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CONFLUENT MAPPINGS ON [0,1] AND INVERSE LIMITS

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ABSTRACT. Let $I = [0,1]$. In this paper, confluent mappings $f : I \to I$ are characterized. The degree, deg(f) of such a mapping is defined as (number of components of $f^{-1}(0)$) + (number of components of $f^{-1}(1)$) − 1. This definition agrees with the usual definition of $\deg(f)$ in the case where f is open. It is shown that, if $f_i : I \to I$ is confluent, $g_i : I \to I$ is open and $\deg(f_i) = \deg(g_i)$ for $i = 1, 2, \ldots$, then $\varprojlim\{I, f_i\}$ is homeomorphic with $\varprojlim\{I, g_i\}$.

The simplest indecomposable continua are those which can be constructed as inverse limits of $I = [0,1]$, with open bonding maps. In this paper we show that inverse limits on I with confluent bonding maps are homeomorphic to inverse limits on I with open bonding maps, and we identify the particular inverse limit of the latter type that a given inverse limit with confluent bonding maps is homeomorphic to. This weakening of the condition on the bonding map is useful: it makes it much easier to construct mappings onto these continua (see [2] for example). To prove this result, we first obtain a characterization of confluent mappings from I onto I which is similar to the characterization of open mappings in [8]. From this characterization it follows that such confluent mappings are uniformly approximated by open mappings. Given an inverse limit, X, on I with confluent bonding maps we may apply a theorem of Morton Brown [1] to obtain a homeomorphism form X onto an inverse limit on I with open bonding maps. The degrees of the corresponding confluent mappings and open mappings are the same, and thus the inverse limit with open bonding maps is identified.

A classification of inverse limits on I with a fixed open bonding map was obtained independently by W. Debski [3] and W.T. Watkins [8]. A classification in the more general case, allowing different bonding maps, was obtained by Debski. It follows as a corollary to the result discussed above that the inverse limits on I with confluent bonding maps are classified in the same way.

All spaces considered in this paper are metric. A *continuum* is a compact connected metric space and a *mapping* is a continuous function. A mapping f from the X onto the space Y is *confluent* provided that, for each subcontinuum K of Y, each component of $f^{-1}(K)$ is mapped by f onto K. We will adopt the following notational conveniences: if $a = b$, we define $[a, b] = \{a\}$; if H_1 and H_2 are mutually exclusive sets of real numbers then we say that H_2 is *to the right of* H_1 and write $H_1 < H_2$ if it is true that $s < t$ for all $s \in H_1$ and $t \in H_2$; if H_1, H_2 and H_3 are mutually exclusive sets of real numbers then we say that H_2 is between H_1 and H_3 if either $H_1 < H_2 < H_3$ or $H_3 < H_2 < H_1$.

Our first lemma follows easily from the Intermediate Value Theorem and the definition of confluent mappings.

Lemma 1. *Suppose that $f : I \to I$ is confluent, a and b are in $I, a < b$ and $f(a) = f(b)$.*

(1) If there is a number x between a and b such that $f(x) > f(a)$ then there is a number c between a and b such that $f(c) = 1$.

(2) If there is a number x between a and b such that $f(x) < f(a)$ then there is a number c between a and b such that $f(c) = 0$.

Lemma 2. *Suppose that $f : I \to I$ is confluent, a and b are in $I, a < b$ and $f(a) = f(b)$.*

(1) If $f(x) > f(a)$ for all x in (a, b) then there is just one component of $f^{-1}(1)$ which is between a and b.

(2) If $f(x) < f(b)$ for all x in (a, b) then there is just one component of $f^{-1}(0)$ which is between a and b.

Proof. This is a straight forward application of Lemma 1.

From Lemma 2 and the uniform continuity of continuous functions on I we obtain the following lemma.

Lemma 3. *Suppose that $f : I \to I$ is confluent. Then there exists $\epsilon > 0$ such that if a and b are in I and $f(a) = f(b) = 0$ or $f(a) = f(b) = 1$ and f is not constant on $[a, b]$ then $|b - a| > \epsilon$.*

Corollary *Suppose that $f : I \to I$ is confluent. Then $f^{-1}(0)$ and $f^{-1}(1)$ each have only finitely many components.*

Lemma 4. *If $f : I \to I$ is confluent then either $f(0) = 0$ or $f(0) = 1$ and either $f(1) = 0$ or $f(1) = 1$.*

Proof. We will prove the first conclusion only.
Suppose $f(0) \neq 0$ and $f(0) \neq 1$. Let

$$x = \text{g.l.b.}(f^{-1}(0) \cup f^{-1}(1))$$

Then either $f(x) = 0$ or $f(x) = 1$. In either case $x > 0$. Suppose that $f(x) = 0$. Let $K = [f(0)/2, 1]$. Let L be the component of $f^{-1}(K)$ which contains 0. Since f is confluent, there is a point c in L such that $f(c) = 1$. Since $f(0)/2 > 0 = f(x)$ we have that $L \subset [0, x]$, and thus $c < x$. This is inconsistent with the definition of x.

Lemma 5. *Suppose that $f : I \to I$ is confluent, $f^{-1}(0)$ has n components, $L_1 < L_2 < L_3 < \ldots < L_n$, and $f^{-1}(1)$ has m components, $H_1 < H_2 < H_3 < \ldots < H_m$. Then the components of $f^{-1}(0)$ and $f^{-1}(1)$ alternate, and*

(1) *if $0 \in L_1$ and $1 \in H_m$ then $m = n$,*
(2) *if $0 \in H_1$ and $1 \in L_n$ then $m = n$,*
(3) *if $0 \in L_1$ and $1 \in L_n$ then $n = m + 1$, and*
(4) *if $0 \in H_1$ and $1 \in H_m$ then $m = n + 1$.*

Proof: That the components of $f^{-1}(0)$ and $f^{-1}(1)$ alternate follows from Lemma 2. Suppose 0 is in L_1. Thus no H_i is to the left of L_1. Since $H_1, H_2, \ldots, H_m$ alternates with $L_1, L_2, L_3, \ldots, L_n$, there are $n - 1$ H_i's to the left of L_n. There can be no more than one H_i to the right of L_n. If 1 is in L_n, there are no H_i's to the right of L_n. Thus if 1 is in L_n, then $m = n - 1$ and we have (3). If 1 is not in L_n , then 1 is in H_m by Lemma

4. Therefore $m = n$ and we have (1). Similar reasoning yields (2) and (4).

Definition. Suppose that $f : I \to I$ is confluent, m is the number of components of $f^{-1}(0)$, and n is the number of components of $f^{-1}(1)$. The *degree* of f, denoted $\deg(f)$, is the integer $m + n - 1$. Note that if f is an open mapping, then $\deg(f)$ as defined in [3, p. 204] agrees with this definition.

Theorem 1. *A mapping $f : I \to I$ is confluent if and only if there is a positive integer n and sequences $a_0, a_1, \dots, a_n$ and $b_0, b_1, \dots, b_n$ with*

$$0 = a_0 \leq b_0 < a_1 \leq b_1 < \dots < a_{n-1} \leq b_{n-1} < a_n \leq b_n = 1,$$

such that, for $i = 1, 2, \dots, n$, f restricted to $[b_{1-1}, a_i]$ is a monotone mapping of $[b_{i-1}, a_i]$ onto I, and , for $i = 0, 1, 2, \dots, n$, f restricted to $[a_i, b_i]$ is constant and equal to 0 or 1.

Proof. To prove the sufficiency of the stated condition suppose that n is a positive integer, $0 = a_0 \leq b_0 < a_1 \leq b_1 \dots < a_{n-1} \leq b_{n-1} < a_n \leq b_n = 1$, and $f : I \to I$ is a mapping satisfying the condition above with respect to $a_0, a_1, \dots, a_n$ and $b_0, b_1, \dots, b_n$. Suppose that $K = [c, d]$ is a subinterval of I, and that L is a component of $f^{-1}(K)$. Suppose $L \cap [b_{i-1}, a_i] \neq \emptyset$ for some i. Since f is monotone on $[b_{i-1}, a_i]$, there is a subinterval, J, of $[b_{i-1}, a_i]$ such that $f(J) = K$. Also by the monotonicity of f on $[b_{i-1}, a_i], J \cap L \neq \emptyset$. Since L is a component of $f^{-1}(K), J \subset L$. Hence $f(L) = K$. If $L \cap [b_{i-1}, a_i] = \emptyset$ for all i, then $L \subset [a_i, b_i]$ for some i. But, since L is a component of $f^{-1}(K), L = [a_i, b_i]$ and thus $L \cap [b_{i-1}, a_i] \neq \emptyset$, a contradiction. Therefore f is confluent.

Now suppose that $f : I \to I$ is confluent. Suppose that $f^{-1}(0)$ has n components, $L_1 < L_2 \dots < L_n$, and $f^{-1}(1)$ has m components, $H_1 < H_2 < \dots < H_m$. Let $k = \deg(f) = m + n - 1$. Either $0 \in L_1$ or $0 \in H_1$ by Lemma 4. Suppose the former. Define a_0 and b_0 to be the right and left endpoints of L_1 respectively; $L_1 = [a_0, b_0]$. For $i > 0$ define a_i and b_i to be the right and left endpoints, respectively, of $H_{(i+1)/2}$ if i is odd, or to be the right and left endpoints, respectively, of $L_{(i/2+1)}$ if i is even. We need only show that f is monotone on $[b_{i-1}, a_i]$

for $0 < i < k$. Suppose that $i > 0$ and that i is odd. Let $j = (i+1)/2$. Then $[a_{i-1}, b_{i-1}] = L_j$ and $[a_i, b_i] = H_j$. Suppose that t is in I and that $f^{-1}(t)$ has two components, C_1 and C_2, lying in $[b_{i-1}, a_i]$. By Lemma 2, there is a component of either $f^{-1}(0)$ or $f^{-1}(1)$ lying between C_1 and C_2. But L_j and H_j are consecutive components of $f^{-1}(0) \cup f^{-1}(1)$, so we have reached a contradiction. A similar contradiction arises if i is even.

For mappings between Peano continua, light confluent mappings are open (see [6, Theorem 13.23]). The special case of this result where the continuum is [0,1] follows as an easy corollary of Theorem 1.

Corollary *If $f : I \to I$ is a light confluent mapping then f is open.*

Proof. Suppose that $f : I \to I$ is confluent. Let $n = \deg(f)$ and let $a_0 \leq b_0 < a_1 \leq b_1 < \ldots < a_n \leq b_n$ be the sequences given by Theorem 1. Since $f^{-1}(0)$ and $f^{-1}(1)$ are discrete, $a_i = b_i$ for $i = 1, 2, \ldots, n$. Since f is monotone and light on $[a_{i-1}, a_i]$, f is a homeomorphism from $[a_{i-1}, a_i]$ onto I for $i = 1, 2, \ldots, n$. Thus, by [7, Lemma 1, p.453], f is open.

Theorem 2. *The uniform closure of the space of all open mappings from I onto I is the space of all confluent mappings from I onto I.*

Proof. Let $\mathcal{C}$ (respectively, $\mathcal{O}$) denote the space of all confluent (resp., open) mappings from I onto I. Denote the supremum norm of a mapping $f : I \to I$ by $||f||$. We first note that $\mathcal{C}$ is uniformly closed. This can be seen from either [4, Theorem 5.48, p. 41] or [5, 3.1]. Thus, $cl(\mathcal{O}) \subset \mathcal{C}$. Suppose that $f : I \to I$ is confluent. Let $n = deg(f)$ and let $a_0 \leq b_0 < a_1 \leq b_1 < \ldots < a_n \leq b_n$ be the sequences given by Theorem 1. Let $g : I \to I$ be the open mapping such that $g(a_0) = f(a_0)$, and $g(b_i) = f(b_i)$ for $1 < i < n$, and such that g is linear on $[a_0, b_1]$ and on $[b_i, b_{i+1}]$ for $1 < i < n-1$. For $k = 1, 2, 3, \ldots$ define $g_k : I \to I$ by

$$g_k(x) = \frac{1}{k}((k-1)f(x) + g(x)).$$

Now g is increasing on each of the intervals $[a_0, b_1]$, $[b_1, b_2], \dots$, $[b_{n-1}, b_n]$ on which f is nondecreasing and g is decreasing on each of those intervals on which f is nonincreasing. Consequently g_k is either increasing or decreasing on those same intervals. Thus g_k is an open mapping for all k. Clearly

$$||g_k - f|| = \frac{1}{k}||g - f||,$$

so $\lim_{n \to \infty} g_k = f$ uniformly. Therefore $cl(\mathcal{O}) = \mathcal{C}$.

A mapping which is the uniform limit of onto homeomorphisms is called a near-homeomorphism (see [1]). Likewise we define a mapping $f : X \to Y$ of continua to be *near-open* provided that it is the uniform limit of open mappings. With this terminology Theorem 2 may be restated as follows:

Theorem 2a. *Every confluent mapping from I onto I is near-open.*

Lemma 6. *Suppose that $f : I \to I$ and $g : I \to I$ are confluent and $||f - g|| < \frac{1}{2}$.*

(1) If $f(a) = f(b) = 0$ then either f is constant on $[a, b]$ or there is a number c between a and b such that $g(c) = 1$.

(2) If $f(a) = f(b) = 1$ then either f is constant on $[a, b]$ or there is a number c between a and b such that $g(c) = 0$.

Proof. To prove (1), suppose that f is not constant on [a,b]. Then by Lemma 1 there is a number c', such that $a < c' < b$, and $f(c') = 1$. Since $||f - g|| < \frac{1}{2}$, it follows that $g(c') > \frac{1}{2}$, $g(a) < \frac{1}{2}$, and $g(b) < \frac{1}{2}$. By the Intermediate Value Theorem, there are numbers a' and b' such that $a < a' < c' < b' < b, g(a') = \frac{1}{2}$. and $g(b') = \frac{1}{2}$. Then, by Lemma 1, there is a number c , with $a' < c < b'$, such that $g(c) = 1$.

Lemma 7. *Suppose that $f : I \to I$ and $g : I \to I$ are confluent and $||f - g|| < \frac{1}{2}$.*

(1) Between consecutive components of $f^{-1}(0)$ there is just one component of $g^{-1}(1)$.

(2) Between consecutive components of $f^{-1}(1)$ there is just one component of $g^{-1}(0)$.

Proof. Suppose that L_1 and L_2 are consecutive components of $f^{-1}(0)$. By Lemma 6 there is at least one component of $g^{-1}(1)$ between L_1 and L_2. Suppose that $[a,b]$ and $[c,d]$ are components of $g^{-1}(1)$ which lie between L_1 and L_2 and that $b < c$. Then, by Lemma 6, there is a component of $f^{-1}(0)$ between b and c, and consequently between L_1 and L_2. This is a contradiction.

Theorem 3. *Suppose that $f : I \to I$ and $g : I \to I$ are confluent and that $\|f - g\| < \frac{1}{2}$. Then $deg(f) = deg(g)$.*

Proof. Suppose that $f^{-1}(0)$ has n_f components, $f^{-1}(1)$ has m_f components, $g^{-1}(0)$ has n_g components and that $g^{-1}(1)$ has m_g components. Suppose that $f(0) = 0$. Since $\|f - g\| < \frac{1}{2}$ and g is confluent, $g(0) = 0$. All components of $g^{-1}(1)$ lie in the complement of $f^{-1}(0)$, again since $\|f - g\| < \frac{1}{2}$. If $f(1) = 0$, then $g(1) = 0$ and all components of $g^{-1}(1)$ lie between components of $f^{-1}(0)$. Thus, in this case, $m_f = m_g$ by Lemmas 2 and 7. Now suppose that $f(1) = 1$. Let $[a,b]$ denote the right most component of $f^{-1}(0)$. There is at least one component of $g^{-1}(1)$ to the right of b since $g(1) = 1$. If there were two components of $g^{-1}(1)$ to the right of b, by Lemma 7 there would be a component of $f^{-1}(0)$ to the right of b. Hence there is just one component of $g^{-1}(1)$ to the right of b. By similar reasoning, there is just one component of $f^{-1}(1)$ to the right b. All components of $g^{-1}(1)$ to the left of a lie between two components of $f^{-1}(0)$. Hence, by Lemmas 2 and 7, $g^{-1}(1)$ and $f^{-1}(1)$ have the same number of components to the left of a. Therefore $m_f = m_g$. Similar reasoning yields that $n_f = n_g$. Therefore $deg(f) = deg(g)$.

An *inverse sequence* is a pair $\{x_i, f_i\}$ whose first term is a sequence of spaces and whose second term is a sequence of mappings $f_i : X_{i+1} \to X_i$, called the *bonding maps* of the sequence. The *inverse limit* of the sequence $\{X_i, f_i\}$ is the set

$$\varprojlim\{X_i, f_i\} = \{(x_1, x_2, x_3, \dots) | \quad x_i \in X_i \text{ and } f_i(x_{i+1}) = x_i\},$$

with the topology induced by the metric

$$d(\underline{x}, \underline{y}) = \sum_{i>0} \frac{d_i(x_i, y_i)}{2^i}$$

where $\underline{x} = (x_1, x_2, x_3, \dots)$, and $\underline{y} = (y_1, y_2, y_3, \dots)$ are in $\varprojlim\{X_i, f_i\}$ and d_i is the metric for X_i.

Theorem 4. *Suppose that $f_i : I \to I$ is confluent for $i = 1, 2, \dots$. Then there is a sequence g_i of open mappings from I onto I such that $deg(g_i) = deg(f_i)$ and $\varprojlim\{I, f_i\}$ is homeomorphic to $\varprojlim\{I, g_i\}$.*

Proof. For each i there exists a sequence $\{h_{ij}\}_{j=1}^{\infty}$ of open mappings of I onto I which converges uniformly to f_i and which has the property that $||f_i - h_{ij}|| < \frac{1}{2}$ for all j. By Theorem 3, $deg(f_i) = deg(h_{ij})$ for all i and j. Let

$$K_i = \{h_{ij} | j = 1, 2, \dots\}.$$

By [1, Theorem 3, p.481], there is a sequence $\{g_i\}_{i=1}^{\infty}$ such that g_i is in K_i and $\varprojlim\{g_i, I\}$ is homeomorphic to $\varprojlim\{f_i, I\}$.

If n is a positive integer, we define the *standard open mapping of degree n*, $w_n : I \to I$, to be the mapping such that, for $i = 1, \dots n$,

$$w_n(i/n) = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd,} \end{cases}$$

and which is linear on the intervals $[(i-1)/n, i/n]$ (see [3,p. 203]).

Corollary 1. *Suppose that $f_i; I \to I$ is confluent for $i = 1, 2, \dots$ and that $n_i = deg(f_i)$. Then $\varprojlim\{f_i, I\}$ is homeomorphic to $\varprojlim\{I, w_{n_i}\}$.*

Proof. This follows from Theorem 4 and [3, Lemma 4, p. 204].

Corollary 2. *If $f : I \to I$ and $g : I \to I$ are confluent then $\varprojlim\{I, f\}$ and $\varprojlim\{I, g\}$ are homeomorphic if and only if $deg(f)$ and $deg(g)$ have the same prime factors.*

Proof. This follows from Corollary 1 and [8, Thm. 5, p. 599].

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