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# COMMENTS ON SEPARATION

by

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# COMMENTS ON SEPARATION

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## 1. Collapsing Wage's Machine

In 1976, Michael Wage [34] invented a machine which takes normal spaces which are not collectionwise normal and makes them not normal. This process preserves, for example, countable paracompactness and Mooreness and so it was used to construct, for example, non-normal countably paracompact Moore spaces under various hypotheses. In this process, what happens is that first the isolated points of two copies of the input space are identified and then each isolated point is split so that each new isolated point can only be in a basic open neighborhood

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of one element of the discrete unseparated family of closed sets from each copy and so that these elements must be different. In this section, we note that this construction can be modified by then collapsing the unseparated closed sets of one of these copies to points and thus can be used to obtain a curious and interesting example.

This section originates with a question of Peg Daniels: Are countably paracompact screenable collectionwise Hausdorff spaces strongly collectionwise Hausdorff?

We need the main theorem of [36].

*Theorem* 1 ([36]). *Let* Z *be a strongly zero-dimensional normal collectionwise Hausdorff space. Let* D *be the family of closed discrete subsets of* Z. *If there is a family* B *of clopen subsets of* Z *and a function* $m: B \to [0,1]$ *such that*

*If* $d \in D$ *and* $\varepsilon > 0$, *then there is* $U \in B$ *such that* $d \subset U$ *and* $m(U) < \varepsilon$

*If* $E \in [B]^{\leq\omega}$ *and* $\Sigma\{m(U): U \in E\} < 1$ *then* $\cup E \neq Z$

*then there is a zero-dimensional normal collectionwise Hausdorff space* Y *which is not collectionwise normal with respect to copies of* Z.

*Corollary* 1. *There is a zero-dimensional normal space* Y *which is not collectionwise normal with respect to copies of a compact zero-dimensional Hausdorff space but which is collectionwise normal with respect to metrizable sets.*

*Proof*. Let $Z$ be the double arrow space on the Cantor set, that is, $Z = 2^{\omega} \times 2$ with the lexicographic order topology. Now $Z$ is a zero-dimensional compact Hausdorff space. Let $CO(Z)$ be the family of clopen subsets of $Z$. Let $f^i\colon 2^{\omega} \times 2 \to 2^{\omega}$ be the projection mappings. We can give $Z$ a measure $m\colon CO(Z) \to [0,1]$ defined by $m(A) = \frac{\mu(f^0(A))+\mu(f^1(A))}{2}$, where $\mu$ is the product measure on $2^{\omega}$. Since closed discrete sets in $Z$ are finite, we can apply Theorem 1 to get $Y$. By a simple back-and-forth argument [35], any normal collectionwise Hausdorff space is collectionwise normal with respect to countable sets, and so since metrizable sets in $Y$ are countable and any element of $Y$ which is not an element of a copy of $Z$ is an isolated point, $Y$ is collectionwise normal with respect to metrizable sets.

*Theorem* 2 (Wage [34]). *There is a countably paracompact screenable zero-dimensional Hausdorff space* $W$ *which is not collectionwise normal with respect to compact sets but which is collectionwise normal with respect to metrizable sets.*

*Proof*. We use Corollary 1 to get a zero-dimensional normal space $Y$ which is not collectionwise normal with respect to copies of a compact zero-dimensional Hausdorff space but which is collectionwise normal with respect to metrizable sets. Without loss of generality, we can assume that $Y$ has an unseparated discrete family $\{F_{\alpha}\colon \alpha \in \kappa\}$ of compact sets so that, letting $F = \cup\{F_{\alpha}\colon \alpha \in \kappa\}$, we have that $G = Y - F$ is a set of isolated points. We do not need to know anything else about the structure of Y.

Let $\mathcal{B}$ be the family of clopen sets in Y, each element of which intersects at most one $F_\alpha$. Topologize

$$W = (F \times 2) \cup (G \times \{(\alpha,\beta) \in \kappa^2 : \alpha \neq \beta\})$$

by letting

$$B^i = ((B \cap F_\gamma) \times \{i\})$$
$$\cup((B \cap G) \times \{(\alpha,\beta) \in \kappa^2 : (\gamma = \alpha \wedge i = 0) \vee$$
$$(\gamma = \beta \wedge i = 1)\})$$

be open where $B \in \mathcal{B}$ is such that $B \cap F_\gamma \neq \emptyset$ and $i \in 2$ and by letting the elements of $W - (F \times 2)$ be isolated points. This is Wage's machine.

Let us argue that W is a countably paracompact screenable zero-dimensional Hausdorff space which is collectionwise normal for metrizable sets but which is not normal with respect to two closed sets, each of which is the free union of compact sets.

Each point is the intersection of its neighborhoods in Y and thus also in W. This means that W is $T_1$.

Suppose B is clopen in Y and $(f,i) \in (F \times 2) - B^j$. If $f \notin B$, then there is a clopen subset D of Y such that $f \in D$ and $B \cap D = \emptyset$ and then $(f,i) \in D^i$ and $D^i \cap B^j = \emptyset$. If $f \in B$, then $i \neq j$ and if $B \cap F_\gamma \neq \emptyset$, then $f \in F_\gamma$. Now $B^i \cap B^j = \emptyset$. We have shown that the assigned base for W consists of clopen sets and so that W is Hausdorff and zero-dimensional.

We show countable paracompactness. Let U be a countable open cover of W. Let $P_i$ be a clopen countable partition of F which refines

$$\{A: (\exists B \in \mathcal{U}) A \times \{i\} = B \cap (F \times \{i\})\}$$

Assume, without loss of generality, that $P_0 = P_1$. Let $\mathcal{B}_i$ be a disjoint clopen family in Y whose restriction to F is $P_i$. Now $\{B^i: B \in \mathcal{B}_i, i \in 2\}$ is a locally finite family which covers W, except for some isolated points (since $B_0^{j_0} \cap B_1^{j_1} \neq \emptyset \Rightarrow B_0 = B_1$).

Let us argue that W is collectionwise Hausdorff and thus, since metrizable sets in W are the union of countably many discrete families of points, collectionwise normal with respect to metrizable sets. Let D be a discrete family of points. We assume that there is a discrete family of points $E \subset F$ such that $D = E \times 2$. Since Y is collectionwise Hausdorff, there is a disjoint subfamily $\{U_e: e \in E\}$ of $\mathcal{B}$ such that $e \in U_e$. We claim that $\{U_e^i: e \in E, i \in 2\}$ separates D. Now $U_e^i \cap U_{e'}^i = \emptyset$ where $e \neq e'$ and $(g,\alpha,\beta) \in U_e^i \cap U_e^{1-i}$ and $U_e \cap F_\gamma \neq \emptyset$ implies $\alpha = \gamma = \beta$ which is a contradiction.

Let us argue that W has two unseparated closed sets $F \times \{0\}$ and $F \times \{1\}$, each of which is the free union of compact sets. Otherwise, let $\{U_\alpha^i: \alpha \in \kappa, i \in 2\}$ be open sets in W such that $U_\alpha^i \supset F_\alpha \times \{i\}$ and $\alpha,\beta \in \kappa$ implies $U_\alpha^i \cap U_\beta^{1-i} = \emptyset$. Let $U_\alpha^i = \cup\{B^i: B \in V_\alpha^i\}$. Assume, without loss of generality, that each $V_\alpha^0 = V_\alpha^1$. Let $U_\alpha = \cup V_\alpha^i$. Now $\{U_\alpha: \alpha \in \kappa\}$ is a family of open sets such that $U_\alpha \supset F_\alpha$. There is $\alpha,\beta \in \kappa$ such that $\alpha \neq \beta$ and $U_\alpha \cap U_\beta \neq \emptyset$. So there is $B_0 \in V_\alpha^i$ and $B_1 \in V_\beta^i$ such that $B_0 \cap B_1 \neq \emptyset$. Therefore $B_0^0 \cap B_1^1 \neq \emptyset$ and so $U_\alpha^0 \cap U_\beta^1 \neq \emptyset$ which is a contradiction.

We show screenable by noting that if each $\{B_\gamma^n : n \in \omega\}$ is a countable open subfamily of $\mathcal{B}$, each element of which intersects $F_\gamma$, then for fixed n, $B_\gamma^n \cap B_\delta^n = \emptyset$. In W, we argue by covering each $F_\gamma \times \{i\}$ by a countable open family of sets of the form $B^i$ where $B \in \mathcal{B}$. Since $B_\gamma^n \cap B_\delta^n = \emptyset \Rightarrow (B_\gamma^n)^i \cap (B_\delta^n)^i = \emptyset$, this shows that we can cover $F \times 2$ with a $\sigma$-disjoint open refinement of any given open cover. One more disjoint family takes care of the isolated points.

*Theorem* 3. *There is a countably paracompact screenable zero-dimensional Hausdorff space* X *which is collectionwise normal with respect to metrizable sets but which is not strongly collectionwise Hausdorff.*

*Proof*. Start with W in Theorem 2. Identify each $F_\gamma \times \{1\}$ to a single point. By Proposition 2.4.9 of [9], the quotient mapping f is closed and thus perfect. Call the quotient space X. By Theorem 3.7.20 and Exercise 5.2.G (a) in [9], perfect mappings preserve Hausdorff and countably paracompact. We must show that X is screenable zero-dimensional and collectionwise Hausdorff but fails to be strongly collectionwise Hausdorff.

We show that X is zero-dimensional. Suppose that $x \in X$ is the image of $F_\gamma$ under the quotient mapping. Suppose that U is an open neighborhood of x in X. Now $f^{-1}(U)$ is an open set in W which contains $F_\gamma$. By the compactness of $F_\gamma$ and the regularity and zero-dimensionality of W, there is a clopen set V in W which lies inside $f^{-1}(U)$, contains $F_\gamma$ and intersects no other $F_\delta$.

Now $f(V)$ is clopen since $f^{-1}(f(V)) = V$. Since $x \in f(V) \subset U$, the proof is complete.

We show that X is screenable. Let $\mathcal{V}$ be an open cover of X. We can first assume that the preimage of each element of $\mathcal{V}$ intersects at most one $F_\gamma \times \{i\}$. Let $\mathcal{U}$ be the open cover $\{f^{-1}(V): V \in \mathcal{V}\}$. We note that in the argument for screenability of W, if each element of $\mathcal{U}$ intersects at most one $F_\gamma \times \{i\}$ and if any element of $\mathcal{U}$ which intersects $F_\gamma \times \{1\}$, actually contains $F_\gamma \times \{1\}$, then the $\sigma$-disjoint refinement $\mathcal{S}$ can be assumed to do the same. Now $\{f(S): S \in \mathcal{S}\}$ is also $\sigma$-disjoint since each element of $\mathcal{S}$ either contains each $F_\gamma \times \{1\}$ or is disjoint from it.

X is not strongly collectionwise Hausdorff because the identified $F \times \{1\}$ (a closed discrete set) cannot be separated from $F \times \{0\}$. Otherwise the preimages would be separated in W.

X is collectionwise Hausdorff, however, because any closed discrete set may be assumed to consist of $\{f(F_\gamma \times \{1\}): \gamma \in \kappa\}$ and a closed discrete subset $D \times \{0\}$ of $F \times \{0\}$. First, apply strong collectionwise Hausdorffness in Y to get a discrete family $\mathcal{K}$ of clopen sets in Y which separate $D \subset F$ in Y. For each $\gamma \in \kappa$, let $U_\gamma$ be a clopen subset of Y which contains $F_\gamma$ and is disjoint from every element of $\mathcal{K}$ except those which contain elements of $D \cap F_\gamma$. We claim that

$$\{f(K^0): K \in \mathcal{K}\} \cup \{f((U_\gamma)^1): \gamma \in \kappa\}$$

separates the closed discrete set. Suppose $K^0 \cap (U_\gamma)^1 \neq \emptyset$

and K intersects $F_\delta$. If $\delta \neq \gamma$, then that contradicts the choice of $U_\gamma$. If $\delta = \gamma$, then that contradicts the definition of the topology on W.

## 2. Graph Theory and Separation

A subtraction technique is used in many arguments where a discrete family of sets is being separated. This technique can be abstracted into a lemma and this lemma used to provide a less technical proof of a theorem of Fleissner and Reed. We need the idea of a graph of a discrete family.

*Definition* 1. Let $F$ be a discrete family of sets in a topological space X and let $\mathcal{U}$ be an open cover of X such that the closure of each $U \in \mathcal{U}$ intersects a unique $F(U) \in F$. Let G be the graph on $\mathcal{U}$ whose edges are $\{(U,V) \in \mathcal{U}^2: U \cap V \neq \emptyset, F(U) \neq F(V)\}$. We say that G is a graph of $F$.

A simple sufficient condition for a discrete family to be separated can be stated:

*Theorem* 4. *If a discrete family* $F$ *has a graph* G *in which each vertex has countable degree, then* $F$ *is separated.*

*Proof*. Any graph G of countable degree is the free union of countable subgraphs. Thus the edges of G can be listed $\{U_{\alpha,n}: \alpha \in \kappa, n \in \omega\}$ where $\alpha \neq \beta$ implies $(U_{\alpha,n}, U_{\beta,m}) \notin G$. Let

$$V_{\alpha,n} = U_{\alpha,n} - \cup\{\overline{U_{\alpha,m}}: m < n, F(U_{\alpha,m}) \neq F(U_{\alpha,n})\}$$

For each $F \in \mathcal{F}$, let $U(V) = \cup\{V_{\alpha,n}: F(U_{\alpha,n}) = F\}$. Now $\{U(F): F \in \mathcal{F}\}$ separates $\mathcal{F}$.

An application of this theorem is:

*Lemma* 1. *If* $X$ *is a regular para-Lindelöf space and* $\mathcal{F}$ *is a discrete family of sets such that at most one element of* $\mathcal{F}$ *is not Lindelöf, then* $\mathcal{F}$ *is separated.*

*Proof.* Let $\mathcal{U}_0$ be an open family such that the closure of each $U \in \mathcal{U}_0$ intersects a unique $F(U) \in \mathcal{F}$. Apply the para-Lindelöf property to $\mathcal{U}_0$ three times. That is, find $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$ such that each $\mathcal{U}_{i+1}$ is a locally countable refinement of $\mathcal{U}_i$ which witnesses the local countability of $\mathcal{U}_i$ as well when $i \geq 1$ (i.e. if $i \geq 1$, then each element of $\mathcal{U}_{i+1}$ intersects countably many elements of $\mathcal{U}_i$). Let $\mathcal{U}$ consist of those elements $U$ of $\mathcal{U}_3$ where $F(U)$ is Lindelöf and those elements $V$ of $\mathcal{U}_2$ where $F(V)$ is not Lindelöf. We show that each vertex of the induced graph of $\mathcal{F}$ has countable degree.

Let $V \in \mathcal{U}_3 \cup \mathcal{U}_2$ and $U \in \mathcal{U}_3$ where $F(U)$ is Lindelöf. If $V \cap U \neq \emptyset$, then $V$ must intersect some $W \in \mathcal{U}_1$ such that $F(W) = F(U)$. There are countably many such $W$ and so countably many possible $F(U)$. For each $F(U)$, there are only countably many $U$ since locally countable families in a Lindelöf set are countable.

If $V \in \mathcal{U}_2 \cup \mathcal{U}_3$, $U \in \mathcal{U}_2$, $V \cap U \neq \emptyset$ and $F(V) \neq F(U)$, then $V \in \mathcal{U}_3$ and there are countably many possibilities for $U$.

Applications of the lemma are:

*Corollary* 2. (Fleissner, Reed [16]). *If* X *is a regular, para-Lindelöf space, then* X *is strongly collectionwise normal for Lindelöf sets (i.e. whenever* $\{A_\alpha : \alpha \in \kappa\}$ *is a discrete family of Lindelöf sets, there is a discrete family of open sets* $\{U_\alpha : \alpha \in \kappa\}$ *such that each* $U_\alpha \supset A_\alpha$*).*

*Corollary* 3. (Burke, Davis). *Para-Lindelöf pseudocompact spaces are compact.*

*Proof.* More, para-Lindelöf pseudo-Lindelöf (i.e. each discrete family of open sets is countable, often called DCCC) spaces are Lindelöf. If such a space is countably compact (or $\aleph_1$-compact) then apply Aquaro's Lemma (see page 302 of [9]). Otherwise take an infinite (uncountable) discrete family of points and apply the previous corollary for points.

*Corollary* 4. *Para-Lindelöf metacompact locally Lindelöf spaces are paracompact.*

*Proof.* Apply the proof that metacompact collectionwise normal spaces are paracompact.

## 3. Normal versus Normalized

The study of normal spaces which are not collectionwise normal has often been seen as a combinatorial problem involving the existence of normalized but unseparated families.

*Definition* 2. If X is a topological space and $\mathcal{F} = \{F_\alpha: \alpha \in \kappa\}$ is a discrete family of sets, then we say that $\mathcal{F}$ is normalized if, for each $A \subset \kappa$, there are disjoint open sets U and V in X such that, for each $\alpha \in A$, $F_\alpha \subset U$ and, for each $\alpha \in \kappa - A$, $F_\alpha \subset V$.

A finer analysis of normality is served by another definition:

*Definition* 3. If X is a topological space and $\mathcal{F} = \{F_\alpha: \alpha \in \kappa\}$ is a discrete family of sets, then we say that $\mathcal{F}$ is strongly normalized if, for any two disjoint closed sets $A, B \subset \cup\{F_\alpha: \alpha \in \kappa\}$, there are disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.

Rudin and Starbird, and Nyikos showed that normality exerts a stronger influence on a discrete family than mere normalization. In [24], they construct, for each cardinal $\lambda < \kappa$, a Moore space $T_\lambda$ so that if there is a first countable normal space which is not collectionwise normal, then some $T_\lambda$ has a normalized discrete family of closed sets which is not separated. In [20], Peter Nyikos showed that these spaces of Rudin and Starbird are never normal unless they are metrizable. The reason that these spaces are not normal is that their unseparated normalized discrete families are not strongly normalized.

Peg Daniels has shown:

*Theorem* 5. ([6]). *There is no locally compact boundedly metacompact space which is normal but not paracompact.*

She also showed that counterexamples to the original problem due to Tall [28] and Arhangel'skiĭ [2] of whether there is a locally compact metacompact normal space which is not paracompact are probably found in Pixley-Roy spaces. We need the notation $A^*$ to represent the set $A$ with the topology in which the proper closed sets are precisely the finite sets.

*Theorem* 6. ([6]). *If there is a normal, locally compact, metacompact space* $Y$ *that is not paracompact, then there is a cardinal* $\kappa$ *and a subspace* $Z \subset PR(\kappa^*)$ *such that* $\kappa$ *is a closed discrete normalized subset of* $Z$ *which cannot be separated in* $Z$. *Furthermore, if* $Y$ *is also zero-dimensional, then there is such a subspace* $Z$ *which is a perfect image of* $Y$, *hence also normal, locally compact, metacompact but not paracompact.*

However, there seems to be no reason to assume that a counterexample should be zero-dimensional. Furthermore, there seems no reason to believe that the existence of a locally compact metacompact space with a discrete normalized unseparated family of *points* implies the existence of a normal example.

In this section, we present an example under $MA_{\aleph_1}$ of a space which satisfies the first part of the conclusion of Theorem 6. Of course, for a discrete family of points, strong normalization and normalization are equivalent. Since Theorem 5 is true in ZFC, we deduce that there is a big difference even between strongly normalized and normal. Here the reason is that local compactness requires the rest of the space to have a complexity which fights against normality.

*Theorem* 7. ($MA_{\aleph_1}$). *There is a locally compact (boundedly) metacompact completely regular space* X *with a discrete family of points which is normalized but not separated. In fact,* X *is a subspace of* $PR(\omega_1^*)$.

*Proof.* Let $A \in [{}^{\omega}2]^{\omega_1}$. For technical reasons, we define X to be a subspace of $PR((\omega_1 \times (\omega + 1))^*)$. Of course $(\omega_1 \times (\omega + 1))^*$ and $\omega_1^*$ are homeomorphic so this makes no difference. Let $X = \{\{(\alpha,\omega)\}: \alpha \in A\} \cup \{\{(\alpha,\omega), (\alpha,n)\}: \alpha \in A, n \in \omega\} \cup \{\{(\alpha,\omega),(\alpha,n),(\beta,\omega),(\beta,n)\}: \alpha,\beta \in A, n \in \omega, \alpha \restriction n = \beta \restriction n\}$. As a subspace of the Pixley-Roy space, X is a metacompact completely regular zero-dimensional Hausdorff space.

*Lemma* 2. X *is locally compact.*

*Proof.* It suffices to show that $(\forall\alpha \in A)\{F \in X: (\alpha,\omega) \in F\}$ is compact. Suppose $\mathcal{U}$ is an open cover of $\{F \in X: (\alpha,\omega) \in F\}$. Let U be a basic open neighborhood of $(\alpha,\omega)$ which lies inside an element of $\mathcal{U}$. Suppose that $H \in [\omega]^{<\omega}$ lists all second coordinates other than $\omega$ used as parameters in U. For each $n \in H$, let $U_n$ be a basic open neighborhood of $\{(\alpha,\omega),(\alpha,n)\}$ which lies inside an element of $\mathcal{U}$. We claim that $\{U\} \cup \{U_n: n \in \omega\}$ cover all but finitely many elements of $\{F \in X: (\alpha,\omega) \in F\}$. Suppose that $G \in [A]^{<\omega}$ lists all first coordinates other than $\alpha$ used as parameters in $U$ or in some $U_n$ where $n \in H$. Suppose $J = \{(\alpha,\omega),(\alpha,n),(\beta,\omega),(\beta,n)\}$ is not covered. We deduce that, since $J \not\subseteq U$, either $\beta \in G$ or $n \in H$. If $n \in H$, then we deduce that, since $J \not\subseteq U_n$, we must have $\beta \in G$. Thus in either case $\beta \in G$. Now, since $J \in X$, we know that $\alpha \upharpoonright n = \beta \upharpoonright n$. Since $\alpha \neq \beta$, that leaves finitely many possibilities for $n$ for each $\beta \in G$.

*Lemma* 3. *The discrete family of points* $\{\{(\alpha,\omega)\}$*:* $\alpha \in A\}$ *is normalized but not separated.*

Let $H$ and $K$ be disjoint subsets of $A$. Since $A \in [2^\omega]^{\omega_1}$, we know (see [23]) that, under $MA_{\aleph_1}$, there is a function $f: A \to \omega$ such that $(\forall h \in H)(\forall k \in K) h \upharpoonright (\max\{f(h), f(k)\}) \neq k \upharpoonright (\max\{f(h),f(k)\})$. We claim that the neighborhood $\{F \in X: (h,\omega) \in F, (\forall i \leq f(h))(h,i) \not\subseteq F\}$ is disjoint from $\{F \in X: (k,\omega) \in F, (\forall i \leq f(k))(k,i) \not\subseteq F\}$ whenever $h \in H$ and $k \in K$.

Otherwise, suppose that $G = \{(h,\omega),(h,n),(k,\omega),(k,n)\}$ lies in both neighborhoods. We deduce that $n > f(h)$ and $n > f(k)$. This means that $h \upharpoonright n \neq k \upharpoonright n$. Since $G \in X$, we deduce that $h \upharpoonright n = k \upharpoonright n$ which is a contradiction.

If $\{U_\alpha: \alpha \in A\}$ were disjoint basic open neighborhoods of $\{\{(\alpha,\omega)\}: \alpha \in A\}$, then suppose that $H_\alpha \in [\omega]^{<\omega}$ lists all second coordinates other than $\omega$ used as parameters in $U_\alpha$. We can find an uncountable $B \subset A$ and $n \in \omega$ such that $\alpha \in B$ implies $H_\alpha \subset n$. Find $\sigma \in 2^n$ and assume, without loss of generality, that $\alpha \in B$ implies $\alpha \restriction n = \sigma$. Suppose that $G_\alpha \in [A]^{<\omega}$ list all first coordinates other than $\alpha$ used as parameters in $U_\alpha$. By the free set lemma, we can find an uncountable $C \subset B$ such that $c,d \in C \Rightarrow c \notin G_d$, $d \notin G_c$. If $c,d \in C$ then we have $\{(c,\omega),(c,n),(d,\omega),(d,n)\} \in U_c \cap U_d$ which is a contradiction.

*Lemma* 4. X *is* 2-*metacompact (any open cover has an open refinement so that each element lies in at most two elements of the refinement).*

*Proof.* Given any open cover by basic open sets, select an element of the cover for each $\{(\alpha,\omega)\}$. Then select a basic open neighborhood of each $\{(\alpha,\omega),(\alpha,n)\}$ which is not yet covered. Finally cover the isolated points which have not yet been covered with singletons. Note that a basic open neighborhood of $\{(\alpha,\omega)\}$ and a basic open neighborhood of $\{(\alpha,\omega),(\alpha,n)\}$ are disjoint unless $\{(\alpha,\omega),(\alpha,n)\}$ is an element of the basic open neighborhood of $\{(\alpha,\omega)\}$. This means that each $\{(\alpha,\omega),(\alpha,n),(\beta,\omega),(\beta,n)\}$ lies in at most one basic open neighborhood of an element of the form $\{(\alpha,\omega)\}$ or $\{(\alpha,\omega),(\alpha,n)\}$ and thus that no element of X is an element of more than two elements of the open refinement which has been constructed.

The property of being countably paracompact has a separation component which has content in non-normal spaces. No established terminology has arisen for this property. We will call it (*) in this section.

*Definition* 4. Let $X$ be a topological space. We say that $X$ is (*) if for any discrete family $\{F_n: n \in \omega\}$ of closed sets, there is a locally finite family $\{U_n: n \in \omega\}$ of open sets such that each $F_n \subset U_n$.

Countably paracompact spaces are (*)-spaces. Furthermore normal spaces are also (*)-spaces. The results of [37] use only (*) and thus can be viewed as generalizations of the corresponding results of [12] which use normality.

The definition of normalized thus leads naturally to another definition:

*Definition* 5. If $X$ is a topological space and $F = \{F_\alpha: \alpha \in \kappa\}$ is a discrete family of sets, then we say that $F$ is (*)-ized if, for each partition $\{A_n: n \in \omega\}$ of $\kappa$ into countably many disjoint subsets, there is a locally finite open family $\{U_n: n \in \omega\}$ such that, for each $\alpha \in A_n$, $F_\alpha \subset U_n$.

The referee has observed that, by way of contrast, Daniels has established:

*Theorem* 8. ([7]) ($MA_{\omega_1}$). *In a locally compact boundedly metacompact space, a discrete (*)-ized family of points of cardinality* $\aleph_1$ *is separated.*

Thus the example in Theorem 7 is normalized but not (*)-ized. This serves to illustrate that although normal implies (*), normalized does not imply (*)-ized.

## 4. An Example of Davies

In 1979, Peter Davies answered a question of Fleissner and Reed by constructing a curious example in ZFC:

*Theorem* 9. ([8]). *There is a completely regular space* Z *of cardinality* $\aleph_1$ *which has a point-countable base but which also has a closed discrete subset* D *which is not a* $G_\delta$.

An examination of Davies' space shows that each element of Z - D is an isolated point and that basic open neighborhoods of elements of D are countable and metrizable.

Nyikos has asked whether, in first countable normal spaces, each closed discrete set must be a $G_\delta$-set. Of course, under V = L, the answer is yes (if a closed discrete set is separated and each point is a $G_\delta$, then the closed discrete set must be a $G_\delta$-set). Shelah has answered this question.

*Theorem* 10. (Shelah [25]). *It is consistent (with* CH*) that there is a first countable normal space with a closed discrete set which is not a* $G_\delta$*-set.*

This leaves Nyikos' question open under $MA_{\omega_1}$ where the classical examples of first countable normal spaces with unseparated closed discrete sets are found.

*Problem* 1. Does $MA_{\omega_1}$ imply that there is a first countable normal space with a closed discrete set which is not a $G_\delta$-set?

In this section, we begin by showing that the fact that Davies' neighborhoods are metrizable but not compact is the best possible in ZFC.

*Theorem* 11. $(MA_{\omega_1})$. *In a Hausdorff space of cardinality* $\aleph_1$ *whose neighborhoods are either points or convergent sequences, closed discrete sets are* $G_\delta$*-sets.*

This theorem also shows that, although the classical examples of normal first countable spaces with unseparated closed discrete sets are Cantor trees and ladder systems, examples of first countable spaces with closed discrete sets which are not $G_\delta$-sets must be more complicated under $MA_{\aleph_1}$.

The theorem is a direct consequence of the following combinatorial result:

*Lemma* 5. $(MA_{\omega_1})$. *If* $\{D_\alpha : \alpha \in \omega_1\}$ *is a family of almost disjoint countable subsets of* $\omega_1$*, then* $\omega_1$ *is the union of countably many sets each of which intersect each* $D_\alpha$ *in a finite set.*

*Proof*. Let $\mathbb{P} = Fn(\omega_1, \omega) \times [\omega_1]^{<\omega}$. Let $(f^1, A^1) \leq (f^2, A^2)$ if and only if $f^1 \supset f^2$ and $A^1 \supset A^2$ and $(\forall \alpha \in A^2)$ $(f^1 - f^2)(D_\alpha) \cap f^2(D_\alpha) = \emptyset$. Now $(\mathbb{P}, \leq)$ is a partial order.

Each $D_\alpha = \{(f,A) \in \mathbf{P}: \alpha \in \mathrm{dom}(f)\}$ is dense. Each $E_\alpha = \{(f,A) \in \mathbf{P}: \alpha \in A\}$ is dense.

We shall show that $\mathbf{P}$ has the countable chain condition. Let $\{(f^\gamma, A^\gamma): \gamma \in \omega_1\}$ be an antichain. Without loss of generality, the $f^\gamma$'s are a compatible $\Delta$-system with root $\Gamma$ and the $A^\gamma$'s are a $\Delta$-system with root $\Delta$. Without loss of generality, $\mathrm{dom}(f^\gamma - \Gamma)$ is disjoint from $D_\alpha$ whenever $\alpha \in \Delta$. Without loss of generality, $\cup\{D_\alpha: \alpha \in A^{\gamma_1}\}$ is disjoint from $\mathrm{dom}(f^{\gamma_2} - \Gamma)$ when $\gamma_1 < \gamma_2$. Incompatibility implies that

$$(\forall \gamma_1 < \gamma_2)(\exists \alpha \in A^{\gamma_2}): (f^{\gamma_1} - \Gamma)D_\alpha \cap f^{\gamma_2}(D_\alpha) \neq \emptyset$$

Without loss of generality $\alpha \notin \Delta$ so we can assume that the $A^\gamma$'s are disjoint. Without loss of generality, $|f^\gamma - \Gamma|$ is fixed at m. Thus,

$$(\forall n \in \omega)(\forall i \leq m)(\exists \alpha \in A^{\omega+i}): (f^n - \Gamma)(D_\alpha) \cap f^{\omega+i}(D_\alpha) \neq \emptyset$$

and so, in particular,

$$(\forall n \in \omega)(\forall i \leq m)(\exists \alpha \in A^{\omega+i}): \mathrm{dom}(f^n - \Gamma) \cap D_\alpha \neq \emptyset$$

Let $\mathcal{U}$ be any free (non-principal) ultrafilter on $\omega$. Each $A^{\omega+i}$ is finite, so,

$$(\forall i \leq m)(\exists \alpha(i) \in A^{\omega+i})\{n: \mathrm{dom}(f^n - \Gamma) \cap D_{\alpha(i)} \neq \emptyset\} \in \mathcal{U}$$

Intersecting those m + 1 sets,

$$(\exists B \in \mathcal{U}): (\forall i \leq m)(\forall n \in B)\mathrm{dom}(f^n - \Gamma) \cap D_{\alpha(i)} \neq \emptyset$$

Geometrically, the $\mathrm{dom}(f^n - \Gamma)$ may be viewed as the rows of an $\omega \times m$ matrix and the $D_{\alpha(i)}$ may be viewed as m + 1 almost disjoint sets, each of which intersect each row of this matrix. That is a contradiction.

By $MA_{\omega_1}$, there is a filter $G \subset \mathbf{P}$ meeting each $D_\alpha$ and each $E_\alpha$. Let $g = \cup\{f: (\exists A \in [\omega_1]^{<\omega})(f,A) \in G\}$. Now g maps $\omega_1$ into $\omega$ and $\omega_1 = \cup\{g^{-1}(n): n \in \omega\}$ is a decomposition of $\omega_1$ into countably many sets.

We claim that $(\forall n \in \omega)(\forall \alpha \in \omega_1)|g^{-1}(n) \cap D_\alpha| < \omega$. Let (f,A) be an element of the generic filter such that $\beta \in D_\alpha, \alpha \in A$ and $f(\beta) = n$. We claim that $D_\alpha \cap g^{-1}(n) = f^{-1}(n) \cap D_\alpha$. Suppose $\gamma \in (g^{-1}(n) - f^{-1}(n)) \cap D_\alpha$. Say $\gamma \in h^{-1}(n)$ when $(h,B) \leq (f,A)$. We know that $(\forall \alpha \in A)$ $(h - f)(D_\alpha) \cap f(D_\alpha) = \emptyset$. Now $h(\gamma) = n$, $f(\beta) = n$, $\gamma \in D_\alpha$ and $\beta \in D_\alpha$ which is a contradiction.

Steprans has observed that this is a modification of Wage's partial order in Theorem 1 of [33].

We can shed more light on the problem by using a new definition:

*Definition* 6. A space X is *badly non-collectionwise Hausdorff* if there is a closed discrete set A such that whenever $\{\{U_a^n: a \in A\}: n \in \omega\}$ is a sequence of open sets such that $a \in U_a^n$, there is a sequence $\{a(n): n \in \omega\} \subset A$ such that $m,n \in \omega, m \neq n$ implies $U_{a(n)}^n \cap U_{a(m)}^m \neq \emptyset$.

*Lemma* 6. *If there is a (normal) first countable space* X *which is badly non-collectionwise Hausdorff, then there is a (normal) first countable space* Y *which contains a closed discrete set which is not a* $G_\delta$*-set.*

*Proof*. Assume $X'' = \emptyset$ and assume that $\{U_a^n: n \in \omega\}$ is a decreasing neighborhood base for each $a \in X'$. Next define a partial function $f: X' \to \omega$ to be *complete* if $|\mathrm{dom} f| = \omega$ and $a,b \in \mathrm{dom} f$ implies $U_a^{f(a)} \cap U_b^{f(b)} \neq \emptyset$. We shall introduce a new isolated point $p(f)$ for each complete $f$. We need only specify which open neighborhoods contain each new point $p(f)$ and do so by stating that $p(f) \in U_a^n$ if and only if $a \in \mathrm{dom} f$ and $n \leq f(a)$. The significant property of these new points is that they preserve disjunction. Suppose $U_a^n \cap U_b^m = \emptyset$ (in X). Let $p(f) \in U_a^n \cap U_b^m$ (in the new space). Thus $a,b \in \mathrm{dom} f$ and $n \leq f(a)$ and $m \leq f(b)$. By assumption, $U_a^{f(a)} \cap U_b^{f(b)} \neq \emptyset$ in X which is a contradiction. Thus the introduction of these new isolated points preserves normality. Now suppose X is badly non-collectionwise Hausdorff. We claim the new space fails to be perfect. Suppose otherwise that $\cap\{\cup\{U_a^{f_n(a)}: a \in X'\}: n \in \omega\} = X'$ where each $f_n: X' \to \omega$. There must be a sequence $\{a(n): n \in \omega\} \subset X'$ such that $m,n \in \omega$, $m \neq n$ implies $U_{a(n)}^{f_n(a(n))} \cap U_{a(m)}^{f_m(a(m))} \neq \emptyset$. Let $f: \{a(n): n \in \omega\} \to \omega$ be defined by $f(a(n)) = f_n(a(n))$. Finally the new $p(f) \in U_{a(n)}^{f_n(a(n))}$ for each $n \in \omega$.

The reason that Davies' example has little to do with point-countable bases is the following:

*Lemma* 7. *If there is a (normal) first countable completely regular space* X *which contains a closed discrete set* A *which is not a* $G_\delta$*-set, then there is a (normal) completely regular space* Y *with a point-countable base which contains a closed discrete set which is not a* $G_\delta$*-set. Furthermore, if* $|A| < \aleph_\omega$*, we can get* $|Y| = |X|$.

*Proof.* Let the set of isolated points be denoted by $G$. We can assume that $A \cup G = X$. We let $\mathcal{H} \subset [A]^\omega$ be so that $(\forall J \in [A]^\omega)(\exists H \in \mathcal{H})\colon H \supset J$. Let $Y = A \cup (G \times \mathcal{H})$. Let $G \times \mathcal{H}$ be a set of isolated points. For any neighborhood $U$ of $a \in A$ in $X$, let $U^* = \{a\} \cup \{(g,H)\colon g \in U,\ a \in H\}$. Since $Y$ has been created by *splitting points* in the sense of Bing [3], it suffices to show that $Y$ has a point-countable base and that $A$ remains not a $G_\delta$-set in $Y$.

Certainly each $a \in A$ lies in countably many elements of the canonical base, since $X$ is first countable. On the other hand, each $(g,H)$ lies in its own singleton as well as possibly countably many basic open neighborhoods of each $a \in H$. Thus each element of $Y$ lies in at most countably many elements of the canonical base.

Suppose that $\{U_n\colon n \in \omega\}$ is a family of open sets in $Y$ whose intersection is $A$. For each $n \in \omega$ and each $a \in A$, let $V(a,n)^* \subset U_n$ be a basic open neighborhood of $(a,n)$ and let $W_n = \cup\{V(a,n)\colon a \in A\}$. Since $A$ is not a $G_\delta$-set in $X$, this means that there is $g \in G \cap \cap\{W_n\colon n \in \omega\}$. Choose $B = \{a_n\colon n \in \omega\} \subset A$ such that $g \in V(a_n,n)$ for each $n \in \omega$. Now $(g,B) \in V(a_n,n)^*$ implies that $(g,B) \in U_n$ for each $n \in \omega$ and that is a contradiction.

Finally, suppose that $|A| = \aleph_n$ and $n$ is minimal for not being able to get $|Y| = |X|$. We can show that there is $\mathcal{H} \subset [A]^\omega$ of cardinality $|A|$ so that $(\forall J \in [A]^\omega)(\exists H \in \mathcal{H})\colon H \supset J$. Just note that $\omega_n$ is the union of $\aleph_n$ many subsets, each of cardinality less than $\aleph_n$, so that each countable subset of $\omega_n$ is a subset of one of these subsets and apply minimality of n.

*Corollary* 5. *The following are equivalent:*

- *There is a normal first countable space which is badly non-collectionwise Hausdorff.*
- *There is a normal space with a point-countable base which contains a closed discrete set which is not a* $G_\delta$*-set.*
- *There is a normal first countable space which contains a closed discrete set which is not a* $G_\delta$*-set.*

Davies gave an elegant geometrical description of his example. For variety, we give a description of his example which is somewhat different. In fact, we will show that Davies' example fails to satisfy a very weak separation property.

*Definition* 7. A space X is said to be discretely metanormal if, whenever $\{K_n: n \in \omega\}$ is a discrete family of closed discrete sets, there are open sets $\{U_{n,k}: n,k \in \omega\}$ such that $K_n \subset U_{n,k}$ for each $n,k \in \omega$ and $\cap\{U_{n,k}: n,k \in \omega\} = \emptyset$.

To put this definition into context, we note the definition of *metanormal* in Section 9 (Definition 8).

*Lemma* 8.

- *Any countably metacompact space is metanormal.*

- *Any metanormal space is discretely metanormal.*
- *Any perfect space is metanormal.*
- *Any normal space is metanormal.*
- *Any space in which each closed discrete set is a $G_\delta$-set must be discretely metanormal.*

*Theorem* 12. (Davies [8]). *There is a regular first countable space* X *of cardinality* $\aleph_1$ *which contains a closed discrete set which is not a* $G_\delta$*-set. In fact,* X *fails to be discretely metanormal.*

*Proof.* Let F be the set of countable limit ordinals. Let G be the set of countable successor ordinals. For each $\alpha \in F$, let $\{\alpha_n : n \in \omega\}$ be a sequence of successor ordinals increasing to $\alpha$. We shall assume for simplicity that $\alpha_0 = 0$. Let $X = F \cup [G]^2$ and declare $X - F$ to be a set of isolated points. We declare a basic open neighborhood of $\alpha \in F$ in parameter $\eta < \alpha$ to be

$$U_\eta(\alpha) = \{\alpha\} \cup \{\{\gamma,\delta\} \in [G]^2 : \quad (\exists n \in \omega)\alpha > \gamma > \alpha_{n+1} \geq \delta > \alpha_n > \eta\}$$

This is a first countable $T_1$ topological space.

Furthermore, we shall show that these neighborhoods $U_\eta(\alpha)$ are closed. Let $\beta \in F - U_\eta(\alpha)$.

- Suppose $\beta < \alpha$. Let n be maximal such that $\alpha_n < \beta$. We claim that $U_{\alpha_n}(\beta) \cap U_\eta(\alpha) = \emptyset$. Suppose otherwise that $\{\gamma,\delta\}$ is in the intersection. Since $\{\gamma,\delta\} \in U_{\alpha_n}(\beta)$, we deduce that $\beta > \gamma,\delta > \alpha_n$. By definition of n, we know that $\alpha_{n+1} \geq \beta$ so $\alpha_{n+1} > \gamma,\delta > \alpha_n$ which contradicts the assumption that $\{\gamma,\delta\} \in U_\eta(\alpha)$.

○ Suppose $\beta > \alpha$. We claim that $U_\alpha(\beta) \cap U_\eta(\alpha) = \emptyset$. Suppose otherwise that $\{\gamma,\delta\}$ is in the intersection. We quickly deduce that $\beta > \gamma,\delta > \alpha$ and $\alpha > \gamma,\delta > \eta$ which is a contradiction.

Thus X is zero-dimensional and thus completely regular.

We now show that X fails to be discretely metanormal. Suppose that $\{F_i: i \in \omega\}$ is a partition of F into stationary subsets. Suppose that, for each $i \in \omega$, $\{U_{i,n}: n \in \omega\}$ is a family of open sets each of which contains $F_i$. Suppose that $\pi: \omega \times F \to \omega_1$ is such that, for each $i,n \in \omega$, $\cup\{U_{\pi(n,\beta)}(\beta): \beta \in F_i\} \subset U_{i,n}$. By the pressing-down lemma, there are stationary sets $S_{n,i} \subset F_i$ and ordinals $\{\eta_{n,i}: n,i \in \omega\}$ such that $(\forall\beta \in S_{n,i})\pi(n,\beta) = \eta_{n,i}$. Let $\eta = \sup\{\eta_{n,i}: n,i \in \omega\}$. For each $n,i \in \omega$, there is $k(n,i) \in \omega$ and $\alpha^{n,i} \in \omega_1$ such that, without loss of generality, $(\forall\alpha \in S_{n,i})\alpha_{k(n,i)} = \alpha^{n,i} > \eta$. Let $\alpha^* = \sup\{\alpha^{n,i}: n,i \in \omega\}$. For each $n,i \in \omega$, there is $r(n,i) \in \omega$ and $\beta^{n,i} \in \omega_1$ such that, without loss of generality, $(\forall\alpha \in S_{n,i})\alpha_{r(n,i)} = \beta^{n,i} > \alpha^*$ and such that $r(n,i)$ is minimal for $\alpha_{r(n,i)} > \alpha^*$. Let $\beta^* = \sup\{\beta^{n,i}: n,i \in \omega\}$. Now, for each $n,i \in \omega$, choose $\gamma^{n,i} \in S_{n,i}$ such that $\gamma^{n,i} > \beta^*$.

We claim that $\{\alpha^*,\beta^*\} \in U_{\pi(n,\gamma^{n,i})}(\gamma^{n,i})$ for each $n,i \in \omega$ as required. We can calculate

$$\pi(n,\gamma^{n,i}) = \eta_{n,i} < \eta < \alpha^{n,i} = (\gamma^{n,i})_{k(n,i)}$$
$$< \alpha^* < \beta^{n,i} = (\gamma^{n,i})_{r(n,i)} < \beta^* < \gamma^{n,i}$$

## 5. Perfectly Normal Suslin Spaces

In this section, we refer to non-separable spaces which have the countable chain condition as *Suslin spaces*. Junnila [18] has constructed a perfectly normal Suslin space of character $2^{\aleph_1}$. In this section, we investigate the possibility of lowering the character in this construction. We observe:

*Theorem* 13. *There is a perfectly normal Suslin space of character* $\kappa$ *if and only if there is either an* L-*space of character* $\kappa$ *or a normal space of character* $\kappa$ *which is not weakly* $\aleph_1$-*collectionwise Hausdorff.*

*Proof*. ($\Leftarrow$) Hereditarily Lindelöf regular spaces are perfectly normal and have the countable chain condition. Any normal space of character $\kappa$ which is not weakly $\aleph_1$-collectionwise Hausdorff can be assumed to consist of a closed discrete set of cardinality $\aleph_1$ and a set of isolated points and Bing's method [3] of constructing Example H can be applied to make it perfectly normal. Now Junnila's method [18] can be applied, iterating the example $\omega$ times until it has the countable chain condition.

($\Rightarrow$) Let X be a perfectly normal space of character $\kappa$ with the countable chain condition which is not separable. If X has an uncountable discrete subset D, then $Y = X - (\overline{D} - D)$ is an open subset of X which also has character $\kappa$ and is normal (since X is hereditarily normal). Now D is closed in Y and Y has the countable chain condition, so Y is not weakly $\aleph_1$-collectionwise Hausdorff. If there is no uncountable discrete subset in X, then we can apply Šapirovskiĭ's result [32] that any space of countable spread has a dense subspace which is hereditarily Lindelöf. This dense subspace cannot be separable since X is not separable, and so must be an L-space whose character is no greater than that of X.

*Corollary* 6. *Perfectly normal Suslin spaces must have character greater than* $2^\omega$ *if and only if* $2^\omega < 2^{\omega_1}$ *and there are no* L-*spaces.*

*Proof.* $2^\omega = 2^{\omega_1}$ implies that Junnila's example has character $2^\omega$.

Tall [30] (using ideas of Šapirovskiĭ [32]) showed that $2^\omega < 2^{\omega_1}$ implies that normal spaces of character $2^\omega$ are weakly $\aleph_1$-collectionwise Hausdorff.

If there is an L-space, then there is an L-space which has weight $\omega_1$ (see 3.6 of [22]).

In 1980, Szentmiklóssy showed that it is consistent with $MA_{\aleph_1}$ (which implies $2^\omega = 2^{\omega_1}$) that there are no first countable L-spaces. The consistency of the non-existence of (first countable) L-spaces with $2^\omega < 2^{\omega_1}$ remains open.

*Corollary* 7. *If it is consistent with* $2^\omega < 2^{\omega_1}$ *that there are no first countable* L-*spaces, then it is consistent that there are no perfectly normal first countable Suslin spaces.*

**6. Fleissner's George**

The purpose of this section is to make three simultaneous observations about George, a space constructed by Fleissner in [11]. First, George need not be constructed on successor cardinals but can be done as well on regular limit cardinals. The significance of this is that it shows that any forcing which attempts to show the consistency of "$(<\kappa,\infty)$-collectionwise normal, $(\infty, <\kappa)$-collectionwise normal, cardinality $\kappa$ and character $\kappa$ implies $(\kappa,\kappa)$-collectionwise normal when $\kappa$ is regular" had better destroy inaccessible cardinals.

Second, George need not be constructed on cardinals at all but can be done as well on many right-separated spaces of order-type $\kappa$ which do not have a discrete subset of cardinal $\kappa$. The significance of this is that it shows that, under the continuum hypothesis, there is a normal space of character $\aleph_1$ which is not collectionwise normal with respect to a family of hereditarily separable sets. This contrasts nicely with Tall's result [29] that it is consistent with the continuum hypothesis that normal spaces of character $\aleph_1$ are collectionwise normal with respect to hereditarily Lindelöf sets.

Third, George has a closed subspace which is locally countable and is still not collectionwise normal. The significance of this lies in Frank Tall's conjecture that if the existence of large cardinals is consistent, then it is consistent that all normal spaces of countable tightness are collectionwise normal. This closed subspace disproves this in a strong fashion.

We shall proceed by presenting a general construction. First we explain the notation: A space is $(\kappa,\lambda)$-collectionwise normal if it is collectionwise normal with respect to any $\kappa$ many sets each of cardinality at most $\lambda$. The notations $(< \kappa,\infty)$-collectionwise normal and $(\infty, < \kappa)$-collectionwise normal are self-explanatory. A topology $\tau$ on $\kappa$ is said to be *right-separated* if each $\alpha \in \kappa$ is open.

*Theorem* 14. *If $\kappa$ is a regular uncountable cardinal and $\tau$ is a right-separated strongly zero-dimensional collectionwise normal topology on $\kappa$ which does not admit an unbounded closed discrete subspace and which does not admit a stationary set of isolated points in any $\kappa - \alpha$, then there is a space* W *of cardinality $\kappa$ and character at most* $\sup\{2^\gamma: \gamma < \kappa\}$ *which is $(< \kappa,\infty)$-collectionwise normal and $(\infty, < \kappa)$-collectionwise normal but not collectionwise normal with respect to $\kappa$ many copies of $(\kappa,\tau)$. Furthermore, if $\kappa$ is the successor of another cardinal $\mu$, then each element of* W *has a neighborhood of cardinality $\mu$.*

*Proof*. Let $X_\alpha = \{(\beta,\alpha) \in \kappa^2: \beta > \alpha\}$. Let $Y_\beta = \{(\gamma,\alpha) \in \kappa^2: \beta \geq \gamma > \alpha\}$. Let $F = \cup\{X_\alpha: \alpha \in \kappa\} = \cup\{Y_\alpha: \alpha \in \kappa\}$ be the subspace of $\kappa \times \kappa$ where the first $\kappa$ has the topology $\tau$ and the second $\kappa$ has the discrete topology.

Let $C(Y_\beta,\lambda)$ be the set of continuous functions from $Y_\beta$ into $\lambda$ where $Y_\beta$ has the subspace topology and $\lambda$ has the discrete topology. That is, $C(Y_\beta,\lambda)$ is the set of partitions of $Y_\beta$ into at most $\lambda$-many clopen sets.

Let $G_\beta = \Pi\{\lambda^{C(Y_\beta,\lambda)} : \lambda \in \kappa\}$. Formally, each of these $C(Y_\beta,\lambda)$'s should be multiplied by $\{\lambda\}$ to "make a note of" the codomain of each function. We do not do this to avoid the burden of notation, but we remind the reader that a consequence of this convention is that $\{C(Y_\beta,\lambda): \lambda \in \kappa\}$ is a disjoint family.

Let $X = \cup\{G_\beta: \beta \in \kappa\}$. Each $(\beta,\alpha) \in F$ can be identified with the $g \in G_\beta$ defined by letting $g(f) = f(\beta,\alpha)$ for each $f \in C(Y_\beta,\lambda)$. Of course, we are again abusing notation here since, really, $g(\lambda)(f) = f(\beta,\alpha)$. Thus $F$ can be considered a subset of $X$ since $C(Y_\beta,\lambda)$ separates the elements of $\{\beta\} \times \beta$.

A neighborhood of $(\beta,\alpha) \in F$ has a parameter

$$D \in [\cup\{C(Y_\beta,\lambda): \lambda \in \kappa\}]^{<\omega}$$

and is

$$D(\beta,\alpha) = \{h \in G_{\beta^*}: \beta^* \leq \beta,\ (\forall f \in D) h(f \restriction Y_{\beta^*}) = f(\beta,\alpha)$$

Again we mean $h(\lambda)(f \restriction Y_{\beta^*})$ where $\lambda$ is such that $f \in C(Y_\beta,\lambda)$. Let $G = X - F$ be a set of isolated points.

First note that $(\beta,\alpha) \in D(\beta,\alpha)$. This is true since $\beta \leq \beta$ and $(\forall f \in D) f \restriction Y_\beta = f$ by the definition of the embedding of $F$ in $X$.

To see that $X$ is a topological space, let $(\beta_1,\alpha_1) \in D(\beta,\alpha)$. We will show that $(\beta_1,\alpha_1) \in E(\beta_1,\alpha_1) \subset D(\beta,\alpha)$

where $E = \{f \upharpoonright Y_{\beta_1}: f \in D\}$. Next, to show that $E(\beta_1,\alpha_1) \subset D(\beta,\alpha)$, let $h \in E(\beta_1,\alpha_1)$. Suppose $h \in G_{\beta^*}$ and so $\beta^* \leq \beta_1 \leq \beta$. If $f \in D$, then $h \in E(\beta_1,\alpha_1)$ implies that $h(f \upharpoonright Y_{\beta_1} \upharpoonright Y_{\beta^*}) = (f \upharpoonright Y_{\beta_1})(\beta_1,\alpha_1)$. Letting $g \in F$ represent $(\beta_1,\alpha_1)$, we know that since $(\beta_1,\alpha_1) \in D(\beta,\alpha)$, $(f \upharpoonright Y_{\beta_1})(\beta_1,\alpha_1) = g(f \upharpoonright Y_{\beta_1}) = f(\beta,\alpha)$. Thus we deduce that $h(f \upharpoonright Y_{\beta^*}) = f(\beta,\alpha)$ as required.

To see that F has the subspace topology, we will show that for each clopen subset U of F, there is a function f such that

$$\cup\{\{f \upharpoonright Y_\beta\}(\beta,\alpha): (\beta,\alpha) \in U\} \cap F = U$$

Let $f: F \to 2$ be the characteristic function of U in F. It suffices to show that, whenever $(\beta,\alpha) \in U$ and $(\beta_0,\alpha_0) \in \{f \upharpoonright Y_\beta\}(\beta,\alpha) \cap F$, $(\beta_0,\alpha_0)$ must be in U. If $(\beta_0,\alpha_0)$ is identified with g, then $g(f \upharpoonright Y_{\beta_0}) = (f \upharpoonright Y_{\beta_0})(\beta_0,\alpha_0)$ while $(\beta_0,\alpha_0) \in \{f \upharpoonright Y_\beta\}(\beta,\alpha)$ means that $\beta_0 \leq \beta$ and so $g(f \upharpoonright Y_\beta \upharpoonright Y_{\beta_0}) = g(f \upharpoonright Y_{\beta_0}) = (f \upharpoonright Y_\beta)(\beta,\alpha) = f(\beta,\alpha)$. Therefore $f(\beta,\alpha) = f(\beta_0,\alpha_0)$ and so by definition of f, since $(\beta,\alpha) \in U$, we have $(\beta_0,\alpha_0) \in U$.

We must also show that for each parameter D, $D(\beta,\alpha) \cap F$ is open in F. Now $D \in [\cup\{C(Y_\beta,\lambda): \lambda \in \kappa\}]^{<\omega}$ so let $K = \cap\{f^{-1}(f(\beta,\alpha)): f \in D\}$. We know that K is an open set since each $Y_\beta$ is open. We calculate that

$$D(\beta,\alpha) \cap F =$$

$$\{(\beta^*,\alpha^*) \in F: \beta^* \leq \beta;\ (\forall f \in D)(f \upharpoonright Y_{\beta^*})(\beta^*,\alpha^*) = f(\beta,\alpha)\}$$

$$= (\cup\{G_{\beta^*}\colon \beta^* \le \beta\}) \cap \cap\{\{(\beta^*,\alpha^*) \in F\colon f(\beta^*,\alpha^*) = f(\beta,\alpha)\}\colon f \in D\}$$

$$= (\cup\{G_{\beta^*}\colon \beta^* \le \beta\}) \cap \cap\{f^{-1}(f(\beta,\alpha))\colon f \in D\}$$

$$= (\cup\{G_{\beta^*}\colon \beta^* \le \beta\}) \cap K$$

which is open as required.

Points are not closed in X but this is best treated with an afterthought--an application of the "perfection" lemma of [36]. To be specific, let $Z = F \cup (G \times \omega)$, let $G \times \omega$ be a set of isolated points and let $(U \cap F) \cup ((U \cap G) \times (\omega - n))$ be open for each open U in X and $n \in \omega$. Now Z is $T_1$ so long as F is $T_1$ as a subspace.

We shall show that X (and thus Z) is $(\lambda,\infty)$-collectionwise normal for each $\lambda < \kappa$. This demonstrates that Z is normal and Hausdorff. It suffices to separate a $\lambda$-partition of F into clopen sets. Let $f\colon F \to \lambda$ be a partition of F into clopen sets. Let $K_\gamma = \cup\{\{f \upharpoonright Y_\beta\}(\beta,\alpha)\colon (\beta,\alpha) \in f^{-1}(\gamma)\}$ for each $\gamma \in \lambda$. Each $K_\gamma$ is an open set which contains $f^{-1}(\gamma)$. We shall show that $\gamma \ne \gamma_0$ implies $K_\gamma \cap K_{\gamma_0} = \emptyset$. Let $f(\beta,\alpha) = \gamma$ and $f(\beta_0,\alpha_0) = \gamma_0$ and suppose $h \in \{f \upharpoonright Y_\beta\}(\beta,\alpha) \cap \{f \upharpoonright Y_{\beta_0}\}(\beta_0,\alpha_0)$ where $h \in G_{\beta^*}$. Thus $h(f \upharpoonright Y_\beta \upharpoonright Y_{\beta^*}) = (f \upharpoonright Y_\beta)(\beta,\alpha)$ and $h(f \upharpoonright Y_{\beta_0} \upharpoonright Y_{\beta^*}) = (f \upharpoonright Y_{\beta_0})(\beta_0,\alpha_0)$. Since $\beta^* < \beta,\beta_0$, we have $\gamma = f(\beta,\alpha) = f(\beta_0,\alpha_0) = \gamma_0$ which is a contradiction.

We now pass to a closed subspace $W \subset Z$. Let $W = F \cup \{(g,n) \in G \times \omega\colon g(\phi) = \alpha \Rightarrow \phi^{-1}(\alpha) \ne \emptyset\}$. Since W is a closed subspace of Z, it is a normal Hausdorff space which is $(\lambda,\infty)$-collectionwise normal for each $\lambda < \kappa$ and whose subspace of non-isolated points may be identified with F.

We show that even the closed subspace W fails to be collectionwise normal.

First we point out that a natural error is to reason as follows: Let $f \in C(F,\kappa)$ be defined by $f(\beta,\alpha) = \alpha$. It seems that $\{f \upharpoonright Y_\beta\}(\beta,\alpha) \cap \{f \upharpoonright Y_{\beta^*}\}(\beta^*,\alpha^*) = \emptyset$ whenever $\alpha \neq \alpha^*$. However this is not true. The reader must ask "for which $\lambda(\beta) \in \kappa$, is $f \upharpoonright Y_\beta \in C(Y_\beta,\lambda(\beta))$?". For the intersection to be empty $\lambda(\beta) = \lambda(\beta^*) = \lambda$ and that cannot be so for all $\beta \in \kappa$ since $\lambda < \kappa$.

We shall show that $\{X_\alpha: \alpha \in \kappa\}$ is a clopen partition of F which cannot be separated in W. Suppose $\{U_\alpha: \alpha \in \kappa\}$ were a separation of $\{X_\alpha: \alpha \in \kappa\}$. For each $\alpha \in \kappa$ and $\beta > \alpha$, there is a parameter $D_{\beta,\alpha}$ such that $D_{\beta,\alpha}(\beta,\alpha) \subset U_\alpha$. Actually $D_{\beta,\alpha}(\beta,\alpha)$ is a subset of X so we mean the set obtained by multiplying each isolated point by some fixed $\omega - n$ and then intersecting with W. We will continue to work in X for ease of exposition. Let $D^*_{\beta,\alpha} = \{\lambda \in \kappa: D_{\beta,\alpha} \cap C(Y_\beta,\lambda) \neq \emptyset\}$. Each $D^*_{\beta,\alpha}$ is finite.

We will now do a $\Delta$-system analysis of this neighborhood assignment. Since we need S and $\{S_\alpha: \alpha \in S\}$ to accomplish several things simultaneously, we will implicitly construct a finite descending sequence $\{S^i: i \in n\}$ of stationary sets and a sequence of families $\{S^i_\alpha: \alpha \in S^i\}$ such that, for any $i < i'$ and $\alpha \in S^{i'}$, $S^i_\alpha \supset S^{i'}_\alpha$. We start with $S^0 = \kappa$ and $(\forall\alpha \in \kappa)S^0_\alpha = \kappa$. We finish with $S^n$ and $\{S^n_\alpha: \alpha \in S^n\}$ but name these sets $S$ and $\{S_\alpha: \alpha \in S\}$. We will search only for facts which can be assumed to be true about unboundedly many $\alpha$ and, for each such $\alpha$, stationarily many $\beta$. That is, we will define an unbounded set $S \subset \kappa$ and, for each $\alpha \in S$, a stationary set $S_\alpha \subset \kappa$ and speak only about $(\beta,\alpha)$ which satisfy $\alpha \in S$ and $\beta \in S_\alpha$. We will not mention any of this (even $S$ or $S_\alpha$) explicitly.

We assume that each $\{D^*_{\beta,\alpha}: \beta \in \kappa\}$ is a $\Delta$-system with root $\Delta_\alpha$. We assume that $\{\Delta_\alpha: \alpha \in \kappa\}$ is a $\Delta$-system with root $\Delta$. Since there does not exist a stationary set of isolated points in any tail, we have

$$(\forall\alpha \in \kappa)(\exists\eta(\alpha) \in \kappa)(\forall\beta \in \kappa)$$
$$(\eta(\alpha),\alpha) \in \cap\{\phi^{-1}(\phi(\beta,\alpha)): \phi \in D_{\beta,\alpha}\}$$

For each $\lambda \in \Delta$, let $D^\lambda_{\beta,\alpha} = D_{\beta,\alpha} \cap C(Y_\beta,\lambda)$. If $\beta$ is not an isolated point in $\kappa - (\alpha + 1)$, then any $\phi \in D^\lambda_{\beta,\alpha}$ is determined by $\phi \upharpoonright \cup\{Y_\delta: \delta < \beta\}$. Thus, in this case, there are ordinals $\eta(\beta,\alpha,\lambda) < \beta$ such that all the elements of $D^\lambda_{\beta,\alpha}$ are distinct when restricted to $Y_{\eta(\beta,\alpha,\lambda)}$. Let us list, with fixed $n_\lambda$, $D^\lambda_{\beta,\alpha} = \{f^{\beta,\alpha,\lambda}_k: k \in n_\lambda\}$ for each $\lambda \in \Delta$. We also assume that, for each $\lambda \in \Delta$, $\eta(\beta,\alpha,\lambda)$ does not depend on $\beta$ and has $\eta^*(\alpha)$ as a maximum, as $\lambda$ ranges over $\Delta$. For each $\lambda \in \Delta$, find $\{v^\lambda_k: k \in n_\lambda\}$ such that, for each $k \in n_\lambda$, $f^{\beta,\alpha,\lambda}_k(\beta,\alpha) = v^\lambda_k$. This is possible since each $\lambda < \kappa$. This completes the $\Delta$-system analysis.

We now do a Ramsey analysis. Choose $\{\alpha_i: i \in \omega\}$ arbitrarily (from S). Let $\eta^* = \sup\{\eta(\alpha_i),\eta^*(\alpha_i): i \in \omega\}$.

Find $\{\beta_i: i \in \omega\}$ (so that $\beta_i \in S_{\alpha_i}$) such that each $\beta_i \geq \eta^*$ and each $D^*_{\beta_i,\alpha_i} \cap D^*_{\beta_j,\alpha_j} = \Delta$ for $i \neq j$. We are invoking a $\Delta$-system principle which says that if $\kappa$ is uncountable and we are given a $\kappa \times \omega$ matrix of finite sets and each n'th column forms a $\Delta$-system with root $\Delta_n$, and if these roots form a $\Delta$-system with root $\Delta$, then we can choose $\{\beta_i: i \in \kappa\}$ such that the finite sets with coordinates $\{(\beta_i,i): i \in \omega\}$ form a $\Delta$-system.

Construct a mapping $h_\lambda: [\omega]^2 \to 2^{n_\lambda \times n_\lambda}$ for each $\lambda \in \Delta$ by defining

$$h_\lambda(\{i,j\})(k,m) = 1 \Leftrightarrow f_k^{\beta_i,\alpha_i,\lambda} \upharpoonright \eta^* = f_m^{\beta_j,\alpha_j,\lambda} \upharpoonright \eta^*$$

Apply Ramsey's theorem to assume, without loss of generality, that each $h_\lambda$ is constant. We are finished the Ramsey analysis.

We claim that the supposedly disjoint neighborhoods remaining actually intersect. We shall show that $D_{\beta_0,\alpha_0}(\beta_0,\alpha_0) \cap D_{\beta_1,\alpha_1}(\beta_1,\alpha_1) \neq \emptyset$. Suppose $\beta_0 < \beta_1$. Let $\lambda \in \Delta$ be fixed. We will find $g \in \lambda^{C(Y_{\beta_0},\lambda)}$ such that

1. $(\forall f \in D^\lambda_{\beta_0,\alpha_0})g(f \upharpoonright Y_{\beta_0}) = f(\beta_0,\alpha_0)$

2. $(\forall f \in D^\lambda_{\beta_1,\alpha_1})g(f \upharpoonright Y_{\beta_0}) = f(\beta_1,\alpha_1)$

3. $g(\phi) = \alpha \Rightarrow \phi^{-1}(\alpha) \neq \emptyset$

We can satisfy (1) and (2) unless $(\exists f_0 \in D^\lambda_{\beta_0,\alpha_0})$ and $(\exists f_1 \in D^\lambda_{\beta_1,\alpha_1})$ such that $f_0 \upharpoonright Y_{\beta_0} = f_1 \upharpoonright Y_{\beta_0}$ and yet

$f_0(\beta_0,\alpha_0) \neq f_1(\beta_1,\alpha_1)$. Let us suppose $f_0$ has index $k_0$ and $f_1$ has index $k_1$. In particular, then, $f_0 \upharpoonright Y_{\eta^*} = f_1 \upharpoonright Y_{\eta^*}$ and so by definition, since

$$f_{k_0}^{\beta_0,\alpha_0,\lambda} \upharpoonright \eta^* = f_{k_1}^{\beta_1,\alpha_1,\lambda} \upharpoonright \eta^*$$

we have $h_\lambda(0,1)(k_0,k_1) = 1$. This implies by the use of Ramsey's theorem that $h_\lambda(1,2)(k_0,k_1) = 1$ and $h_\lambda(0,2)(k_0,k_1) = 1$. This means, by definition of $h_\lambda$, that

$$f_{k_0}^{\beta_1,\alpha_1,\lambda} \upharpoonright \eta^* = f_{k_1}^{\beta_2,\alpha_2,\lambda} \upharpoonright \eta^*$$

and

$$f_{k_0}^{\beta_0,\alpha_0,\lambda} \upharpoonright \eta^* = f_{k_1}^{\beta_2,\alpha_2,\lambda} \upharpoonright \eta^*$$

These equations imply, by transitivity of equality, that

$$f_{k_0}^{\beta_1,\alpha_1,\lambda} \upharpoonright \eta^* = f_{k_1}^{\beta_1,\alpha_1,\lambda} \upharpoonright \eta^*$$

Since $\eta^* < \eta(\beta_1,\alpha_1,\lambda)$, we deduce that $k_0 = k_1 = k$. This means that $f_0(\beta_0,\alpha_0) = f_k^{\beta_0,\alpha_0,\lambda}(\beta_0,\alpha_0) = v_k^\lambda = f_k^{\beta_1,\alpha_1,\lambda}(\beta_1,\alpha_1) = f_1(\beta_1,\alpha_1)$ which is a contradiction.

We can also satisfy (3) unless, for example, there is $f_1 \in D^\lambda_{\beta_1,\alpha_1}$ such that $(f_1 \upharpoonright Y_{\beta_0})^{-1} f_1(\beta_1,\alpha_1) = \emptyset$. The definition of $\eta(\alpha)$ implies that

$$(\eta(\alpha_1),\alpha_1) \in f_1^{-1}(f_1(\beta_1,\alpha_1))$$

Since $\eta(\alpha) < \eta^* \leq \beta_0$, we have a contradiction. The proof that W fails to be collectionwise normal is complete.

We shall show that W is also collectionwise normal with respect to sets of cardinality less than $\kappa$. The space Z is quite adequate to illustrate all the properties which have so far been demonstrated for W. The reason we have taken W as the space is the fact that

*Lemma* 9. *For any* $\gamma < \alpha < \beta$, *there is a neighborhood of* $(\beta,\alpha)$ *which misses* $\cup\{G_\delta: \delta \leq \gamma\}$.

*Proof.* Let $K = \cup\{X_\alpha: \alpha \in \gamma\}$. Let $\chi$ be the characteristic function of K. Suppose that $h \in \{\chi \restriction Y_\beta\}(\beta,\alpha) \cap G_\delta$ and $\delta \leq \gamma$. We deduce that $h(\chi \restriction Y_\delta) = h(\chi \restriction Y_\beta \restriction Y_\delta) = (\chi \restriction Y_\beta)(\beta,\alpha) = 0$. Meanwhile, by the definition of the subspace W, $h(\phi) = \alpha \Rightarrow \phi^{-1}(\alpha) \neq \emptyset$. In this context, we deduce that $(\chi \restriction Y_\delta)^{-1}(0) \neq \emptyset$. By the definition of $\chi$, we must have $Y_\delta \cap (X - K) \neq \emptyset$ which is a contradiction to $Y_\delta \subset Y_\gamma \subset K$.

*Proof of Theorem continued:* Any discrete family D of closed sets each of cardinality less than $\kappa$ intersects each $X_\alpha$ in a bounded set. Let us define a function $\eta$ such that this bounded set is contained in $\{(\beta,\alpha): \beta < \eta(\alpha)\}$. We can find a closed unbounded set C in $\kappa$ including 0 such that $\alpha < \beta$ and $\beta \in C$ implies $\eta(\alpha) < \beta$. Let $\Delta = \{(\beta,\alpha) \in F: \beta > \gamma > \alpha \Rightarrow \gamma \notin C\}$. Now $\cup D \cap F \subset \Delta$. For each $\gamma \in C$, let $\Delta_\gamma = \{(\beta,\alpha) \in \Delta: \gamma$ is the greatest element of C such that $\gamma \leq \alpha\}$. We have $\Delta = \cup\{\Delta_\gamma: \gamma \in C\}$. By Lemma 9, each element of $\Delta_\gamma$ has a neighborhood which does not intersect any basic open neighborhood of any $\Delta_\delta$ whenever $\delta < \gamma$ and whenever there is an element of $C$ between $\delta$ and $\gamma$. This means that $\{\Delta_\gamma : \gamma \in C\}$ is a separated family. To separate $D$, it suffices to separate each $\{F \cap \Delta_\gamma : F \in D\}$ and, since each of these families has cardinality less than $\kappa$, this is possible.

The character of each point $(\beta,\alpha)$ can be calculated as $\Sigma_{\lambda\in\kappa} \lambda^{|\beta|} \leq \sup\{2^\gamma : \gamma < \kappa\}$ (since $|Y_\beta| = |\beta|$).

We shall reduce the (local) size of the space. Define $G^*_\beta$ to be a dense subspace of $\Pi\{\lambda^{C(Y_\beta,\lambda)} : \lambda \in \kappa\}$ in the product topology. Then, for each $g \in G^*_\beta$, we define a new function $g^* \in G_\beta$. For each $\phi$ such that $\phi^{-1}(g(\phi)) \neq \emptyset$, define $g^*(\phi) = g(\phi)$. For each $\phi$ such that $\phi^{-1}(g(\phi)) = \emptyset$, choose $\alpha^*$ such that $\phi^{-1}(\alpha^*) \neq \emptyset$ and define $g^*(\phi) = \alpha^*$. Now replace $G_\beta$ with $\{g^* : g \in G^*_\beta\}$. In this closed subspace, we can still find the counterexample to collectionwise normality.

Let $\mu$ be a cardinal defined to be $\kappa$ if $\kappa$ is a limit cardinal and defined to be the predecessor of $\kappa$ if $\kappa$ is a successor cardinal. We can calculate

$$d\left(\prod_{\lambda\in\kappa} \lambda^{C(Y_\beta,\lambda)}\right) \leq d\left(\prod_{\lambda\in\kappa} \mu^{C(\mu,\mu)}\right) = d(\mu^{\kappa\cdot 2^\mu}) = \mu$$

*Corollary* 8. (ZFC). *For each cardinal* $\kappa$, *there is a* $(<\kappa,\infty)$-*collectionwise normal,* $(\infty,<\kappa)$-*collectionwise normal space which has cardinality* $\kappa$ *and has character*

$\sup\{2^\gamma : \gamma < \kappa\}$ *but which fails to be* $(\kappa,\kappa)$*-collectionwise normal. If* $\kappa$ *is an inaccessible cardinal, then the character is* $\kappa$.

*Proof.* Use the order topology on $\kappa$ as $\tau$. No stationary set has the discrete subspace topology. There are no infinite closed discrete sets. Compact zero-dimensional spaces are strongly zero-dimensional and all separation takes place in a compact initial segment.

*Corollary* 9. (CH). *There is a normal, collectionwise Hausdorff locally countable space of character* $2^\omega$ *which fails to be collectionwise normal with respect to* $\aleph_1$ *many hereditarily separable sets.*

*Proof.* Any hereditarily separable topology fails to have a stationary set of isolated points in a tail or an unbounded closed discrete subspace so it suffices to exhibit, under the continuum hypothesis, a strongly zero-dimensional collectionwise normal hereditarily separable right-separated space. The Kunen line [17] is such a space. To see that the Kunen line is strongly zero-dimensional, note that, in the proof of normality of $\tau$ in [17], the $\tau$-open sets and $\rho$-open sets can be taken to be $\tau$-clopen and $\rho$-clopen respectively since $\tau$ is zero-dimensional and, if we start with the Cantor set, so is $\rho$.

*Corollary* 10. (ZFC). *There is a normal, collectionwise Hausdorff locally countable space of cardinality* $\aleph_1$ *and character* $2^\omega$ *which fails to be collectionwise normal.*

*Corollary* 11. (ZFC). *There is a normal space of countable tightness which fails to be collectionwise normal.*

**7. Singular Compactness**

For certain cardinals $\kappa$ and certain properties P, if a structure of cardinality $\kappa$ exhibits property P, then some substructure of cardinality less than $\kappa$ must also exhibit property P. We say that property P *reflects downward* at $\kappa$. For first-order properties, this is just the downward Löwenheim-Skolem theorem. The question of whether the second-order separation properties considered in this article reflect downward has gathered a fair amount of interest. In this section, we consider two properties, both of which are the failure of certain families of points to be separated. The referee has established that the failure of collectionwise Hausdorff reflects downward at strong limits of countable cofinality when the character is less than $\kappa$. We have established that the failure of 'weakly collectionwise Hausdorff' reflects downward at singular strong limits $\kappa$ when the character is less than $\kappa$.

We say that a space X is weakly $\kappa$-collectionwise Hausdorff if any discrete family of points of cardinality $\kappa$ contains a separated subfamily of cardinality $\kappa$. If a space is weakly $\kappa$-collectionwise Hausdorff for all cardinals $\kappa$, then we say that X is weakly collectionwise Hausdorff. This property was introduced by Frank Tall [30]. He characterized this property as a "consolation prize," when neither collectionwise normal nor collectionwise Hausdorff could be obtained but it has proven surprisingly interesting.

*Theorem* 15. *If* $\kappa$ *is a singular strong limit cardinal, then character* $< \kappa$ *and weakly* $< \kappa$*-collectionwise Hausdorff imply weakly* $\kappa$*-collectionwise Hausdorff.*

*Proof.* Suppose X is a space of character $< \kappa$ which has a discrete family of points A such that $|A| = \kappa$ and yet which contains no separated subfamily of cardinality $\kappa$. The notation "X has character $< \kappa$" means that there is a cardinal $\lambda < \kappa$ such that the character of each point of X is at most $\lambda$. We thus assume that each point $x \in X$ has a neighborhood basis $\{U_x(\alpha): \alpha \in \lambda\}$. Let $\{\kappa_\alpha: \alpha < cf(\kappa)\}$ be an increasing sequence of successor cardinals such that $\kappa_0 > 2^{\lambda \cdot cf(\kappa)}$ and, for each $\alpha \in cf(\kappa)$, $2^{\Sigma\{\kappa_\beta: \beta<\alpha\}} < \kappa_\alpha$. Express A as the union of a partition $\{A_\alpha: \alpha \in cf(\kappa)\}$ where $|A_\alpha| = \kappa_\alpha$. For each $\alpha$, let $A_{<\alpha} = \cup\{A_\beta: \beta < \alpha\}$ and, for each $x \in A_\alpha$, let $f_x: A_{<\alpha} \times \lambda^2 \to 2$ be defined by $f_x(y,\gamma,\delta) = 1 \Leftrightarrow U_x(\gamma) \cap U_y(\delta) \neq \emptyset$. There are precisely $2^{|A_{<\alpha}| \cdot \lambda} = 2^{\Sigma\{\kappa_\beta: \beta<\alpha\}} \cdot 2^\lambda < \kappa_\alpha$ many possibilities for $f_x$ and so we can find, for each $\alpha \in cf(\kappa)$, a subset $B_\alpha$ of $A_\alpha$ of cardinality $\kappa_\alpha$ and a function $f_\alpha: A_{<\alpha} \times \lambda^2 \to 2$ such that $(\forall x \in B_\alpha) f_x = f_\alpha$.

Now, for each $\alpha \in cf(\kappa)$ and each $x \in B_\alpha$, let $g_x: (cf(\kappa) - (\alpha + 1)) \times \lambda^2 \to 2$ be defined by $g_x(\beta,\gamma,\delta) =$

$f_\beta(x,\gamma,\delta)$. There are precisely $2^{cf(\kappa)\cdot\lambda} < \kappa_\alpha$ many possibilities for $g_x$ and so we can assume, without loss of generality, that for each $\alpha \in cf(\kappa)$, we can find $g_\alpha$: $(cf(\kappa) - (\alpha + 1)) \times \lambda^2 \to 2$ such that $(\forall x \in B_\alpha) g_x = g_\alpha$. Choose $x_\alpha \in B_\alpha$ for each $\alpha \in cf(\kappa)$. The discrete family of points $\{x_\alpha: \alpha \in cf(\kappa)\}$ has cardinality $cf(\kappa)$. If X were weakly $< \kappa$-collectionwise Hausdorff, then this family would have a separated subfamily $\{x_\alpha: \alpha \in C\}$ of cardinality $cf(\kappa)$. Let $h: C \to \lambda$ witness this fact. We can now define a separation for $\cup\{B_\alpha: \alpha \in C\}$. Take an assignment which witnesses the fact that each $B_\alpha$ is separated (we can do this without loss of generality since each $B_\alpha$ has cardinality less than $\kappa$ and X is weakly $< \kappa$-collectionwise Hausdorff) and combine it with the assignment $j: \cup\{B_\alpha: \alpha \in C\} \to \lambda$ defined by $j(x) = h(\gamma) \Leftrightarrow x \in B_\gamma$. We need to show that

$$\alpha,\beta \in C,\ \alpha < \beta,\ x \in B_\alpha,\ y \in B_\beta \Rightarrow U_x(j(x)) \cap U_y(j(y)) = \emptyset$$

To see this, first deduce that

$$U_x(j(x)) \cap U_y(j(y)) = \emptyset \Leftrightarrow f_y(x,j(y),j(x)) = 0$$
$$\Leftrightarrow f_\beta(x,j(y),j(x)) = 0 \Leftrightarrow g_x(\beta,j(y),j(x)) = 0$$
$$\Leftrightarrow g_\alpha(\beta,j(y),j(x)) = 0$$

This deduction is true for all choices of x and y, including $x_\alpha, x_\beta$. Meanwhile, $U_{x_\alpha}(h(\alpha)) \cap U_{x_\beta}(h(\beta)) = \emptyset$. Since $j(x_\alpha) = h(\alpha)$ and $j(x_\beta) = h(\beta)$, we deduce by the forward implication that

$$g_\alpha(\beta,j(x_\beta),j(x_\alpha)) = 0$$

Since $j(x_\beta) = h(\beta) = j(y)$ and $j(x_\alpha) = h(\alpha) = j(x)$, we have $g_\alpha(\beta, j(y), j(x)) = 0$. The backwards implication shows that we have identified a subfamily $\cup\{B_\alpha: \alpha \in C\}$ of cardinality $\kappa$ which is separated.

Note that no separation axioms (not even $T_0$) are assumed in this argument. I thank the referee for pointing out that the original proof of this theorem was half-baked.

*Problem* 2. If $\kappa$ is not a singular strong limit cardinal, then is it consistent that there is a first countable space which is weakly $< \kappa$-collectionwise Hausdorff but which fails to be weakly $\kappa$-collectionwise Hausdorff?

The fact that the failure of collectionwise Hausdorff does not often reflect downward has a long history. In 1972, Blair [4] established, under GCH, that, for each regular uncountable cardinal $\kappa$, there is a regular space which is $< \kappa$-collectionwise Hausdorff but not $\kappa$-collectionwise Hausdorff. In 1975, Fleissner [14] removed the assumption of GCH and weakened the assumption of regularity to uncountable cofinality. In 1976, Fleissner [13] solved the remaining case by showing that, for any uncountable cardinal $\kappa$, there is a regular space which is $< \kappa$-collectionwise Hausdorff but not $\kappa$-collectionwise Hausdorff.

Restricting our attention to spaces of small character makes the existence of such spaces a more difficult question. In 1977, Fleissner [15] showed that, if $\kappa$ is a regular cardinal which is not weakly compact, then it is consistent that there is a first countable regular space which is $< \kappa$-collectionwise Hausdorff but not $\kappa$-collectionwise Hausdorff. On the other hand, if $\kappa$ is weakly compact, then every first countable regular space which is $< \kappa$-collectionwise Hausdorff must be $\kappa$-collectionwise Hausdorff. In 1977, Shelah [26] showed that, if it is consistent that there is a weakly compact cardinal, then it is consistent that every locally countable first countable space which is $\aleph_1$-collectionwise Hausdorff must be $\aleph_2$-collectionwise Hausdorff. Furthermore, he showed that, if it is consistent that there is a supercompact cardinal, then it is consistent that every locally countable first countable space which is $\aleph_1$-collectionwise Hausdorff must be collectionwise Hausdorff.

The referee has succeeded in proving an important result which we include with their kind permission. The author is impressed by this proof as a hybrid of Fleissner's proof [12] that, under V = L, normal Moore spaces are collectionwise Hausdorff and Pol's proof [21] of Arhangel'skiĭ's result [1] that first countable compact Hausdorff spaces have cardinality at most the continuum.

*Theorem* 16. (The Referee). *If $\kappa$ is a strong limit cardinal of cofinality $\omega$, then any space of character less than $\kappa$ which is $< \kappa$-collectionwise Hausdorff must be $\kappa$-collectionwise Hausdorff.*

*Proof.* Suppose $\kappa$ is an unseparated closed discrete set in a space X of character $\chi$ less than $\kappa$. Let $\kappa_n \nearrow \kappa$. We claim that, for each $n \in \omega$, there is a neighborhood assignment $f_n: \kappa_n \to \chi$ which separates $\kappa_n$ such that

$$(\forall\alpha \in \kappa - \kappa_n)\alpha \notin \overline{\cup\{U(\beta,f_n(\beta)): \beta \in \kappa_n\}}$$

Suppose not. For every neighborhood assignment $f: \kappa_n \to \chi$, let $\alpha_f \in \kappa - \kappa_n$ be such that

$$\alpha_f \in \overline{\cup\{U(\beta,f(\beta)): \beta \in \kappa_n\}}$$

There are $\chi^{\kappa_n} < \kappa$ many possible f and so, by $(< \kappa)$-collectionwise Hausdorff, let $U(\beta,g(\beta))$ be a neighborhood assignment which separates $\kappa_n \cup\{\alpha_f: f \in \chi^{\kappa_n}\}$. Now let $f = g \upharpoonright \kappa_n$. We know that

$$\alpha_f \in \overline{\cup\{U(\beta,f(\beta)): \beta \in \kappa_n\}} = \overline{\cup\{U(\beta,g(\beta)): \beta \in \kappa_n\}}$$

and so $U(\alpha_f,g(\alpha_f)) \cap U(\beta,g(\beta)) \neq \emptyset$ for some $\beta \in \kappa_n$ which is a contradiction. For each $\alpha \in \kappa$, let $n_\alpha$ be minimal such that $\alpha \in \kappa_n$. For $m < n_\alpha$, let $n(\alpha,m) \in \omega$ be such that

$$U(\alpha,n(\alpha,m)) \cap \cup\{U(\beta,f_m(\beta)): \beta \in \kappa_m\} = \emptyset$$

A separation is given by assigning, to each $\alpha \in \kappa$, the neighborhood

$$U(\alpha,f_{n_\alpha}(\alpha)) \cap \cap\{U(\alpha,n(\alpha,m)): m < n_\alpha\}$$

*Problem* 3. Suppose $\kappa$ is not a strong limit cardinal of cofinality $\omega$. Is it consistent that there is a first countable space which is $< \kappa$-collectionwise Hausdorff but not $\kappa$-collectionwise Hausdorff?

## 8. Navy's Spaces

This section will attempt to clarify the interrelationship of base properties in regular topological spaces. In this section, a base property is the property of having a base with a 'nice' local behaviour. The 'simple' examples are the following:

- There is a $\sigma$-discrete base
- There is a $\sigma$-locally finite base
- There is a $\sigma$-locally countable base
- There is a $\sigma$-disjoint base
- There is a $\sigma$-point finite base
- There is a $\sigma$-point countable base

Of course, the last property is equivalent to having a point countable base. Regular spaces with a locally countable base are precisely the spaces which are the free union of separable metric spaces.

The question arises whether there are any non-trivial implications among these properties. Certainly there is the classical metrization theorem:

*Theorem* 17. (Bing, Nagata, Smirnov, Stone). (See Theorems 4.4.7 and 4.4.8 of [9]).

*If* X *is a regular space, then the following are equivalent:*

- X *is metrizable*
- X *has a $\sigma$-discrete base*
- X *has a $\sigma$-locally finite base*

In this section, we observe that the para-Lindelöf not paracompact spaces of Caryn Navy complete the assertion that there are no other non-trivial relationships among these base properties (even among Moore spaces).

Four counterexamples are needed to demonstrate this fact:

*Theorem* 18. 1.

1. *There is a Moore space with a $\sigma$-locally countable base and a $\sigma$-disjoint base which fails to be metrizable* (Fleissner, Reed [16]).
2. *There is a Moore space with a $\sigma$-disjoint base which fails to have a $\sigma$-locally countable base* (Bing's example B [3]).
3. *There is a Moore space with a $\sigma$-locally countable base which fails to have a $\sigma$-point finite base* (Navy [19]).
4. *There is a Moore space with a $\sigma$-locally countable base and a $\sigma$-point finite base which fails to have a $\sigma$-disjoint base* (Navy [19]).

Burke asked in the Problems Section of the 1979 issue of Topology Proceedings (Vol. 4, No. 2) whether there is a regular space with a $\sigma$-locally countable base which fails to have a $\sigma$-disjoint base. Navy's counterexamples (3) and (4) in the theorem answer this question affirmatively.

The construction of counterexamples (3) and (4) uses ideas of Fleissner [10] and are obtained by jettisoning some of what makes Navy's examples so extraordinary. We will refer to counterexample (3) as P and counterexample (4) as Q.

*Proof*. Let $X = \omega_1^{\omega}$. Let $BO(X)$ be the set of basic open sets of X represented by nonempty finite partial functions $\sigma: \omega \to \omega_1$. Let $L \subset [BO(X)]^2$ be the set of unordered pairs $\{\rho,\tau\}$ such that $\mathrm{dom}(\rho) = \mathrm{dom}(\tau)$ and

$$\rho(0) < \tau(0) < \rho(1) < \tau(1) < \ldots$$

Let $M \subset [BO(X)]^{\omega}$ be the set of *unordered* countable sets $\{\rho_i: i \in \omega\}$ such that $(\forall i,j \in \omega)\mathrm{dom}(\rho_i) = \mathrm{dom}(\rho_j)$ and such that

$$\rho_0(0) < \rho_1(0) < \rho_2(0) < \ldots < \rho_0(1) < \rho_1(1) < \rho_2(1) < \ldots$$

We define a topology on $Q = X \cup L$ as follows: Let L be a set of isolated points. A neighborhood of $f \in X$ is given in parameters $n \geq 2$ and a finite set $K \subset f(0)$ by

$$U_{n,K}(f) = \{g \in X: g \upharpoonright n = f \upharpoonright n\} \cup \{\{\rho,\tau\} \in L: \rho \supset f \upharpoonright n,\ \tau(0) \notin K\}$$

We define a topology on $P = X \cup M$ as follows: Let M be a set of isolated points. A neighborhood of $f \in X$ is given in parameters $n \geq 2$ and a finite set $K \subset f(0)$ by

$$U_{n,K}(f) = \{g \in X: g \upharpoonright n = f \upharpoonright n\} \cup \{\{\rho_i: i \in \omega\} \in M: \rho_0 \supset f \upharpoonright n,\ (\forall i \geq 1)\rho_i(0) \notin K\}$$

*Claim*. This defines a base. Let $f_0 \in U_{n,K}(f) \cap U_{n_1,K_1}(f_1) \cap X$. That means that $f_0 \upharpoonright n = f \upharpoonright n$ and $f_0 \upharpoonright n_1 = f_1 \upharpoonright n_1$. Suppose $n_1 > n$. Thus $f_0 \in U_{n_1,K\cup K_1}(f_0)$ $U_{n,K}(f) \cap U_{n_1,K_1}(f_1)$ as required.

*Claim.* This is a clopen base and points are closed. Every point is the intersection of its neighborhoods and $X$ is closed in the space. Let $g \in X - U_{n,K}(f)$. Thus $g \restriction n \neq f \restriction n$. If $U_{n,K}(f) \cap U_{n,\emptyset}(g) \neq \emptyset$, then $f(0) \neq g(0)$ and so,

- (For Q) If $U_{n,\{f(0)\}}(g) \cap U_{n,K}(f)$ contains $\{\rho,\tau\}$ where $\mathrm{dom}(\rho) = \mathrm{dom}(\tau) = m \geq n$, then, say $\rho \restriction n = g \restriction n$ and $\tau \restriction n = f \restriction n$. Now $\tau(0) = f(0) \in \{f(0)\}$ and so $\{\rho,\tau\} \not\subseteq U_{n,\{f(0)\}}(g)$ which is a contradiction.
- (For P) If $U_{n,\{f(0)\}}(g) \cap U_{n,K}(f)$ contains $\{\rho_i: i \in \omega\}$ where $(\forall i \geq 0)\mathrm{dom}(\rho_i) = m \geq n$, then, say $\rho_0 \restriction n = g \restriction n$ and $\rho_1 \restriction n = f \restriction n$. Now $\rho_1(0) = f(0) \in \{f(0)\}$ and so $\{\rho_i: i \in \omega\} \not\subseteq U_{n,\{f(0)\}}(g)$ which is a contradiction.

*Claim.* This is a $\sigma$-locally countable base. For each $n$, $f \restriction n = g \restriction n$ implies $U_{n,K}(f) = U_{n,K}(g)$. So we consider $U_K(\sigma)$ for $\sigma$ basic open where $\mathrm{dom}(\sigma) = n$ for simplicity of notation. For each $\sigma$, there are countably many finite $K \subset \sigma(0)$. Let them be enumerated as $\{K(\sigma,i): i \geq 0\}$. Claim that $\{U_{K(\sigma,i)}(\sigma): \mathrm{dom}(\sigma) = n\}$ is a locally countable family of open sets for each fixed $i \geq 0$ and $n \geq 2$. It is a disjoint family on $X$.

(For Q) Any isolated point $\{\rho,\tau\}$ is in at most two elements, namely those of $\rho \restriction n$ and $\tau \restriction n$.

(For P) Any isolated point $\{\rho_i: i \in \omega\}$ is in at most countably many elements, namely those of $\rho_i \restriction n$ for $i \in \omega$.

Let $g \in X$. We claim that $U_\emptyset(g \upharpoonright (n + 1))$ intersects countably many elements of $\{U_{K(\sigma,i)}(\sigma)\colon \sigma \neq g \upharpoonright n, \mathrm{dom}(\sigma) = n\}$. If it intersects in $\{\rho,\tau\}$ (for example), then $g \upharpoonright (n + 1) = \rho \upharpoonright (n + 1)$ and $\sigma = \tau \upharpoonright n$. Since $\rho$ and $\tau$ are 'entwined', $\mathrm{rng}(\sigma) \subset g(n)$. Thus there are only countably many possible such $\sigma$ and so we are done.

*Claim.* Q has a $\sigma$-point finite base. Any isolated point is in at most two elements of $\{U_{K(\sigma,i)}(\sigma)\colon \mathrm{dom}(\sigma) = n\}$ for each $i \geq 0$ and $n \geq 2$. These families are disjoint on X.

*Claim.* P and Q are Moore spaces. For each $\alpha \in \omega_1$, let $\{F_n^\alpha\colon n \in \omega\}$ list $[\alpha]^{<\omega}$. For each $n \geq 2$ and $r \in \omega$, let $\mathcal{W}_{n,r}$ be the open cover which is obtained by taking $\{U_{n,K}(f)\colon K = F_r^{f(0)}, f \in X\}$ and adding all isolated points which are not covered by that family. We must show that for every element x of P (or Q) and for every neighborhood U of x, there are $i \geq 2$, $r \in \omega$ such that $\mathrm{St}(x,\mathcal{W}_{i,r}) \subset U$ (see page 408 of [9]).

If x is an isolated point, thus an unordered set of partial functions whose common domain is some $n \in \omega$, then, for each $i > n$, there is exactly one element of $\mathcal{W}_{i,r}$ which contains x, namely $\{x\}$.

If $x \in X$, then there is some $U_{n,K}(x) \subset U$. Let $i = n$ and let r be such that $K = F_r^{x(0)}$. Suppose $x \in U_{j,L}(f) \in \mathcal{W}_{i,r}$ and $y \in U_{j,L}(f)$. Now $U_{j,L}(f) \in \mathcal{W}_{i,r}$ implies that $j = i$ and $L = F_r^{f(0)}$. Since $x \in U_{j,L}(f)$, we know that

$x \restriction j = f \restriction j$ which means that, since $i \geq 2$, $x(0) = f(0)$ which means that $K = L$. This implies that $y \in U_{n,K}(f) = U_{n,K}(x) \subset U$ as required.

*Claim.* There does not exist a $\sigma$-disjoint base for Q and there does not exist a $\sigma$-point finite base for P. Suppose $\mathcal{B} = \cup\{\mathcal{B}_i: i \in \omega\}$ were such a base. Let $\{U_{K(\sigma)}(\sigma): \sigma \in A\}$ be a family of basic open sets each contained in an element $B_\sigma$ of $\mathcal{B}$ such that $B_\sigma \cap \{f: f(0) \neq \sigma(0)\} = \emptyset$ such that A is a cover of X by elements of BO(X). Here we need Fleissner's combinatorial notion of an n-full set. We say that $B \subset BO(X)$ is an n-full set if each element has domain n and if

$$(\forall k < n)(\forall \sigma \in B)|\{\alpha \in \omega_1: (\exists \tau \in B)\tau \supset \sigma \restriction k \cup (k,\alpha)\}| = \aleph_1$$

These n-full sets have nice combinatorial properties: any cover of X by elements of BO(X) contains an n-full set for some $n \in \omega$; furthermore, if an n-full set is partitioned into countably-many subsets, at least one of these subsets contains an n-full set. These facts are proved in Fleissner's paper [10] and were later used in Fleissner's solution of the normal Moore space problem.

Find an n-full set $B \subset A$, and an i,k such that, for each $b \in B$, $U_{K(b)}(b)$ is contained in an element of $\mathcal{B}_i$ and $|K(b)| = k$. Define $\{\sigma(i): i \geq 0\} \subset B$ such that

$$\sigma_0(0) < \sigma_1(0) < \sigma_2(0) < \ldots < \sigma_0(1) < \sigma_1(1) < \sigma_2(1) < \ldots$$

We define a mapping with domain $[\omega]^2$ as follows: Consider $\{i,j\}$ where $i < j$. Map it to $\emptyset$ if neither $\sigma_j(0) \in K(\sigma_i)$ nor $\sigma_i(0) \in K(\sigma_j)$. Map it to $(0,m)$ if $\sigma_j(0) \in K(\sigma_i)$ and is the m-th element of $K(\sigma_i)$. Map it to $(1,m)$ if $\sigma_i(0) \in K(\sigma_j)$ and is the m-th element of $K(\sigma_j)$ and if the previous case does not apply. This mapping has $2k + 1$ possible outcomes. By Ramsey's theorem, we may assume, without loss of generality, that each unordered pair is mapped to a fixed point. We claim that the fixed point is $\emptyset$. Suppose for example that the fixed point is $(1,m)$. In this case, both $\sigma_1(0)$ and $\sigma_2(0)$ are the m-th element of $K(\sigma_3)$ which is impossible. The other case is similar. The claim is proved.

- (For Q) $\{\sigma_0,\sigma_1\} \in U_{K(\sigma_0)}(\sigma_0) \cap U_{K(\sigma_1)}(\sigma_1)$ but $U_{K(\sigma_0)}(\sigma_0)$ and $U_{K(\sigma_1)}(\sigma_1)$ are subsets of distinct elements of $\mathcal{B}_i$ and this is a contradiction to the disjointness of $\mathcal{B}_i$.
- (For P) $\{\sigma_i: i \in \omega\} \in \cap\{U_{K(\sigma_i)}(\sigma_i): i \in \omega\}$ but the $U_{K(\sigma_i)}(\sigma_i)$ are contained in distinct elements of $\mathcal{B}_i$ and this is a contradiction to the point finiteness of each $\mathcal{B}_i$.

The contradiction has been reached.

## 9. Meta-Normality of Box Products

In 1980, Eric van Douwen [31] investigated weak separation properties of box products. A very weak separation property, within the realm of completely regular spaces, is *metanormal:*

*Definition* 8. (van Douwen [31]). A space X is said to be *metanormal* if, whenever $\{F_n : n \in \omega\}$ is a discrete family of closed sets and $\{U_{n,k} : n,k \in \omega\}$ are open sets such that $F_n \subset U_{n,k}$ for each $n,k \in \omega$, then $\cap\{U_{n,k} : n,k \in \omega\} = \emptyset$.

This property is weaker than all of perfect, countably metacompact and normal (see pages 62 and 63 of [31]). Stone showed in 1948 [27] that ${}^{\omega_1}\omega$ is not normal. The generalized box product $< \kappa - \square^{\kappa^+} \kappa$ is defined as the product $\kappa^{\kappa^+}$ with the topology generated by declaring $\{f \in \kappa^{\kappa^+} : f \supset \sigma\}$ to be open for each partial function $\sigma$ from $\kappa^+$ to $\kappa$ of cardinality less than $\kappa$. Of course, when $\kappa = \omega$, this is just ${}^{\omega_1}\omega$ with the Tychonoff topology. Borges noted in 1969 [5] that Stone's proof showed that the generalized box product $< \kappa - \square^{\kappa^+} \kappa$ is not normal when $\kappa$ is regular and asked if the assumption of regularity is necessary. van Douwen showed that the assumption is unnecessary and asked if non-normality could be improved to non-metanormality. He obtained this improvement under the assumption $\Sigma\{\kappa^\lambda : \lambda < \kappa\} = \kappa$. In this section, we show that, in ZFC, $< \kappa - \square^{\kappa^+} \kappa$ is not metanormal no matter what $\kappa$ is. Moreover, we show the truth of a combinatorial proposition raised by van Douwen which implies the non-metanormality in a simple manner (see remark 13.12 of [31]).

*Lemma* 10. *Let* INJ = $\{f \in \kappa^\alpha : \alpha \in \kappa^+$, f is one-to-one$\}$. *Let* $g_f \in [f]^{<\kappa}$ *be defined for each* $f \in$ INJ. *There are* $\{f_\alpha : \alpha \in \kappa^+\} \subset$ INJ *with increasing domains such that* $\{g_{f_\alpha} : \alpha \in \kappa^+\}$ *are compatible.*

*Proof.* Let $\{A_\alpha : \alpha \leq \kappa\}$ be a disjoint subfamily of $[\kappa]^\kappa$. Construct $\{h_\alpha \in r_\alpha^{d_\alpha} : \alpha \in \kappa\}$ continuous increasing injections such that

- $h_0 = \emptyset$
- each $d_\alpha$ is an ordinal in $\kappa^+$
- each $r_\alpha \subset \kappa$
- $r_{\alpha+1} - r_\alpha \subset A_\alpha$
- $f \in \mathrm{INJ},\ f \supset h_\alpha \Rightarrow \alpha(\mathrm{dom}(g_f)) < d_{\alpha+1}$

Note that $\alpha(A)$ is the $\alpha$-th element of A. If the construction fails after $h_\alpha$ is constructed then $\{\alpha(\mathrm{dom}(g_f)) : f \supset h_\alpha,\ f \in \mathrm{INJ}\}$ is unbounded in $\kappa^+$. So construct $\{f_\gamma : \gamma \in \kappa^+\} \subset$ INJ such that

- each $f_\gamma \supset h_\alpha$
- $\gamma < \gamma_1 \Rightarrow \alpha(\mathrm{dom}(g_{f_{\gamma_1}})) > \sup(\mathrm{dom}(g_{f_\gamma}))$

We claim that $\{g_{f_\gamma} : \gamma \in \kappa^+\}$ are compatible and thus that we succeed in achieving the conclusion of the lemma.

Suppose otherwise that $g_{f_\gamma}(\beta) \neq g_{f_{\gamma_1}}(\beta)$ where $\gamma < \gamma_1$. Let $\beta = \delta(\mathrm{dom}(g_{f_{\gamma_1}}))$.

- If $\delta \geq \alpha$ then $\delta(\mathrm{dom}(g_{f_{\gamma_1}})) \geq \alpha(\mathrm{dom}(g_{f_{\gamma_1}})) > \sup(\mathrm{dom}(g_{f_\gamma}))$ which means that $g_{f_\gamma}(\beta)$ isn't even defined.

- If $\delta < \alpha$, then since $f_{\gamma_1} \supset h_\alpha \supset h_\delta$, we have, by construction of $d_{\delta+1}$, $\beta = \delta(\text{dom}(g_{f_{\gamma_1}})) < d_{\delta+1} \leq d_\alpha$. This means that $\beta \in \text{dom}(h_\alpha)$ which is a contradiction since we can calculate

$$g_{f_\gamma}(\beta) = f_\gamma(\beta) = h_\alpha(\beta) = f_{\gamma_1}(\beta) = g_{f_{\gamma_1}}(\beta)$$

Thus the construction can be carried out.

Let $h = \cup\{h_\alpha : \alpha \in \kappa\}$. We claim that $f \in$ INJ and $f \supset h$ implies that $g_f \subset h$. To see this, suppose $\beta \in \text{dom}(g_f)$. We must show $\beta \in \text{dom}(h)$. Say $\beta = \delta(\text{dom}(g_f))$. Now $f \supset h_\delta$ implies that $\beta = \delta(\text{dom}(g_f)) < d_{\delta+1} = \text{dom}(h_{\delta+1}) < \text{dom}(h)$. Now there are $\{f_\alpha : \alpha \in \kappa^+\} \subset$ INJ with increasing domains, each of which contains h since $A_\kappa$ is always available for the range. Now $\{g_{f_\alpha} : \alpha \in \kappa^+\}$ are all subsets of h and thus are compatible.

*Corollary* 12. (ZFC). *For any infinite cardinal* $\kappa$, $< \kappa - \square^{\kappa^+} \kappa$ *is not metanormal.*

Eric van Douwen has informed us that S. Todorčević has independently obtained this result.

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