countable type [1] are bi-k-spaces. A space
X is a *countably bi-k-space* [17] if, whenever (A_n) is a

decreasing sequence accumulating at x in X, then there

exists a k-sequence (B_n) in X such that $x \in \overline{A_n \cap B_n}$ for

any $n \in N$.   Strongly Fréchet spaces and bi-k-spaces are

countably bi-k-spaces.

   Definition 1.4.   According to E. Michael [18], a

space X is an *inner-one* A-*space*, if whenever (A_n) is a

decreasing sequence accumulating at x in X, then there

exists a non-closed subset $\{x_n; \, n \in N\}$ of X with $x_n \in A_n$.

Some properties of inner-one A-spaces and related spaces

are investigated in [19].   Countably bi-k-spaces are inner-

one A-spaces, but the converse does not hold.

   We conclude this section by recording some elementary

facts which will be often used later on.   These are well-

known, or easily proved.   (4)(ii) is due to [13] or [20].

   Proposition 1.5.   (1)   *Let* X *be determined by*

$\{X_\alpha; \, \alpha \in A\}$.   *For each* $\alpha \in A$, *let* $X_\alpha \subset Y_\alpha \subset X$.   *Then* X *is*

determined by $\{Y_\alpha; \, \alpha \in A\}$.

   (2)   *Let* X *be determined by* $\{X_\alpha; \, \alpha \in A\}$.   *If each* X_α

is determined by $\{X_{\alpha\beta}; \, \beta \in B\}$, *then* X *is determined by*

$\{X_{\alpha\beta}; \, \alpha \in A, \, \beta \in B\}$.

   (3)   *Let* f: X → Y *be a quotient map.   If* X *be deter-*

mined by C, *then* Y *is determined by* $\{f(C); \, C \in C\}$.

   (4)   (i)   *If* X *is determined by* k-*spaces; sequential*

spaces; c-spaces, then so is X *respectively*.

      (ii)   *If* X *is dominated by paracompact spaces,*

then so is X.

2. k-ness of X × Y

The following theorem is essentially proved in [29],
but let us give a direct proof.

*Theorem 2.1. Let X be a c-space, and let Y be a
bi-k-space. If X × Y is determined by* {C × Y; C *is counta-*
bly compact in X} (in particular, X × Y is a k-space), *then
X is an inner-one* A-*space, or Y is locally countably com-
pact.*

Proof. Suppose that X is not inner-one A. Then there
exist a point $p \in X$ and a decreasing sequence (A_n) with
$p \in \overline{A_n - \{p\}}$ satisfying (*) below.

(*) For any $x_n \in A_n$, $\{x_n; n \in N\}$ is closed in X.

Let $A_n' = A_n - \{p\}$, and let $A = \cap\{A_n'; n \in N\}$.
Suppose A is not closed in X. Since X is a c-space,
there exists a countable, non-closed subsets of A. Then
the sequence (A_n) does not satisfy (*). This is a contra-
diction. Hence A is closed in X. Let $B_n = A_n' - A$. Then
the sequence (B_n) satisfies (*) with $p \in \overline{B_n}$ and
$\cap\{B_n; n \in N\} = \emptyset$. Since X is a c-space, there exists a
countable $C_n \subset B_n$ with $p \in \overline{C_n}$ for each $n \in N$. Moreover,
suppose that Y is not locally countably compact. Then
for some point y in Y, $F = \{Y - K; K$ is countably compact
in Y} is a filter base accumulating at y. Since Y is
bi-k, there exists a k-sequence (E_n) such that $y \in \overline{E_n \cap F}$
for any $n \in N$ and $F \in F$. Thus E_n is not countably compact.
Then there exists a sequence $\{D_n; n \in N\}$ of pairwise

disjoint, closed discrete, and infinite countable subsets in Y such that $D_n \subset E_n$, $D_n \cap L = \emptyset$, where $L = \cap\{E_n; n \in N\}$. Let $Z = (\cup\{D_n; n \in N\} \cup L)$, and let $Z* = Z/L$. Since Z is closed in Y, $X \times Z$ is determined by $\{C \times Z; C$ is countably compact in $X\}$. But $Z*$ is the perfect image of Z, so $X \times Z^*$ is the perfect image of $X \times Z$. Thus $X \times Z^*$ is determined by $\{C \times Z*; C$ is countably compact in $X\}$ by Proposition 1.5(3).

Now, let $p = [L]$ in Z^*. Then any point except p is isolated in Z^*, and any nbd of p contains all D_n except finitely many D_n. Since the C_n are infinite countable sets, we can assume that for each $n \in N$, $D_n = C_n$ (as a set) in Z^*. Let $S = \{(x,x); x \in C_n$ for some $n \in N\}$. Then $(p,p) \in \overline{S} - S$ in $X \times Z^*$. Thus S is not closed in $X \times Z^*$. But S is closed in $X \times Z^*$. Indeed, let C be a countably compact subset of X. Then, by (*) it follows that there exists $n \in N$ such that $\cup\{C_m; m \geqq n\} \cap C$ is finite. Also, $\{C_n; n \in N\}$ is point-finite. Thus we see that $S \cap (C \times Z^*)$ is closed in $C \times Z^*$. This implies that S is closed in $X \times Z^*$. This is a contradiction. Thus X is inner-one A, or Y is locally countably compact. (In particular, if $X \times Y$ is a k-space, then the cover of compact subsets of $X \times Y$ is a refinement of $C = \{C \times Y; C$ is countably compact in $X\}$. Thus, by proposition 1.5(1), $X \times Y$ is determined by C. Hence X is inner-one A, or Y is locally countably compact.)

In the previous theorem, the property "X is a c-space"
is essential under (CH); see Remark 2.16(1).

Concerning necessary and sufficient conditions for
the product X × Y to be a k-space, the following question
is posed in view of Theorem 2.1.

Question 2.2.   Let X be a k-space and c-space (in
particular, let X be a sequential space).   Let Y be a
paracompact, bi-k-space.

(1)   If X is an inner-one A-space, then X × Y is a
k-space?

(2)   If X × Y is a k-space, then X is a countably
bi-k-space, or Y is locally compact?

If (1); or (2) is affirmative, then a characterization
for the product X × Y to be a k-space is respectively
"X is inner one A, or Y is locally compact;" or "X is
countably bi-k, or Y is locally compact" by Theorem 2.1
or [28: Proposition 4.6].

We will give some affirmative answers to Question 2.2
for fairly general types of k-spaces.   We recall that every
k-space is precisely the quotient image of a paracompact,
locally compact space ([6]).   Here we can replace "locally
compact space" by "bi-k-space" in view of [17].   In terms
of this, first, let us consider the following contions (C)
and (Q).

(C)   Closed image of a paracompact bi-k-space.

As modifications of (C), we shall also consider (C_0) stronger than (C), and (C_1) weaker than (C).

(C_0) Closed image of a locally compact paracompact space.

(C_1) Closed image of a countably bi-k-space.

(Q)   Quotient Lindelöf image of a paracompact bi-k-space.   Here a Lindelöf image denotes the image under a map with the pre-image of each point Lindelöf.

Every paracompact M-space X, as well as every closed image of X satisfies (C).   In particular, every Lašnev space (= closed image of a metric space) satisfies (C). A characterization of closed images of paracompact M-spaces (resp. metric spaces) is given in [22] (resp. [7]). Every Fréchet space dominated by paracompact bi-k-spaces (resp. paracompact locally compact spaces) satisfies (C) (resp. (C_0)); see [35].

Because k-and-$\aleph_0$-spaces are precisely the quotient images of separable metric spaces [14], they satisfy (Q). Because k_ω-spaces (resp. spaces determined by a point-countable cover of compact subsets) are precisely the quotient (resp. quotient Lindelöf) images of locally compact Lindelöf (resp. locally compact paracompact) spaces [21], they satisfy (Q).

Lemma 2.3.   The following are equivalent.

(1)   X *satisfies* (C_0).

(2)   X *has a hereditarily closure-preserving cover of compact subsets.*

(3) X *satisfies* (C) *and the condition* (*): *Any closed subset of* X, *which is a paracompact* M-*space, is locally compact.*

Proof. The equivalence (1) ⟷ (2) is known, or easy. Indeed, (1) ⟹ (2) is easy. For (2) ⟹ (1), let L be the topological sum of the compact subsets. Then X is the closed image of a paracompact locally compact space L, hence X satisfies (C_0). The equivalence (1) ⟷ (3) is due to [33; Theorem 1.1].

We recall two canonical quotient spaces S_ω and S_2. For $\alpha \geqq \omega$, let S_α be the quotient space obtained from the topological sum of α convergent sequences by identifying all the limit points with a single point $\{\infty\}$ (in particu-. lar, S_ω is called the sequential fan). Let $S_2 = (N \times N) \cup N \cup \{0\}$ with each point of $N \times N$ isolated. A local base of $n \in N$ consists of all sets of the form $\{n\} \cup \{(m,n);$ $m \geqq m_0\}$, and U is a neighborhood of 0 if an only if $0 \in U$ and U is a neighborhood of all but finitely many $n \in N$ (S_2 is called the Arens' space).

Lemma 2.4. *Let* X *be an inner-one* A-*space. Then each of the following conditions implies that* X *is a countably* bi-k-*space. Indeed,* X *is a metric space for* (a); *strongly* Fréchet *space for* (b), (c), *and* (d); *countably* bi-k-*space for* (e); *and* bi-k-*space* (resp. locally compact space) *for* (f) *and* (g).

(a) Lašnev *space.*

(b) Fréchet *space.*

(c) k-*space in which every point is a* G_δ-*set.*

(d) *Hereditarily normal, sequential space.*

(e) *Space satisfying* (C_1).

(f) *Space satisfying* (C) (resp. (C_0)).

(g) c-*space satisfying* (Q).

Proof. For (a), X is a metric space by [17; Corollary 9.10]. For (b), X is a strongly Fréchet space by [24; Theorem 5.1]. For (c), X is sequential by [17; Theorem 7.3]. While, any inner-one A-space contains no closed copy of S_ω and no S_2. Then, for (c) and (d), a sequential space X is strongly Fréchet by [34; Theorem 3.1]. For (e), X is a countably bi-k-space by Theorem 6.3 and Proposition 2.4 in [19], and [17; p. 114]. For (f) and (g), X is a bi-k-space (resp. locally compact space) by Theorems 9.5 and 9.9, and Proposition 3.E.4 in [17].

Let $\{X_\alpha;\ \alpha < \gamma\}$ be a cover of X. For each $\alpha < \gamma$, let $L_0 = X_0$, $L_\alpha = X_\alpha - \cup\{X_\beta;\ \beta < \alpha\}$, and $X_\alpha{}^* = $ cl L_α. We will use these notations.

Lemma 2.5. Let X be dominated by $\{X_\alpha;\ \alpha < \gamma\}$. *Then the following hold.*

(1) X *is determined by* $\{X_\alpha{}^*;\ \alpha < \gamma\}$.

(2) *Let* A *be a subset of* X. *For each* $\alpha < \gamma$, *let* B_α *be a subset of* L_α *such that* A $\cup$ B_α *is closed in* X, *and let* B = $\cup\{B_\alpha;\ \alpha < \gamma\}$. *Then* S = A $\cup$ B *is closed in* X.

(3) *If $F_\alpha \subset L_\alpha$ is finite for each $\alpha < \gamma$, then*
$D = \cup\{F_\alpha; \ \alpha < \gamma\}$ *is closed and discrete in X.*

Proof. (1) For $F \subset X$, let $F \cap X_\alpha^*$ be closed in X_α^*
for each $\alpha < \gamma$. Then $F \cap X_0$ is closed in X_0. Suppose
that $F \cap X_\alpha$ is closed in X_α for each $\alpha < \delta$. Let $F_\delta =$
$(F \cap X_\delta) \cap \cup\{X_\alpha; \ \alpha < \delta\}$. Then $F_\delta \subset \cup\{X_\alpha; \ \alpha < \delta\}$, and
$F_\delta \cap X_\alpha = (F \cap X_\alpha) \cap X_\delta$ is closed in X_α for each $\alpha < \delta$.
Then F_δ is closed in X. While $F \cap X_\delta = F_\delta \cup (F \cap X_\delta^*)$.
Thus, since $F \cap X_\delta^*$ is closed in X_δ, $F \cap X_\delta$ is closed in
X_δ. Hence, by induction, $F \cap X_\alpha$ is closed in X_α for each
$\alpha < \gamma$. Then F is closed in X.

(2) $S \cap X_0 = (A \cap X_0) \cup (B \cap X_0) = ((A \cup B_0) \cap X_0)$.
Then $S \cap X_0$ is closed in X_0. Suppose that $S \cap X_\alpha$ is
closed in X_α for each $\alpha < \delta$. Let $E_\delta = (S \cap X_\delta) \cap$
$\cup\{X_\alpha; \ \alpha < \delta\}$. Then as is seen, E_δ is closed in X. While
$S \cap X_\delta = (A \cap X_\delta) \cup (B \cap L_\delta) \cup E_\delta = ((A \cup B_\delta) \cap X_\delta) \cup E_\delta$.
Thus $S \cap X_\delta$ is closed in X_δ. Hence, by induction, $S \cap X_\alpha$
is closed in X_α for each $\alpha < \gamma$. Then S is closed in X.

(3) In (2), putting $A = \emptyset$ and $B_\alpha = F_\alpha$, we see that
any subset of D is closed in X. Hence D is closed and
discrete in X.

*Lemma 2.6. Let X be a c-space dominated by a cover
(or determined by a point-countable cover). If X is an
inner-one A-space, then each point of X has a neighborhood
which is contained in a finite union of elements of the
cover.*

Proof. First, we show the parenthetic part. Sup-
pose that the assertion does not hold. Let X be

determined by a point-countable cover C. For each
countable subset A of X, let $\{P_n(A); n \in N\} = \{C \in C;$
$C \cap A \neq \emptyset\}$. Then, since X is a c-space, for some $x \in X$
there exists a sequence $\{B_n; n \in N\}$ of countable subsets
of X such that $\{x\} = B_1$, $x \in \overline{B_n}$, and $B_n \cap P_i(B_j) = \emptyset$ if
$i,j < n$. We note that any element of C meets only
finitely many B_n. Let $D_n = \cup\{B_i; i \geqq n\}$ for each $n \in N$.
Then (D_n) is a decreasing sequence accumulating at x.
Since X is inner-one A, there exists a non-closed subset
$D = \{x_n; n \in N\}$ with $x_n \in D_n$. Thus there exists $C_0 \in C$
such that $C_0 \cap D$ is not closed in C. Hence C_0 meets
infinitely many B_n. This is a contradiction. The
parenthetic part holds.

Now, let X be dominated by $\{X_\alpha; \alpha < \gamma\}$. Suppose
that $\{X_\alpha*; \alpha < \gamma\}$ is not point-finite. Then, for some
$x \in X$, there exists an infinite sequence $\{X_{\alpha_n}*; n \in N\}$ with
$X_{\alpha_n}* \ni x$. Let $B_n = \cup\{L_{\alpha_n}; n \in N\} - \cup\{L_{\alpha_i}; i \leqq n\}$. Then
(B_n) is a decreasing sequence accumulating at x. Since X
is inner-one A, there exists a non-closed subset B =
$\{x_n; n \in N\}$ with $x_n \in B_n$. But $B \cap L_\alpha$ is finite for each
$\alpha < \gamma$. Then B is closed discrete in X by Lemma 2.5(3).
This is a contradiction. Thus $\{X_\alpha*; \alpha < \gamma\}$ is point-
finite. But, by Lemma 2.5(1) X is determined by
$\{X_\alpha*; \alpha < \gamma\}$. Thus the lemma holds by the parenthetic
part.

Lemma 2.7.   *Each of the following conditions implies*
that X × Y *is a* k-*space.*

(a)   X *is a* k-*space, and* Y *is locally compact* ([6]).

(b)   X *is a sequential space, and* Y *is a locally*
countably compact, sequential space ([2]).

(c)   X *is a countably* bi-k-*space, and* Y *is a* bi-k-
space ([28]).

(d)   X *and* Y *are locally* k_ω-*spaces* (cf. [16]).

(e)   (MA).   X *is locally* < k_c, *where* c = 2^ω, *and* Y *is*
a locally k_ω-*space each of whose compact set is metric*
(cf. [35]).

Now we give some partial answers to Question 2.2.

Theorem 2.8.   *Let* X *be dominated by spaces satisfying*
one of the following conditions (in particular, let X be
a La$\check{s}$nev space, CW-complex, or the quotient s-image of a
metric space).   *Let* Y *be a paracompact* bi-k-*space* (resp.
let Y be a sequential, bi-k-space).

(a)   Fréchet *space.*

(b)   k-*space in which every point is a* G_δ-*set.*

(c)   *Hereditarily normal, sequential space.*

(d)   c-*space* (resp. sequential space) *satisfying*
(C_1) *or* (Q).
Then the following are equivalent.

(1)   X × Y *is a* k-*space.*

(2)   X *is a countably* bi-k-*space, or* Y *is locally*
compact (resp. locally countably compact).

(3)   X *is an inner-one* A-*space, or* Y *is locally com-pact* (resp. locally countably compact).

Proof. We note that each of (a) $\sim$ (d) (resp. paren-thetic part of (d)) implies that X is a c-space (resp. sequential space) by Proposition 1.5(4)(i). (1) $\Rightarrow$ (3) follows from Theorem 2.1. (3) $\Rightarrow$ (2) follows from Lemmas 2.4 & 2.6, and the fact every locally countably bi-k-space is countably bi-k. (2) $\Rightarrow$ (1) follows from Lemma 2.7.

Lemma 2.9.   (1) *Let* X *be determined by* $\{X_\alpha; \alpha \in A\}$. *Let* γ *be a regular cardinal with* $\gamma \geq \omega_1$. *For any* $\alpha \in A$, *suppose that* $\{\beta \in A; X_\alpha \cap X_\beta \neq \emptyset\}$ *has cardinality* $< \gamma$. *Then* X *is the topological sum of* $\{T_d; d \in D\}$ *such that each* T_d *is determined by* $\{X_\alpha; \alpha \in A_d\}$, *where* $A_d \subset A$ *has cardinality* $< \gamma$.

(2) *A space* X *is a paracompact locally* k_ω-*space if and only if it is determined by countably many locally compact paracompact, closed subsets* (equivalently, X is the topological sum of k_ω-spaces).

Proof. (1) This can be proved by the same method as in the proof of (a) $\Rightarrow$ (c) and (d) in [32; Theorem 1].

(2) For the "if" part, let X be determined by locally compact paracompact, closed subsets X_n ($n \in N$). Since each X_n is paracompact, it is determined by a locally finite cover $\{X_{n\alpha}; \alpha \in A\}$ of compact subsets. Then by Proposition 1.5(2), X is determined by a cover $C = \{X_{n\alpha}; n \in N, \alpha \in A\}$ of compact subsets. But any elements of C meets only countably many members. Thus X is the

topological sum of k_ω-spaces by (1). Then X is a para-compact locally k_ω-space. For the "only if" part, X has a locally finite closed cover $\{K_\alpha; \ \alpha \in A\}$ of k_ω-spaces. Each K_α can be dominated by an increasing countable cover $\{K_{\alpha n}; \ n \in N\}$ of compact subsets. Let $L_n = \cup\{K_{\alpha i}; \ \alpha \in A,$ $i \leq n\}$ for each $n \in N$. Then each compact subset of X is contained in some L_n by means of Lemma 2.5(3). But X is a k-space, for it is a locally k-space. Thus, by Proposition 1.5(1), X is determined by a closed cover $\{L_n; \ n \in N\}$ of locally compact paracompact, closed subsets.

We obtained a characterization for X × Y to be a k-space if X is various kinds of k-spaces and Y is a bi-k-space (Theorem 2.8). Next, let us consider the k-ness of X × Y if Y is not necessarily a bi-k-space.

First, when X and Y are dominated by countably many certain bi-k spaces, we have the following theorem.

Theorem 2.10. *Let* X *and* Y *be dominated by countably many paracompact* bi-k, *and* c-*spaces* (in particular, let X and Y be dominated by countably many metric spaces). *Then* X × Y *is a* k-*space if and only if one of the following properties holds.*

(a) X *or* Y *is locally compact.*

(b) X *and* Y *are* bi-k-*spaces.*

(c) X *and* Y *are locally* k_ω-*spaces.*

Proof. The "if" part holds by Lemma 2.7. We prove the "only if" part. Note that X and Y are c-spaces by

Proposition 1.5(4), and that every locally bi-k-space is
bi-k. Thus, if X × Y is a k-space, by Theorem 2.1 and
Lemma 2.6, (a) and (b) hold, otherwise X and Y are domi-
nated by countably many locally compact paracompact
spaces. The last property implies (c) holds by Lemma
2.9(2).

Second, when X and Y are dominated by (not necessar-
ily countably many) certain Fréchet spaces, the assertion
of the previous theorem is equivalent to a certain set-
theoretic axiom weaker·than (CH). We will show this.

Lemma 2.11. Let X be dominated by Fréchet *spaces
satisfying* (C). *If X contains no closed copy of* S_ω *and
no* S_2, *then X is a* bi-k-*space.*

Proof. Let X be dominated by $\{X_\alpha;\ \alpha \in A\}$, where
each X_α is Fréchet. Since any X_α contains no closed copy
of S_2, X is Fréchet in view of the proof of [34; Theorem
2.1(a)]. But X contains no closed copy of S_ω. Then, by
[26; p. 31] X is strongly Fréchet, and so is each X_α.
Then X is a bi-k-space by Lemmas 2.4 and 2.6, and the fact
that every locally bi-k-space is bi-k.

*Lemma 2.12. Let X be dominated by c-spaces satisfy-
ing* (C). *Let Y be a space satisfying* (C). *If X × Y is a
k-space, then X is a* bi-k-*space, or Y has a hereditarily
closure-preserving cover of compact subsets.*

Proof. Suppose X is not a bi-k-space. Then X is
not an inner-one A-space by Lemmas 2.4 and 2.6. Then, by

Theorem 2.1, Y satisfies ($*$) in Lemma 2.3(3). Then, by
Lemma 2.3, Y has a hereditarily closure-preserving cover
of compact subsets.

Let F be the set of all functions from N to N. For
f, g $\in$ F, we define f $\geq$ g if {n $\in$ N; f(n) < g(n)} is
finite. Let b = min {γ; there exists an unbounded family
A $\subset$ F with cardinality γ}. By BF(α), we mean "b $\geq$ α".
It is well-known that (MA) implies "b = c".

The following lemma is due to G. Gruenhage [10].

Lemma 2.13.   (1) $S_{\omega_1} \times S_{\omega_1}$ *is not a k-space.*
   (2) $S_\alpha \times S_\omega$ *is a k-space if and only if* BF(α+) *holds,*
where α+ *means the least cardinal greater than* α.

Now, we show that the assertion of Theorem 2.10 is
equivalent to "BF(ω_2) is false" if X and Y are dominated
by certain Fréchet spaces.

Theorem 2.14.   Let X and Y be dominated by Fréchet
spaces satisfying (C). *Then the following* (1) *and* (2) *are*
equivalent. When X = Y, the assertion (2) *holds; that is,*
X^2 *is a k-space if and only if X is a bi-k-space, or a*
locally k_ω*-space.*
   (1)   BF(ω_2) *is false.*
   (2)   X $\times$ Y *is a k-space if and only if one of the*
properties (a), (b) *and* (c) *in Theorem 2.10 holds.*

Proof.   (2) $\Rightarrow$ (1). Any S_α is dominated by compact
metric spaces. But S_ω is neither bi-k nor locally

compact, and S_{ω_1} is not locally k_ω. Then $S_{\omega_1} \times S_\omega$ is

not a k-space by the "only if" part of (2). Thus $BF(\omega_2)$

is false by Lemma 2.13(2).

(1) $\Rightarrow$ (2). The "if" part holds by Lemma 2.7. We

prove the "only if" part. Let X be dominated by $\{X_\alpha;$

$\alpha < \gamma\}$, where each X_α is a Fréchet space satisfying (C).

First, suppose that one of X and Y is bi-k, but another is

not bi-k. We can assume that X is bi-k, but Y is not bi-k.

Since each $X_\alpha \times Y$ is closed in a k-space $X \times Y$, $X_\alpha \times Y$ is

a k-space. Then, by Lemma 2.12, each X_α is dominated by

a cover of compact subsets. Since each X_α is bi-k, by

Lemma 2.6 X_α is locally compact. Then, a bi-k-space X is

locally compact by Lemma 2.6.

Next suppose that neither X nor Y is bi-k. Since

each $X_\alpha^* \times Y$ is closed in a k-space $X \times Y$, $X_\alpha^* \times Y$ is a

k-space. But each X_α^* is a space satisfying (C). Then,

by Lemma 2.12, X_α^* has a hereditarily closure-preserving

cover $\{C_{\alpha\beta}; \beta < \delta_\alpha\}$ of compact subsets. Hence X_α^* is

dominated by this cover. Let $C_\alpha = \{C_{\alpha\beta}^*; \beta < \delta_\alpha\}$ for each

$\alpha < \gamma$, and $C = \cup\{C_\alpha; \alpha < \gamma\}$. Suppose that for some

$C \in C$, $\Lambda = \{\alpha; C \cap X_\alpha^* \neq \emptyset\}$ has cardinality $\zeta \geq \omega_1$. Let

$x_\alpha \in C \cap X_\alpha^*$ for each $\alpha \in \Lambda$. Note that if $p_\alpha \in L_\alpha$ for

$\alpha \in \Lambda$, $\{p_\alpha; \alpha \in \Lambda\}$ is closed, discrete in X by Lemma

2.5(3). Then, since C is compact, we can assume that

$L_\alpha \cap C = \emptyset$ for any $\alpha \in \Lambda$. For each $\alpha \in \Lambda$, since $x_\alpha \in X_\alpha^*$

and X_α is Fréchet, there exists an infinite sequence A_α in

L_α converging to $x_\alpha \in C$ with $A_\alpha \cap C = \emptyset$. Let

$S = C \cup \cup\{A_\alpha; \ \alpha \in \Lambda\}$, and let T be the quotient space
obtained from S by identifying all the points of C. Then
S is closed in X, and T is a copy of S_ζ by Lemma 2.5(2)
and (3). Now, since Y is not bi-k, Y contains a closed
copy S' of S_ω or S_2 by Lemma 2.11. Since S × S' is a
closed subset of a k-space X × Y, it is a k-space. But
S_ζ is the perfect image of S, and similarly S_ω is the
perfect image of S_2. Then $S_\zeta \times S_\omega$ is the perfect (hence
quotient) image of a k-space S × S'. Thus $S_\zeta \times S_\omega$ is a
k-space, for every quotient image of a k-space is a k-
space. But $BF(\omega_2)$ is false. Then $\zeta < \omega_1$ by Lemma 2.13(2).
This is a contradiction. Then any $C \in C$ meets only count-
ably many X_α*. Besides, for any $C \in C'$ and for any $\alpha < \gamma$,
a compact subset $C \cap X_\alpha$* of X meets only countably many
elements of C_α in view of the above arguments. Then any
$C \in C$ meets only countably many elements of C. While, by
Proposition 1.5(2) and Lemma 2.5(1), it follows that X is
determined by the cover C of compact subsets. Thus X is
locally k_ω by Lemma 2.9(2). Similarly, Y is also locally
k_ω. Therefore the "only if" part holds. When X = Y,
using Lemma 2.13(1), the assertion (2) holds by the same
way as in the above.

   Remark 2.15. In the following, (2) is a generaliza-
tion of Theorem 1.6 in [35], where $X = S_c$ and Y is domi-
nated by metric spaces.

   (1) Let X and Y be dominated by Fréchet spaces
satisfying (C). If X × Y is a k-space, then X or Y is a

locally k_ω-space (equivalently, X or Y is the topological
sum of k_ω-spaces), otherwise X and Y are bi-k-spaces.

   (2)   Let X be a Fréchet space satisfying (C), but is
not locally $< k_c$. Let Y be dominated by Fréchet space
satisfying (C). Then X × Y is a k-space if and only if
Y is locally compact, otherwise X and Y are bi-k-spaces.

   Indeed, note that neither $S_{\omega_1} \times S_{\omega_1}$ nor $S_c \times S_\omega$ is a
k-space by Lemma 2.13. Then (1) holds by the same way as
in the proof of (1) ⇒ (2) of Theorem 2.14. For (2),
suppose that Y is not bi-k. Then, in view of the proof of
(1) ⇒ (2) of Theorem 2.14, X has a hereditarily closure-
preserving cover C by compact subsets such that for each
$x \in X$, $\{C \in C;\ x \in C\}$ has cardinality $< c$. Then X is
locally $< k_c$. This is a contradiction. Thus Y is a bi-k-
space. Hence, (2) holds in view of the proof of Theorem
2.14.

   Remark 2.16.   (1)   In Theorem 2.14, the property
"each piece is Fréchet" is essential. In Theorem 2.1
(resp. Theorems 2.8 and 2.10), the property "X is a c-
space" (resp. "each piece is a c-space") is essential
under (CH).

   Indeed, quite recently Chen Huaipeng [5] showed that,
under (CH), there exists a k_ω-space X satisfying (C_0) such
that X^ω is a k-space, but X is not locally compact. Since
X is not countably compact, X contains a closed copy of N.
Let Y be N^ω, and let Z be the topological sum X + Y of X

and Y. Then Z is dominated by countably many paracompact
M-spaces. Since X × Y is a closed subset of X^ω, it is a
k-space. Then Z × Z is a k-space. X satisfies (C), but
it is not locally compact. Then X is not an inner-one
A-space by Lemma 2.4, hence not a bi-k-space. Y is a
metric space, but not locally compact. Then Y is not
locally k_ω by Lemma 2.6. Then Z is neither bi-k nor
locally k_ω. Then (1) (indeed, (CH)) ⇒ (2) in Theorem 2.14
does not hold if we omit the Fréchet-ness of each piece.
The space Z is also the desired space for the latter part.

  (2) In Theorem 2.14, the condition (C) of each
Fréchet piece is essential.

  Indeed, G. Gruenhage [9] showed that, under (MA),
there exists a countable, Fréchet space X such that any
finite product X^n is Fréchet, but the countable product
X^ω is not Fréchet. Thus X^ω is not even a k-space by
Theorem 3.1(2) in the next Section. Then X is not a bi-k-
space. Because any countable product of bi-k-spaces is a
bi-k-space by [17; Proposition 3.E.4], hence a k-space.
Now, X^2 is Fréchet. Then X is strongly Fréchet in view
of the proof of Proposition 4.E.4 in [17]. But X is not
locally compact. Then X is not locally k_ω by Lemma 2.6.
Hence, (1) (indeed, (CH)) ⇒ (2) in Theorem 2.14 does not
hold if we omit the condition (C).

  In view of Theorem 2.14 and Remark 2.16, the author
has the following question.

Question 2.17. The equivalence (1) ⇔ (2) in Theorem
2.14 holds under case (a) or (b) below?

(a) X and Y are dominated by compact (or compact
sequential) spaces.

(b) X and Y are determined by point-countable covers
of compact (or compact metric) spaces.

As for case (b), we note that if X and Y are k'-
spaces (in the sense of [1]) determined by point-
countable covers of compact spaces, then X × Y is a k-space
in view of [4]. But the property "X and Y are k'-spaces"
is essential even if X = Y and X is a paracompact space
determined by a point-finite cover of compact metric
spaces; see [32; Example 3].

In the following corollary, (A) is a generalization
for cases "X and Y are Lašnev spaces ([10])" and "X and Y
are CW-complexes ([31]), more generally they are closed
images of CW-complexes ([36])". Note that every closed
image of a CW-complex is dominated by compact metric
spaces, but not every Lašnev space is dominated by metric
spaces [37]. The latter half of (A) is also a generaliza-
tion of Theorem 1.7 (1) in [35], where X = Y and X is a
Fréchet space dominated by metric spaces.

Corollary 2.18. *Let X and Y be dominated by* Lašnev
spaces.

(A) *The following* (1) *and* (2) *are equivalent. When*
X = Y, *the assertion* (2) *holds; that is,* X^2 *is a k-space
if and only if X is metric, or locally* k_ω.

(1) $BF(\omega_2)$ *is false.*

(2) $X \times Y$ *is a k-space if and only if one of the*
following holds.

(a)   X *or* Y *is locally compact metric.*

(b)   X *and* Y *are metric.*

(c)   X *and* Y *are locally* k_ω.

(B)   (MA). $X \times Y$ *is a k-space if and only if one of*
the following holds.   When X = Y, *we can omit* (MA).

(a)   X *or* Y *is locally compact metric.*

(b)   X *and* Y *are metric.*

(c)   *One of* X *and* Y *is locally* k_ω, *and another is*
locally < k_c.

Proof.   (A)   Since X and Y are dominated by para-
compact spaces, they are paracompact by Proposition 1.5(4).
But, by Lemmas 2.4 and 2.6, every bi-k-space dominated by
Lašnev spaces is locally metric.   While every paracompact,
locally metric space is metric.   Hence if X is a bi-k-
space, it is metric.   Thus (1) $\Leftrightarrow$ (2) holds by Theorem 2.14.

(B)   From the above, every compact subset of X and Y
is metric.   Then the "if" part holds by Lemma 2.7.   For
the "only if" part, suppose that neither (a) nor (b) holds.
Then neither X nor Y is a bi-k-space in view of the proof
of (1) $\Rightarrow$ (2) of Theorem 2.14.   We note that neither
$S_{\omega_1} \times S_{\omega_1}$ nor $S_c \times S_\omega$ is a k-space by Lemma 2.13, and the
cardinal c is regular under (MA).   Then using Lemma 2.9(1),
X or Y is locally k_ω, and both X and Y are locally < k_c

by the same way as in the proof of (1) $\Rightarrow$ (2) of Theorem
2.14.   Then (c) holds.

We shall say that a space X is *locally α-compact* if
each point of X has a neighborhood whose closure is α-
compact.   Here a space is α-compact if any subset of with
cardinality α has an accumulation point.

*Lemma 2.19.   Let X be dominated by a cover C of com-
pact subsets.   Then for an infinite regular cardinal γ, X
is locally γ-compact if and only if it is locally $< k_\gamma$.*

Proof.   The "if" part is easily proved.   For the
"only if" part, for x $\in$ X, let V(x) be a neighborhood of
x such that $\overline{V(x)}$ is γ-compact.   Then by Lemma 2.5(3),
$\overline{V(x)} \subset \cup C'$ for some $C' \subset C$ with cardinality $< γ$.   Then
$\overline{V(x)}$ is determined by a cover $\{\overline{V(x)} \cap C; C \in C'\}$ of com-
pact subsets, with cardinality $< γ$.   This implies that X
is locally $< k_\gamma$.

We have the following by Corollary 2.18(B) and
Lemma 2.19.

*Corollary 2.20.   (MA).   Let X and Y be dominated by
compact metric spaces.   Then the following are equivalent.
When X = Y, we can omit (MA).*

(1)   *X × Y is a k-space.*

(2)   *X or Y is locally compact metric, otherwise one
of X and Y is locally k_ω and another is locally $< k_c$.*

(3)   X or Y *is locally compact metric, otherwise one*
of X and Y is locally ω_1*-compact and another is locally*
c-compact.

Lemma 2.21.   Let X be dominated by a cover C of com-
pact c-spaces, and let Y have the same property.   Then
X × Y is a k-space if and only if it is a c-space.

Proof.   "If":   Since X × Y is a c-space, it is deter-
mined by countable subsets of X × Y.   Then, by Proposi-
tion 1.5(1), X × Y is determined by $G = \{\overline{D} \times \overline{E}$; D and E
are countable}.   Let D be a countable subset of X.   Since
X is dominated by C, $\overline{D} \subset \cup C'$ for some countable $C' \subset C$.
Then $\overline{D}$ is determined by a countable cover $\{\overline{D} \cap C$; $C \in C'\}$
of compact subsets.   Hence $\overline{D}$ is a k_ω-space.   Similarly,
any separable closed subset of Y is a k_ω-space.   Then any
element of G is a k-space by Lemma 2.7.   Thus X × Y is a
k-space by Proposition 1.5(4).

"Only if":   Since X × Y is a k-space, it is deter-
mined by $H = \{K \times L$; K and L are compact}.   But by
Proposition 1.5(4), X and Y are c-spaces, then so is any
compact subset of X and Y.   Thus, by [12; Theorem 4],
any element of H is a c-space.   Then X × Y is a c-space.

Lemma 2.22.   Let X be dominated by $C = \{X_\alpha$; $\alpha \in A\}$,
and let Y be dominated by $D = \{Y_n$; $n \in N\}$ *with*
$Y_n \subset Y_{n+1}$ $(n \in N)$. *If X × Y is a k-space, then it is*
dominated by $F = \{X_\alpha \times Y_n$; $\alpha \in A, n \in N\}$.

Proof.   Since C and D are closure-preserving closed
covers, so is F in X × Y.   Let F' be any subcollection of

F, and let $S = \cup\, F'$. Since S is closed in a k-space
$X \times Y$, S is also a k-space. Besides, any element of F'
is closed in $X \times Y$. Then, to show S is determined by F',
it suffices to show that each compact subset of S is con-
tained in a finite union of elements of F'. For each
$n \in N$, let $A_n = \cup\{X_\alpha;\ X_\alpha \times Y_n \in F'\}$ (if there are no
$X_\alpha \times Y_n \in F'$, let $A_n = \emptyset$), $B_n = \cup\{A_i;\ i \geq n\}$, and let
$L_n = Y_n - Y_{n-1}$, $Y_0 = \emptyset$. Then $S = \cup\{A_n \times Y_n;\ n \in N\} =$
$\cup\{B_n \times L_n;\ n \in N\}$. Let C be a compact subset of S. Then
there exist compact subsets K in X and L in Y such that
$C \subset K \times L$. Note that the compact set L meets only finitely
many L_n by Lemma 2.5(3). Let $m = \text{Max }\{n \in N;\ L \cap L_n \neq \emptyset\}$.
Then $C \subset (K \times L) \cap S \subset \cup\{B_n \times Y_n;\ n \leq m\}$. Now, each B_n
is dominated by $G_n = \{X_\alpha;\ X_\alpha \times Y_i \in F',\ i \geq n\}$. Then each
compact subset of B_n is contained in a finite union of
elements of G_n by Lemma 2.5(3). Then each compact subset
of $B_n \times Y_n$ is contained in a finite union of elements of
$H_n = \{X_\alpha \times Y_n;\ X_\alpha \in G_n\}$, in particular, so is a compact
subset $(K \times L) \cap (B_n \times Y_n)$ of $B_n \times Y_n$. Hence $(K \times L) \cap S$
is contained in a finite union of elements of $\cup\{H_n;\ n \leq m\}$,
hence so is C. But $Y_n \subset Y_{n+1}$ for each $n \in N$. Then C is
contained in a finite union of elements of F'. Hence
each compact subset of S is contained in a finite union
of elements of F'.

For spaces X and Y dominated by compact metric spaces,
let us give equivalent properties to the k-ness of $X \times Y$.

We note that $S_\omega \times Q$, Q is the rationals, is a c-space, but is not a k-space. Thus the compactness of the metric piece is essential in (2) $\Rightarrow$ (1) of the following theorem.

Theorem 2.23. *Let X and Y be dominated by compact metric spaces. Then the following are equivalent.*

(1) X $\times$ Y *is a k-space.*

(2) X $\times$ Y *is a c-space.*

(3) X $\times$ Y *is dominated by compact metric spaces.*

Proof. (1) $\Leftrightarrow$ (2) follows from Lemma 2.21. (3) $\Rightarrow$ (1) follows from Proposition 1.5(4). We prove (1) $\Rightarrow$ (3). We note that if X is bi-k, by Lemma 2.6, X is locally compact, hence is locally k_ω. Then, in view of Remark 2.15(1), X or Y is the topological sum of k_ω-spaces. Then, using Lemma 2.22, we show that X $\times$ Y is dominated by compact spaces. But any compact subset of X and Y is metric. Then (3) holds.

3. k-ness of X^ω

Theorem 3.1. (1) *Let X^ω be a k-space with X a c-space. Then X^ω is an inner-one A-space.*

(2) *Let X be dominated by $\{X_\alpha; \alpha \in A\}$, where every point of X_α is a G_δ-set in X_α. Then X^ω is a k-space if and only if it is strongly* Fréchet.

(3) *Let X be dominated by c-spaces satisfying (C) or* (Q). *Then X^ω is a k-space if and only if it is a bi-k-space (quivalently, X is a bi-k-space).*

Proof. (1) Let $Y = X^{\omega}$. Since Y is a k-space with
X a c-space, using Remark 3 in [12], Y is a c-space by the
same way as in the proof of the "only if" part of Lemma
2.19. Every countably compact space is inner-one A. Then,
to show that Y is inner-one A, let Y be not countably
compact. Then Y contains a closed copy of N. Then X^{ω},
which is homeomorphic to $Y \times Y^{\omega}$, contains a closed copy
of $Y \times N^{\omega}$. But $Y \times N^{\omega}$ is a k-space with Y a c-space.
Then Y is an inner-one A-space by Theorem 2.1.

(2) Since every strongly Fréchet space is a k-space,
it suffices to prove the "only if" part. Using Lemma
2.5(3), any compact or separable subset S of X is covered
by countably many X_{α}. Then every point of S is a G_{δ}-set
in X. Hence every point of a compact or separable sub-
set of $Y = X^{\omega}$ is a G_{δ}-set in Y. Then any compact subset
of Y is first countable. Thus a k-space Y is sequential
hence a c-space. Then, to show Y is strongly Fréchet, by
[17; Proposition 8.7] it suffices to show that any count-
able subset A of Y is strongly Fréchet. Let $B = \overline{A}$.
Since B is a separable closed subset of Y, B is a sequen-
tial space in which every point is a G_{δ}-set. But B is an
inner-one A-space by means of (1). Thus B is strongly
Fréchet by Lemma 2.4. Hence A is strongly Fréchet.

(3) For the "if" part, we note that every product
of countably many bi-k-spaces is a bi-k-space [17], hence
a k-space. For the "only if" part, since X is a c-space,
X is an inner-one A-space by (1). Thus the "only if" part
follows from Lemmas 2.4 and 2.6.

Remark 3.2.   (1)   In Theorem 3.1(1) (resp. Theorem 3.1(3)), the property "X is a c-space" (resp. "each piece is a c-space") is essential under (CH).

Indeed, let us consider the space $Z = X + N^{\omega}$ in Remark 2.16(1). $X^{\omega} \times N^{\omega}$ is closed in $X^{\omega} \times X^{\omega}$, and $X^{\omega} \times X^{\omega}$ is homeomorphic to a k-space X^{ω}. Then $X^{\omega} \times N^{\omega}$ is a k-space. Thus Z^{ω} is a k-space. Then Z is the desired space in view of Remark 2.16(1).

(2)   The condition (C) or (Q) in Theorem 3.1(3) is essential under (CH).

Indeed, T. Nogura [23] showed that, under (CH), there exists a countable Fréchet space X such that X^{ω} is Fréchet (hence strongly Fréchet), but X is not a bi-sequential space (in the sense of [17; 3D]). Then X is not a bi-k-space, because any b-k-space in which every point is a G_{δ}-set is bi-sequential by [17; Theorem 7.3].

In view of Theorem 3.1 and Remark 3.2, the author has the following question.

Question 3.3.   Let X^{ω} be a k-space with X a c-space. Then X is a countably bi-k-space (equivalently, X^{ω} is countably bi-k)?

In the following corollary, (1) is a generalization of Theorem 1.7 in [35], where X is a Fréchet space dominated by metric spaces.

Corollary 3.4.   *Let* X *be dominated by* Lašnev *spaces. Then the following hold.*

(1) *For* n $\geq$ 2, X^n *is a k-space if and only if X is metric or locally* k_ω.

(2) X^ω *is a k-space if and only if X is metric.*

Proof. For (1), the "only if" part follows from Corollary 2.17(A). For the "if" part, note that every finite product of locally k_ω-spaces is locally k_ω by means of [16; (7.5)]. Then each X^n is a locally k-space, hence is a k-space. For (2), note that if X is a bi-k-space, then it is metric as in the proof of Corollary 2.17(A). Thus (2) follows from Theorem 3.1(2) or (3).

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