TOPOLOGY PROCEEDINGS

Volume 15, 1990

Pages 11–28

http://topology.auburn.edu/tp/

SOME REMARKS ON INITIAL α-COMPACTNESS, $< \alpha$-BOUNDEDNESS AND p-COMPACTNESS

by

Salvador Garcia-Ferreira

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

SOME REMARKS ON INITIAL α-COMPACTNESS, $< \alpha$-BOUNDEDNESS AND p-COMPACTNESS

SALVADOR GARCIA-FERREIRA

ABSTRACT. The basic relationships among initial α-compactness, $< \alpha$-boundedness and p-compactness are established. Our principal results are the following: it follows from GCH that every initially α-compact space is $< \alpha$-bounded, we prove that there is a model M of ZFC in which $M \models$ there exist an initially ω_1-compact (initially $\aleph_\omega$-compact) topological group which is not $< \omega_1$-bounded ($< \aleph_\omega$-bounded); if $\beta_\alpha(\alpha)$ is the α-boundification of α and α is a strong limit singular cardinal, then there is $p \in \cup(\alpha) \cap \beta_\alpha(\alpha)$ such that p-compactness coincides with $< \alpha-$boundedness; a result of Saks is improved by proving that X^γ is initially α-compact for all cardinal $\gamma \Leftrightarrow \exists\, p \in \cup(\alpha)$ (p is decomposable $\wedge$ X is p-compact); we know that GCH implies that $|\beta(\alpha) \setminus \cup(\alpha)| = |\beta_\alpha(\alpha)|$ for each cardinal α, and if α is a strong limit singular cardinal then $|\beta(\alpha) \setminus \cup(\alpha)| = |\beta_\alpha(\alpha)| = 2^\alpha$; we show in ZFC, that if α is singular then $|\beta_\alpha(\alpha)| = |\beta(\alpha) \setminus \cup(\alpha)|^\alpha$, and a model M of ZFC is defined so that $M \models |\beta(\aleph_\omega) \setminus \cup(\aleph_\omega)| < |\beta_{\aleph_\omega}(\aleph_\omega)|$.

0. INTRODUCTION

The authors of [18] introduced the concept of $\alpha-$boundedness in their study of linearly ordered spaces: A space X is α-bounded if $Cl_X(A)$ is compact for every $A \subseteq X$ with $|A| \leq \alpha$. W. W. Comfort [28] and J. E. Vaughan [37] slightly modified this concept as follows: A space X is $<$ *α-bounded* if $Cl_X(A)$ is compact for each $A \subseteq X$ with $|A| < \alpha$. Observe that $< \alpha^+$-boundedness coincides with the original definition given in [18].

In this paper, we study the relations among $< \alpha$-boundedness, initial α-compactness and p-compactness. We present (in section 2) the basic results. It is shown that, assuming GCH, every initially α-compact space is $< \alpha$-bounded, and two examples are given to see that this conclusion can not be established in ZFC. Nevertheless, we prove that if α is singular then every $< \alpha$-bounded space is initially α-compact; if α i-a a strong limit cardinal then initial α-compactness implies $< \alpha$-boundedness; and if α ia a strong limit singular cardinal then there is $p \in \mathrm{U}(\alpha)$ such that p-compactness = $< \alpha$-boundedness = initial α-compactness. The author of [35] asked whether α is a strong limit singular cardinal whenever initial α-compactness is productive. In this direction, we show that if initial α-compactness is productive, then there is p $\in \mathrm{U}(\alpha)$ such that initial α-compactness coincides with p-compactness. O'Callaghan [28] pointed out that a cardinal α is regular iff $\beta_\alpha(\alpha) = N(\alpha)$. In our joint paper [16], we observed that if α is a strong limit singular then $|\beta(\alpha)\setminus \mathrm{U}(\alpha)| = |\beta(\alpha)| = 2^\alpha$, and GCH implies that $|\beta(\alpha)\setminus \mathrm{U}(\alpha)| = |\beta(\alpha)| = 2^\alpha$ for each cardinal α: these two results are direct consequences from Theorem 1.4 below. This makes it natural to ask whether the equality $|\beta_\alpha(\alpha)| = |N(\alpha)|$ can be established by using only the axioms of ZFC. In section 3. we show that there is a model M of ZFC in which $M \models |\beta(\aleph_\omega)\setminus \mathrm{U}(\aleph_\omega)| < |\beta_{\aleph_\omega}(\aleph_\omega)|$ answering this question in the negative.

1. Preliminaries.

All spaces are assumed to be completely regular Hausdorff (Tychonoff). The Greek letters α and γ stand for infinite cardinal numbers and the Greek letters ξ and δ stand for ordinals. If α is a cardinal, then α denotes the space whose underlying set is α with the discrete topology. If $f : X \to Y$ is a continuous function, the Stone extension of f is denoted by $\overline{f} : \beta(X) \to \beta(Y)$. The remainder of $\beta(X)$ is the space $X^* = \beta(X)\setminus X$. For a cardinal α, the set of uniform ultrafilters on α is $\mathrm{U}(\alpha) = \{p \in \omega^* : \forall\ A \in p\ \ (|A| = \alpha)\ \}$ and its complement is denoted by $N(a) = \beta(\alpha)\setminus \mathrm{U}(\alpha)$. If $A \subseteq \alpha$ then the closure of A in $\beta(\alpha)$ is $\hat{A} = \{p \in \beta(\alpha) : A \in p\}$.

A function $f : \gamma \to \beta(\alpha)$ is a *strong embedding* if there is a partition $\{A_\xi : \xi < \gamma\}$ of α such that $f(\xi) \in \hat{A}_\xi$ for each $\xi < \gamma$. The *Rudin-Keisler* order on α^* is defined by $p \leq_{RK} q$ if there is $f : \alpha \to \alpha$ such that $\overline{f}(q) = p$ for p,q $\in \alpha^*$ (see [5]). For p,q $\in \alpha^*$, we say that $p \approx q$ if there is a permutation σ of α with $\overline{\sigma}(p) = q$. Clearly, $\approx$ is an equivalence relation on α^*. If p $\in \alpha^*$, then T(p) $= \{q \in \alpha^* : p \approx q\}$ is called the *type* of p: the types of ultrafilters were introduced by W. Rudin [29]. An ultrafilter p on α is *decomposable* if $\forall\, \omega \leq \gamma \leq \alpha\ \exists\, q \in \cup(\gamma)\ (q \leq_{RK} p)$. For p,q $\in \alpha^*$, their *tensor product* is defined by

$$p \otimes q = \{A \subset \alpha \times \alpha : \{\xi < \alpha : \{\zeta < \alpha : (\xi, \zeta) \in A\} \in q\} \in p\}.$$

Notice that $p \otimes q$ is an ultrafilter on $\alpha \times \alpha$ and can be considered as an ultrafilter on α via a fixed bijection between α and $\alpha \times \alpha$ (for background and historical notes see [5]).

Clearly, compact spaces are trivial examples of $< \alpha$-bounded spaces for any cardinal α. Another important compact-like property is given in the next definition given by Saks [31] and Woods [38]: this is a generalization of Bernstein's concept of p-compactness introduced in [1], for $p \in \omega^*$.

Definition 1.1. (Saks-Woods) *Let $\emptyset \neq M \subseteq \alpha^*$. A space X is M-compact if $\forall\, f \in {}^\alpha X\ \forall\, p \in M (\overline{f}(p) \in X)$.*

If $M = \{p\}$ for $p \in \alpha^*$, we simply write p-compact instead of {p}-compact. In Bernstein's terminology [1], we have that a space X is p-compact if every sequence has a p-limit. It should be mentioned that the p-limit concept of Bernstein was also introduced, in a different form, by Frolík [10], [11], Katětov [22], [23], and Saks [30], independently.

The basic property of M-compactness is stated in the following Theorem.

Theorem 1.2. *Let $\emptyset \neq M \subseteq \alpha^*$ and let X be a space. Then the space*

$\beta_M(X) = \cap\{Y : X \subseteq Y \subseteq \beta(X)$ and Y is M-compact $\}$ satisfies

(1) *X is a dense subspace of $\beta_M(X)$;*

(2) *$\beta_M(X)$ is M-compact;*

(3) *If* $f : X \to Z$ *is continuous and* Z *is M-compact, then* $\overline{f}(\beta_M(X)) \subseteq Z$;

(4) *Up to a homeomorphism fixing* X *pointwise the space* $\beta_M(X)$ *is the only space satisfying (1), (2) and (3).*

This space $\beta_M(X)$ is precisely the (M-compact) reflection considered and studied, in a more general context, by Herrlich and Van der Slot [19],[33], Franklin [8], and Woods [38]: this space $\beta_M(X)$ can be also obtained by an application of the adjoint functor Theorem of Freyd [9] (see [25]).

If $M = \{p\}$ for $p \in \alpha^*$ then the space $\beta_M(X)$ is denoted by $\beta_p(X)$ and it is called the *p-compactification* of X.

It follows directly from the definition that if $\emptyset \neq M \subseteq \alpha^*$ then $M \subseteq \beta_M(\alpha)$ and $T(p) \subseteq \beta_M(\alpha)$, for $p \in \alpha^*$.

Bernstein [1] proved that a (Tychonoff) space X is ω-bounded if and only if X is p-compact for all $p \in \omega^*$. Saks [31] generalized Bernstein's result by establishing that X is $< \alpha^+$-bounded iff X is p-compact for all $p \in \alpha^*$. For $< \alpha$-boundedness, we have that:

Theorem 1.3. *A space* X *is* $< \alpha$*-bounded if and only if* X *is* $(N(\alpha)\backslash\alpha)$*-compact.*

Proof. $\Rightarrow$ Assume that X is $< \alpha$-bounded, let $p \in N(\alpha)\backslash\alpha$ and $f \in {}^{\alpha}X$. Without loss of generality, we may suppose that $p \in \cup(\gamma)$ for some $\gamma < \alpha$. Since X is $< \alpha$-bounded then $\overline{f}(\beta(\gamma)) = Cl_X(f(\gamma)) \subseteq X$. In particular, we have that $\overline{f}(p) \in X$. This shows that X is $(N(\alpha)\backslash\alpha)$-compact.

$\Leftarrow$ If $f \in {}^{\gamma}X$, for some $\omega \leq \gamma < \alpha$, then $Cl_X(Img(f)) \subseteq \overline{f}(\beta(\gamma)) \subseteq X$ (since X is $(N(\alpha)\backslash\alpha)$-compact). Hence, X is $< \alpha$-bounded.

Thus, we have that $< \alpha$-boundedness and $(N(\alpha)\backslash\alpha)$-compactness are the same topological property. For a space X, we write $\beta_\alpha(X)$ in place of $\beta_{(N(\alpha)\backslash\alpha)}(X)$ and $\beta_\alpha(X)$ is called the *α-boundification* of X. By using Theorem 1.3 and elementary cardinal arithmetic we have (see [12] or [16]):

Theorem 1.4. *For every cardinal* α, *we have that*

$$|N(\alpha)| = \alpha^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^\gamma} \leq |\beta_\alpha(\alpha)| \leq (\sum_{\gamma<\alpha} 2^{2^\gamma})^\alpha.$$

We remind the reader the definition of initially α-compact space ([35] offers a good survey on initially α-compact spaces):

1.5 Definition. (Smirnov [34]) *A space X is* initially α-compact *if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq \alpha$ has a finite subcover.*

Saks [31] (see [35]) classified those spaces X whose product X^γ is initially α-compact for all cardinal γ as follows:

Theorem 1.6. (Saks) *For a space X the following conditions are equivalent.*

(1) *X^γ is initially α-compact for each cardinal γ:*
(2) *for each $\gamma \leq \alpha$ there is $p_\gamma \in \mathrm{U}(\gamma)$ such that X is $\{p_\gamma : \gamma \leq \alpha\}$-compact.*

The *Comfort (pre-)order* on α^*, introduced by W. W. Comfort in [12], [13], is defined by $p \leq_c q$ if every q-compact space is p-compact, for $p, q \in \alpha^*$. The basic properties of Comfort order are stated in the following Lemma (a proof is available in [12] and [13]).

Lemma 1.7. *Let $p, q \in \alpha^*$. Then,*

(1) *if $p \leq_{RK} q$ then $p \leq_c q$;*
(2) *$p \leq_c q$ if and only if $p \in \beta_q(\alpha)$;*
(3) *if $\emptyset \neq M \subseteq \alpha^*$, then $\beta_M(\alpha)$ is p-compact if and only if there is $r \in \beta_M(\alpha) \backslash \alpha$ such that $p \leq_c r$.*

2. On Initial α-Compactness and α-Boundedness.

In [18, Lemma 3], the authors noticed that an $< \alpha^+$-bounded spaces is initially α-compact. Clearly, if α is regular then

$$G = \{x \in \prod_{\xi<\alpha} G_\xi : |\{\xi < \alpha : x_\xi \neq e_\xi\}| < \alpha\},$$

where G_ξ is a non-trivial, compact topological Abelian group with identity e_ξ for $\xi < \alpha$, is an example of an $< \alpha$-bounded topological Abelian group which is not initially α-compact. The next Theorem shows that for α singular there is no such example. This Theorem is a direct application of the following Lemma, due to Stephenson and Vaughan [36].

Lemma 2.1. (Stephenson-Vaughan) *Let α be a singular cardinal. If X is initially α-compact for all $\omega \leq \gamma < \alpha$, then X is initially α-compact.*

Theorem 2.2. *If α is singular, then every $< \alpha$-bounded space is initially α-compact.*

Proof. Let α be singular and let X be an $< \alpha$-bounded space. In virtue of Lemma 2.1, it is enough to verify that X is initially γ-compact for all $\omega \leq \gamma < \alpha$. Indeed, if $\omega \leq \gamma < \alpha$, then X is $< \gamma^+$-bounded: hence, by Lemma 3 of [18] quoted above, X is initially γ-compact.

For α is regular, the space α with the order topology is an $< \alpha$-bounded, non-compact space of cardinallity α. But for singular cardinals the situation is quite different. In fact, we have:

Corollary 2.3. *Let α be singular. If X is an $< \alpha$-bounded, non-compact space, then $\alpha < |X|$ and $\alpha < w(X)$.*

Proof. Let X be a space satisfying our conditions. By Theorem 2.2, X is initially α-compact. It is then evident that $\alpha < |X|$ and $\alpha < w(X)$.

Gulden, Fleischman and Weston [18] asked whether there is an initially α-compact space, $\alpha > \omega$, which is not $< \alpha^+$-bounded. In [32], the authors showed that for every regular cardinal $\alpha \geq \omega$ there is an initially α-compact topological group which is not $< \alpha^+$-bounded. It is then natural to ask whether every initially α-compact space is α-bounded. The answer is in the positive if α is a strong limit cardinal. In fact, Saks and Stephenson [32] showed:

Lemma 2.4. (Saks-Stephenson) *If X is an initially α-compact space and $2^\gamma \leq \alpha$ for some cardinal $\omega \leq \gamma < \alpha$, then X is $< \gamma^+$-bounded. In addition,*

(1) *if α is a strong limit cardinal, then every initially α-compact space is $< \alpha$-bounded; and*
(2) *if α is a strong limit singular cardinal, then initial α-compactness and $< \alpha$-boundedness are the same concept.*

The next two Theorems prove that initial $< \alpha$-compactness and α-boundedness agree and disagree depending on the model of ZFC.

Theorem 2.5. *If GCH holds, every initially α-compact space is $< \alpha$-bounded for each cardinal α.*

Proof. Assume GCH. let X be an initially α-compact space. If α is a strong limit cardinal, then the conclusion follows from Lemma 2.4 (1). Suppose that α is not a strong limit cardinal. Then, by GCH, we have that $\alpha = 2^\gamma = \gamma^+$ for some cardinal $\gamma < \alpha$. Since $\alpha = \gamma^+ = 2^\gamma$, for $\gamma < \alpha$, then X is $< \alpha$-bounded by Lemma 2.4.

We need the next result discovered by Saks [32], [35, 3.5].

Lemma 2.6. (Saks) *Let X be an initially α-compact space and $A \in [X]^{\leq 2^\alpha}$. Then there is an initially α-compact subspace G of X such that $A \subseteq G$ and $|G| \leq 2^\alpha$. In case X is a topological group, then G may be taken to be a subgroup of X.*

Theorem 2.7. *There is a model of ZFC in which there exists an initially ω_1-compact (initially $\aleph_\omega$ − compact) topological group which is not $< \omega_1$-bounded ($< \aleph_\omega$-bounded).*

Proof. Let M be a model of ZFC in which Lusin's hypothesis holds, that is $M \models 2^\omega = 2^{\omega_1}$. Then, by Theorem 1.4 we have that $M \models |\, N(\omega_1)| = 2^{2^{\omega_1}}$. We know that $\beta(\omega_1)$ can be embedded as a subspace of the topological group $H = \mathbb{S}^{2^{\omega_1}}$ (see, for instance, [5, Corollary 2.11]), where $\mathbb{S}$ is the unitary circle. Set $F = Cl_H(< \beta(\omega_1) >)$, where $< \beta(\omega_1) >$ denotes the subgroup of H generated by $\beta(\omega_1)$. By Lemma 2.6, we can find an initially ω_1-compact topological subgroup G of F such that $\omega_1 \subseteq G$ and $|G| \leq 2^\omega$. Assume that G is $< \omega_1$-bounded. Since $\omega_1 \subseteq \beta(\omega_1) \cap G$ then $\beta_{\omega_1}(\omega_1) = N(\omega_1) \subseteq G$ and so $M \models |\, N(\omega_1)| = 2^{2^\omega} = |G|$, which is a contradiction. Thus G is an initially ω_1-compact, non-$< \omega_1$- bounded topological group. To obtain the later example, we will use Easton Forcing (for a complete treatment of Easton Forcing see [20] and [26]).

Indeed, let N be a countable transitive model of ZFC + GCH. We define a function E by dom(E)= $\{\omega, \aleph_{\omega+1}\}, E(\omega) =$

$\aleph_{\omega+2}$, and $E(\aleph_{\omega+1}) = \aleph_{\omega+2}$. Clearly, E is an Easton function. Then there is a generic extension M of N such that M and N have the same cardinals and $M \models \forall\, \kappa \in dom(E)\ (E(\kappa) = 2^{\kappa})$. Thus, we have that

$$M \models 2^{\aleph_\omega} \leq 2^{\aleph_{\omega+1}} = E(\aleph_{\omega+1}) = \aleph_{\omega+2} < 2^{2^{\omega}} = 2^{E(\omega)} = 2^{\aleph_{\omega+2}}.$$

Hence, by Theorem 1.4, $M \models 2^{\aleph_\omega} < 2^{2^{\omega}} \leq |\beta_{\aleph_\omega}(\aleph_\omega)|$. Now, to show that $Cl_K(<\beta(\aleph_\omega)>)$, where $K = \mathbb{S}^{2^{\aleph_\omega}}$ and $<\beta(\aleph_\omega)>$ is the subgroup of K generated by $\beta(\aleph_\omega)$, contains a subgroup with the required properties, we proceed as in the previous example.

Theorem 1.3 shows that $<\alpha$-boundedness implies p-compactness for each $p \in N(\alpha)\backslash\alpha$. Conversely, it is evident that if $2^{\alpha} < 2^{\gamma}$ for some $\omega \leq \gamma < \alpha$ then there is no $p \in \cup(\alpha)$ such that p-compactness implies $<\alpha$-boundedness since $|N(\alpha)| > 2^{\alpha} \geq |\beta_p(\alpha)|$ for each $p \in \cup(\alpha)$. Nevertheless, we have:

Theorem 2.8. *If $\emptyset \neq M \in [\alpha^*]^{\leq 2^{\alpha}}$, then there is $p \in \cup\,(\alpha)$ such that p-compactness implies M-compactness. In particular,*

(1) *if α is a strong limit cardinal, then $\exists\ p \in \cup(\alpha)$ (p-compactness $\Rightarrow$ $<\alpha$-boundedness);*
(2) *assuming GCH, for each cardinal α we have that $\exists\, p \in \cup(\alpha)$ (p-compactness $\Rightarrow$ $<\alpha$-boundedness).*

The conclusion will be a direct consequence of the following Lemma due to Comfort and Negrepontis [4], [5, 10.9-10.13] and [3, Theorem 6.4].

Lemma 2.9. (Comfort-Negrepontis) *Let κ be a regular cardinal such that $\omega \leq \kappa \leq \alpha = \alpha^{<\kappa}$, and suppose that every κ-complete filter on α extends to a κ-complete ultrafilter. Then $\forall\ A \in [\{p \in \beta(\alpha) : p \text{ is } \kappa\text{-complete}\}]^{\leq 2^{\alpha}}\ \exists\ q \in \cup(\alpha)\ \forall\ p \in A(p <_{RK} q)$.*

Proof. (This alternative proof is due to Alan Dow). It is a result of Hausdorff, and Engelking and Karlowicz (see [5, Theorem 3.16]) that for each α there is a $((2^{\alpha}, \alpha)$-independent matrix) family $\{A^{\xi}_{\zeta} : \xi < \alpha \text{ and } \zeta < 2^{\alpha}\}$ of subsets of α such that

(1) $A^\xi_\zeta \cap A^\eta_\zeta = \emptyset$ for $\xi < \eta < \alpha$ and $\zeta < 2^\alpha$;
(2) if $I \in [2^\alpha]^{<\kappa}$ and $\psi \in {}^I\alpha$, then $\cap_{\zeta\in I} A^{\psi(\zeta)}_\zeta \neq \emptyset$.

We may assume that $\cup_{\xi<\alpha} A^\xi_\zeta = \alpha$ for each $\zeta < 2^\alpha$. Let $\{p_\zeta : \zeta < 2^\alpha\}$ be a set of κ- complete ultrafilters on α. Define $\mathcal{F} = \{\cup_{\xi\in B} A^\xi_\zeta : B \in p_\zeta \text{ and } \zeta < 2^\alpha\}^+$. It is evident that $\mathcal{F}$ is κ-complete, and if $\mathcal{F} \subseteq p \in \beta(\alpha)$ (p can be taken to be κ-complete) then $\overline{f}_\zeta(p) = p_\zeta$, where $f_\zeta \in {}^\alpha\alpha$ is defined by $f^{-1}_\zeta(\{\xi\}) = A^\xi_\zeta$ for $\xi < \alpha$ and $\zeta < 2^\alpha$.

Proof of Theorem 2.8. Let $\emptyset \neq M \in [\alpha^*]^{\leq 2^\alpha}$. According to Lemma 2.9, there is $r \in \alpha^*$ such that $q \leq_{RK} r$ for all $q \in M$. Fix $s \in \cup(\alpha)$ and let $N = M \cup \{s\}$. Applying Lemma 2.9 again, there is $p \in \alpha^*$ for which $q \leq_{RK} p$ for all $q \in N$. It is not then hard to show that $p \in \cup(\alpha)$ and p-compactness implies M-compactness (by Lemma 1.7. (1)). If α is a strong limit cardinal, then the conclusion to (1) follows from Theorem 1.3, Theorem 1.4 and Lemma 2.9 since $|N(\alpha)| = 2^\alpha$.

Next, we will show that if α is a strong limit singular cardinal then $<\alpha$-boundedness coincides with p-compactness for some $p \in \beta_\alpha(\alpha) \cap (\alpha)$ (Corollary 2.14). In fact, this will be a particular case of a more general result (Theorem 2.13). We need the following two Lemmas: A proof of clauses (1) and (3) of Lemma 2.10 is available in [5], clause (2) of the same Lemma is shown in [14], and lemma 2.11 is a slight generalization of a result proved by Blass [2].

Lemma 2.10. *Let $p, q \in \alpha^*$. Then*

(1) *([5, Lemma 7.21. (b)]) $p <_{RK} p \otimes q$ and $q <_{RK} p \otimes q$;*
(2) *([14]) if $\emptyset \neq M \subseteq \alpha^*$ and $p, q \in \beta_M(\alpha)$, then $p \otimes q \in \beta_M(\alpha)\backslash\alpha$; and*
(3) *(Blass [5, Lemma 16.5]) if $f : \alpha \to T(q) \subseteq \alpha^*$ is a strong embedding, then $\overline{f}(p) \approx p \otimes q$.*

Lemma 2.11. *Let $p \in \gamma^*$. If $e, f : \gamma \to \alpha^*$ are functions such that f is a strong embedding and $\{\xi < \gamma : e(\xi) \leq_{RK} f(\xi)\} \in p$, then $\overline{e}(p) \leq_{RK} \overline{f}(p)$.*

Theorem 2.12. *Let $\emptyset \neq M \subseteq \alpha^*$ be such that $\emptyset \neq M \cap \mathsf{U}(\gamma)$ for $\gamma \leq \alpha$. Then $\forall\ A \in [\beta_M(\alpha)\backslash\alpha]^{\leq\alpha}\ \exists\ p \in \beta_M(\alpha) \cap \mathsf{U}(\alpha)\ \forall\ q \in A(q <_{RK} p)$.*

Proof. Let $A = \{p_\xi : \xi < \alpha\} \subseteq \beta_M(\alpha)\backslash\alpha$. We will define $q_\xi \in \alpha^*$, for each $\xi \leq \alpha$, such that

(1) $q_\xi \in \beta_M(\alpha)\backslash\alpha$ for each $\xi \leq \alpha$
(2) $p_\xi \leq_{RK} q_\xi$ for $\xi < \alpha$: and
(3) $q_\xi \leq_{RK} q_\zeta$ whenever $\xi < \zeta < \alpha$.

We proceed by transfinite induction. For $\xi = 0$ we let $q_0 = p_0$. Assume that q_ξ has been defined so that (1), (2) and (3) hold for each $\xi < \delta \leq \alpha$ (where δ denotes an ordinal number). We consider two cases:

(a) If $\delta = \xi + 1$ then we define $q_\delta = p_\delta \otimes q_\xi$. Conditions (1), (2) and (3) follow from Lemma 2.10.

(b) Assume that δ is a limit ordinal. We may identify δ with the cardinal number $|\delta|$. Let $f : \delta \rightarrow \beta_M(\alpha)\backslash\alpha$ be a strong embedding such that $f(\xi) \in T(q_\xi)$, for each $\xi < \delta$ (this embedding can be achieved by choosing a partition $\{A_\xi : \xi < \delta\}$ of α with $|A_\xi| = \alpha$ and picking $f(\xi) \in \hat{A}_\xi \cap T(q_\xi)$, for $\xi < \delta$). Fix $r \in M \cap \mathsf{U}(\delta) \subseteq \mathsf{U}(\delta) \cap \beta_M(\alpha)$. For each $\xi < \delta$, let $e_\xi : \delta \rightarrow T(q_\xi) \subseteq \alpha^*$ be a strong embedding. By Lemma 2.10 (3), we obtain that $\overline{e}_\xi(r) \approx r \otimes q_\xi$, for each $\xi < \delta$. It follows from the induction hypothesis and (3) above that $q_\xi \approx e_\xi(\zeta) \leq_{RK} q_\zeta \approx f(\zeta)$ for $\xi \leq \zeta < \delta$. Hence, by Lemma 2.11, $q_\xi \leq_{RK} \overline{e}_\xi(r) \approx r \otimes q_\xi \leq_{RK} \overline{f}(r)$ for $\xi < \delta$. The r-compactness of $\beta_M(\alpha)$ implies that $\overline{f}(r) = s \in \beta_M(\alpha)$. Thus, we define $q_\delta = s \otimes p_\delta$. According to Lemma 2.10, we have that $q_\delta \in \beta_M(\alpha)\backslash\alpha$ and conditions (1), (2) and (3) hold.

Finally, choose $t \in \beta_M(\alpha) \cap \mathsf{U}(\alpha)$ and define $p = q_\alpha \otimes t$. It is then evident that p is the required ultrafilter.

As an immediate Corollary of Theorem 2.12 we have:

Corollary 2.13. *A cardinal number α is singular if only if $\forall\ A \in [\beta_\alpha(\alpha)]^{\leq\alpha}\ \exists\ \ p \in \beta_\alpha(\alpha) \cap \mathsf{U}(\alpha)\ \forall\ q \in A(q <_{RK} p)$. In particular, if α is singular then there is $p \in \beta_\alpha(\alpha) \cap \mathsf{U}(\alpha)$ such that $\beta_\alpha(\alpha)$ is initially α-compact, and if α is strong limit* *singular then p can be taken so that $\beta_p(X) = \beta_\alpha(X)$ for all spaces X; that is, p-compactness $=$ $<\alpha$-boundedness $=$ initial α-compactness.*

Proof. $\Rightarrow$) If α is singular then it is evident that $\beta_\alpha(\alpha) \cap \mathsf{U}(\alpha) \neq \emptyset$ (see [28]). Since $N(\alpha) \subseteq \beta_\alpha(\alpha)$ then $\beta_\alpha(\alpha)$ satisfies the conditions of Theorem 2.12. Thus, the conclusion follows from Theorem 2.12.

$\Leftarrow$) It is not hard to see that $\beta_\alpha(\alpha) = N(\alpha)$ iff α is regular (see [28]); hence, α has to be singular.

The last statement follows from Theorem 1.6, Lemma 1.7 and Lemma 2.4 (2).

For regular cardinals we have the following characterization.

Theorem 2.14. *Let α be a cardinal. Then*

(1) *α is regular $\Leftrightarrow$ $\forall\, A \in [N(\alpha)]^{<\alpha}\ \exists\, p \in N(\alpha)\ \forall\, q \in A(q <_{RK} p)$; and*

(2) *$\alpha = \gamma^+$ for some γ $\Leftrightarrow$ $\forall\, A \in [N(\alpha)]^{\leq\alpha}\ \exists\, p \in N(\alpha)\ \forall\, q \in A(q <_{RK} p)$*

Proof. We only prove clause (1). $\Rightarrow$) Let $A = \{p_\xi : \xi < \gamma\} \subseteq N(\alpha)$ with $\gamma < \alpha$. For every $\xi < \gamma$, let $A_\xi \in p_\xi$ such that $|A_\xi| < \alpha$. Set $A = \cup_{\xi<\gamma} A_\xi$. Since α is regular then $|A| = \delta < \alpha$. By Hewitt-Pondiczery-Marczewski Theorem, we have that $d(\prod_{\xi<\gamma} \hat{A}_\xi) \leq \max\,(|A|, \gamma) = \lambda < \alpha$. By adding points to A if it is necessary, we may assume that $|A| = \lambda$. Applying an argument similar to that in the proof of Lemma 2.3 of [3], we can define a function $f : A \to \prod_{\xi<\gamma} A_\xi$ such that $\overline{f}(\hat{A}) = \prod_{\xi<\gamma} \hat{A}_\xi$, and we can also prove that if $p' \in \hat{A}$ satisfies $\overline{f}(p') = (p_\xi)_{\xi<\gamma}$, then $\forall\, \xi < \gamma\ (p_\xi \leq_{RK} p')$. Thus $p = p' \otimes p'$ is the required point.

$\Leftarrow$ Assume that α is singular. Then $\alpha = \sum_{\xi<\gamma} \alpha_\xi$, where $\gamma = cf(\alpha) < \alpha$ and $\alpha_\xi < \alpha$ for $\xi < \gamma$. Choose $p_\xi \in \alpha_\xi^*$ for $\xi < \gamma$. By assumption, there is $p \in N(\alpha)$ such that $p_\xi \leq_{RK} p$ for each $\xi < \gamma$. Hence, if $A \in p$ then $|A| \geq \alpha_\xi$ for each $\xi < \alpha$, and so $|A| = \alpha$; that is, $p \in \mathsf{U}(\alpha)$ which is a contradiction.

Theorem 2.12 suggests the following improvement of Theorem 1.6 above and Theorem 2.6 of [17].

Corollary 2.15. *Let X be a space. The following are equivalent.*

(1) *X^γ is initially α-compact for all cardinal γ;*
(2) *$X^{2^{2^\alpha}}$ is initially α-compact;*
(3) *$X^{|X|^\alpha}$ is initially α-compact;*
(4) *there is $p \in \mathrm{U}(\alpha)$ decomposable such that X is p-compact.*

Proof. The proof of the equivalence of (1), (2), (3) and (2) of Theorem 1.6 is completely similar to that of Theorem 2.6 of [17].

(1) $\Rightarrow$ (4) According to Theorem 1.6, there is $p_\gamma \in \mathrm{U}(\gamma)$ such that X is p_γ-compact, for each $\gamma \leq \alpha$. Set $M = \{p_\gamma : \gamma \leq \alpha\}$. Now, we verify that X is $(\beta_M(\alpha)\backslash\alpha)$-compact. In fact, we know that $Y = \beta_M(\alpha) = \cup_{\xi<\alpha^+} Y_\xi$, where $Y_0 = \alpha$, and $Y_\eta = \{\overline{f}(p_\gamma); \gamma \leq \alpha \text{ and } f : \alpha \to \cup_{\xi<\eta} Y_\xi\}$ for $\eta < \alpha^+$. We proceed by transfinite induction. Let $q \in Y$. If $q \in Y_1$ then there is $\gamma \leq \alpha$ such that $q \leq_{RK} p_\gamma$; hence, X is q-compact (by Lemma 1.7 (1)). Assume that for every $r \in (\cup_{\xi<\eta} Y_\xi)\backslash\alpha$ we have that X is r-compact and that $q \in Y_\eta$. Let $f : \alpha \to X$ be a function and let $g : \alpha \to \cup_{\xi<\eta} Y_\xi$ such that $\overline{g}(p_\delta) = q$ for some $\delta \leq \alpha$. By induction hypothesis, we have that $\overline{f}(g(\zeta)) \in X$ for each $\zeta < \alpha$. Put $h = \overline{f} \circ g$. Since X is p_δ-compact and $h(\alpha) \subseteq X$, we obtain that $\overline{h}(p_\delta) = \overline{f}(\overline{g}(p_\delta)) = \overline{f}(q) \in X$. This proves our claim. By Theorem 2.12, there is $p \in \beta_M(\alpha)\backslash\alpha$ such that $p_\gamma \leq_{RK} p$ for $\gamma \leq \alpha$. Then p is decomposable and X is p-compact.

(4) $\Rightarrow$ (1) This follows from Theorem 1.6 and Lemma 1.7 (1).

The question whether "there is $p \in \mathrm{U}(\alpha)$ such that X is p-compact" implies "every power of X is initially α-compact" is answered in the affirmative in the core model (H. D. Donder [6] has shown that in the core model every uniform ultrafilter is, regular, decomposable), and it is false whenever p is an indecomposable ultrafilter on a strong limit cardinal α ($p \in \mathrm{U}(\alpha)$ is indecomposable if there is not $r \in \mathrm{U}(\gamma)$ with $r \leq_{RK} p$ for each $\omega < \gamma < \alpha$), moreover, the statement does not even imply that X^1 is initially α-compact: for a proof see [15] and more detailed information concerning indecomposable ultrafilters can be found in [21]. In [37], the author showed that if X is $< \alpha$-bounded and p-compact, for $p \in \mathsf{U}(\alpha)$, then every power of X is initially α-compact.

It is still unknown in ZFC whether the product of initially α-compact spaces is initially α-compact for a regular cardinal α. In the negative fashion, van Douwen [7] (see [35]), assuming GCH, showed that there are two initially α-compact subspaces of $\beta(\alpha)$ whose product is not initially α-compact. Nyikos and Vaughan [27] proved that if α is a cardinal such that $\alpha^{++} \leq 2^{\omega}$, then there is a family of α^{++} initially α-compact spaces whose product is not countably compact. In [35], the author proposed the following question:

2.16 Question. (Stephenson) If initial α-compactness is productive, must α be a strong limit singular cardinal?

In this connection we have:

Theorem 2.17. *If initial α-compactness is productive, then there is $p \in \mathsf{U}(\alpha)$ decomposable such that initial α-compactness and p-compactness are the same concept.*

Proof. Assume that initial α-compactness is productive. Let $M = \bigcap\{Y \subseteq \beta(\alpha) : \alpha \subseteq Y$ and Y is initially α-compact$\}$. Then M is the initial α-compact reflection of α (see [19] or [33]); in particular, M is the smallest initially α-compact subspace of $\beta(\alpha)$ containing α. According to Corollary 2.15, there is $p \in \mathsf{U}(\alpha)$ decomposable such that M is p-compact. It is evident that $p \in M$. We claim that a space X is initially α-compact if and only if X is p-compact. In fact, let X be p-compact. Since $\forall \gamma \leq \alpha \ \exists \ p_\gamma \in \mathsf{U}(\gamma)(p_\gamma \leq_{RK} p)$ then X is initially α-compact (by Theorem 1.6 and Lemma 1.7 (1)). Conversely, let X be a initially α-compact space. By Theorem 1.6, for each $\gamma \leq \alpha$ there is $q_\gamma \in \mathsf{U}(\gamma)$ such that X is q_γ-compact. Set $I = \{q_\gamma : \gamma \leq \alpha\}$. As in the proof of Corollary 2.15, we have that X is $(\beta_I(\alpha)\backslash\alpha)$-compact, since $I \subseteq \beta_I(\alpha)\backslash\alpha$. By Theorem 1.6 and Lemma 1.7, $\beta_I(\alpha)$ is initially α-compact.

Thus, $p \in M \subseteq \beta_I(\alpha)$ and so X is p-compact. This proves our claim

Theorem 2.17 suggests the following question:

2.18 Question. If initial α-compactness is productive, must initial α-compactness coincide with $< \alpha$-boundedness?

3. The α-Boundification of α

Our aim in this section is to produce a model M of ZFC in which $M \models |\ N(\aleph_\omega)| < |\beta_{\aleph_\omega}(\aleph_\omega)|$. The following sequence of results are needed.

Lemma 3.1. *Let $\omega \leq \lambda \leq \alpha$ be cardinals, $p \in \mathrm{U}(\lambda)$, $\{A_\xi : \xi < \lambda\}$ a partition of α with $|A_\xi| = \alpha$ for $\xi < \lambda$, and $f, g : \lambda \to \beta(\alpha)$ functions satisfying $f(\xi), g(\xi) \in \hat{A}_\xi$ for $\xi < \lambda$. The $\overline{f}(p) = \overline{g}(p)$ if and only if $\{\xi < \lambda : f(\xi) = g(\xi)\} \in p$.*

Proof. $\Leftarrow$) This is evident.

$\Rightarrow$) Assume that $A = \{\xi < \lambda : f(\xi) \neq g(\xi)\} \in p$. For every $\xi \in A$ choose β_ξ and C_ξ disjoint subsets of A_ξ such that $f(\xi) \in \hat{B}_\xi, g(\xi) \in \hat{C}_\xi$ and $A_\xi = B_\xi \cup C_\xi$. Define $B = \cup_{\xi \in A} B_\xi$ and $C = \cup_{\xi \in A} C_\xi$. Then $\overline{f}(p) \in \hat{B}$ and $\overline{g}(p) \in \hat{C}$, but this is a contradiction because $\hat{B} \cap \hat{C} = \emptyset$.

In the next results we use the concept of ultraproduct: the reader may consult [5, p. 186] for the definition of ultraproducts.

Lemma 3.2. *Let $\omega \leq \lambda \leq \alpha$ be cardinals, let $\emptyset \neq M \subseteq \alpha^*$ and let γ_ξ be a cardinal such that $\omega \leq \gamma_\xi \leq |\beta_M(\alpha)|$ for each $\xi < \lambda$. Then*

$$|\textstyle\prod_{\xi<\lambda} \gamma_\xi / p| \leq |\beta_M(\alpha)| \text{ for every } p \in \mathrm{U}(\lambda) \cap \beta_M(\alpha).$$

Proof. Let $\{A_\xi : \xi < \lambda\}$ be a partition of α with $|A_\xi| = \alpha$ for $\xi < \lambda$. Since $\beta_M(A_\xi)$ (A_ξ with the discrete topology) is homeomorphic to $\beta_M(\alpha)$ and $\beta_M(A_\xi) \subseteq \beta_M(\alpha) \cap \hat{A}_\xi$, we have that $|\beta_M(\alpha) \cap \hat{A}_\xi| = |\beta_M(\alpha)|$, for $\xi < \lambda$. Hence, for each $\xi < \lambda$ we can choose a subset $S_\xi = \{a(\xi, \zeta) : \zeta < \gamma_\xi\}$

(faithfully indexed) of $\beta_M(\alpha) \cap \hat{A}_\xi$. Fix $p \in \mathsf{U}(\lambda) \cap \beta_M(\alpha)$. Then define $\Phi : \prod_{\xi<\lambda} \gamma_\xi/p \to \beta_M(\alpha)$ by $\Phi(f/p) = \overline{\phi}_f(p)$, where $\phi_f : \lambda \to \beta_M(\alpha)$ is defined by $\phi_f(\xi) = a(\xi, f(\xi))$ for all $f \in \prod_{\xi<\lambda} \gamma_\xi$. Observe from Lemma 3.1 that Φ is well-defined, and since $\beta_M(\alpha)$ is p-compact, for all $\xi, \zeta < \lambda$, then the image of Φ is contained in $\beta_M(\alpha)$. By Lemma 3.1, Φ is one-to-one. Therefore, $|\prod_{\xi<\lambda} \gamma_\xi/p| \leq |\beta_M(\alpha)|$.

The following Lemma was proved by Keisler [24, Theorem A] (a proof is available in [5, Theorem 12.18 (b)]).

Lemma 3.3. (Keisler) *For $p \in \mathsf{U}(\alpha)$ we have that*

$$|^\alpha(\gamma^{<\alpha})/p| = |^\alpha\gamma/p|^\alpha = \gamma^\alpha.$$

Theorem 3.4. *Let $\omega \leq \kappa \leq \alpha$ be cardinals and let $\emptyset \neq M \subseteq \alpha^*$ such that $\mathsf{U}(\gamma) \cap \beta_M(\alpha) \neq \emptyset$ for each cardinal $\omega \leq \gamma \leq \kappa \leq \alpha$. Then*

$$|\beta_M(\alpha)|^\kappa = |\beta_M(\alpha)|.$$

Proof. Let $\theta = |\beta_M(\alpha)|$. According to Lemma 3.2 and Lemma 3.3, it is enough to prove that $\theta^{<\kappa} \leq |\beta_M(\alpha)|$ since $\mathsf{U}(\kappa) \cap \beta_M(\alpha) \neq \emptyset$, which is equivalent to show that $\theta^\lambda \leq \theta$ for all $\lambda < \kappa$. Indeed, we proceed by transfinite induction. Assume that $\theta^\lambda \leq \theta$ for all $\lambda < \gamma < \kappa$. Since $\alpha \leq \theta$ we have that $\theta^{<\gamma} \leq \theta$. Now, choose $q \in \mathsf{U}(\gamma) \cap \beta_M(\alpha)$. It then follows, from Lemma 3.2 and Lemma 3.3, that $\theta^\gamma = |^\gamma(\theta^{<\gamma})/q| \leq |\beta_M(\alpha)|$. Therefore, $\theta^{<\kappa} \leq \theta$.

As a Corollary of Theorem 3.4 we have that:

Corollary 3.5. *If α is a singular cardinal, then $|\beta_\alpha(\alpha)| = |N(\alpha)|^\alpha$.*

Proof. Let α be a singular cardinal. It is clear that $\mathsf{U}(\gamma) \subseteq N(\alpha) \subseteq \beta_\alpha(\alpha)$, for each $\omega \leq \gamma < \alpha$, and $\beta_\alpha(\alpha) \cap \mathsf{U}(\alpha) \neq \emptyset$. Applying Theorem 3.4, we have that $|\beta_\alpha(\alpha)|^\alpha = |\beta_\alpha(\alpha)|$. The inequality of Theorem 1.4 implies that $|\beta_\alpha(\alpha)| \leq (\sum_{\gamma<\alpha} 2^{2^\gamma})^\alpha \leq |N(\alpha)|^\alpha \leq |\beta_\alpha(\alpha)|^\alpha$, since $|N(\alpha)| = \alpha^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^\gamma}$. Therefore, $|\beta_\alpha(\alpha)| = |N(\alpha)|^\alpha$.

Next, we show the main result of this section which is a consequence of Corollary 3.5 and Easton forcing Theorem.

Theorem 3.6. *There is a model M of ZFC in which*

$$M \models |N(\aleph_\omega)| < |\beta_{\aleph_\omega}(\aleph_\omega)|.$$

Proof. Let N be a countable transitive model of ZFC and assume that GCH holds in N. Define a function $E \in N$ as follows:

$$E(\aleph_n) = \aleph^+_{\omega_n} \text{ for } n < \omega, \text{ and } E(\aleph^+_{\omega_n}) = \aleph^+_{\omega_{n+1}} \text{ for } n < \omega.$$

Clearly, E is an Easton index function. According to Easton Theorem (see [26]), there is a generic extension M of N such that

(1) M and N have the same cardinals; and
(2) $M \models \forall\kappa \in dom(E) \ \ (E(\kappa) = 2^\kappa)$.

Then $M \models 2^{2^{\aleph_n}} = 2^{E(\aleph_n)} = 2^{\aleph^+_{\omega_n}} = E(\aleph^+_{\omega_n}) = \aleph^+_{\omega_{n+1}}$ for $n < \omega$; hence, $M \models 2^{2^{\aleph_n}} < 2^{2^{(\aleph_{n+1})}}$ for $n < \omega$. Thus $M \models cf(\sum_{n<\omega} 2^{2^{\aleph_n}}) = \omega$. Since $M \models 2^{\aleph_0} = E(\aleph_0) = \aleph_{\omega_1} > \aleph_\omega$ then $M \models |N(\aleph_\omega)| = \sum_{n<\omega} 2^{2^{\aleph_n}}$, by Theorem 1.4.

In virtue of Corollary 3.5, we have that

$$M \models |N(\aleph_\omega)| < |N(\aleph_\omega)|^\omega \leq |N(\aleph_\omega)|^{\aleph_\omega} = |\beta_{\aleph_\omega}(\aleph_\omega)|.$$

References

[1] A. R. Bernstein, *A new kind of compactness for topological spaces*, Fund. Math. **66** (1970), 185-193.

[2] A. R. Blass, *Kleene degree of ultrafilters*, in: Recursion Theory week (Oberwolfach, 1984), Lecture Notes in Math. 1141, (Springer-Verlag), 1985, 29-48.

[3] W. W. Comfort, *Ultrafilters: some old and some new results*, Bull. Amer. Math. Soc. **83** (1977), 417-455.

[4] W. W. Comfort and S. Negrepontis, *On families of large oscillation*, Fund. Math. **75** (1972), 275-290.

[5] ———, *The Theory of Ultrafilters*, Grudlehren der Mathematischen Wissenschaften Vol. 211, Springer-Verlag, 1974.

[6] H. D. Donder, *Regularity of ultrafilters and the core model*, Israel J. Math. **63** (1988), 289-322.

[7] E. K. van Douwen, *The product of two normal initially κ-compact spaces*, to appear.

[8] S. P. Franklin, *On epi-reflective hulls*, Topology Appl. **1** (1971), 29-31.

[9] P. J. Freyd, *Functor Theory*, Doctoral Dissertation, Princeton University, 1960.

[10] Z. Frolík, *Types of ultrafilters on countable sets*, in: General Topology and its Relations to Modern Analysis and Algebra II (Proc. Second Prague topological Symposium, 1966) (Academia, Prague, 1967), 142-143.

[11] ———, *Sums of ultrafilters*, Bull. Amer. Math. Soc. **73** (1967), 87-91.

[12] S. García-Ferreira, *Various Orderings on the Space of Ultrafilters*, Doctoral Dissertation, Wesleyan University, 1990.

[13] ———, *Three orderings on $\beta(\omega)\backslash\omega$*, preprint.

[14] ———, *On free p-compact groups*, preprint.

[15] ———, *Comfort types of ultrafilters*, preprint.

[16] S. García-Ferreira and A. Tamariz-Mascarua, *The α-boundification of α*, preprint.

[17] J. Ginsburg and V. Saks, *Some applications of ultrafilters in topology*, Pacific J. Math. **57** (1975), 403-418.

[18] S. L. Gulden, W. M. Fleischman and J.H. Weston, *Linearly ordered topological spaces*, Proc. Amer. Math. Soc. **24** (1970), 197-203.

[19] H. Herrlich and J. Van der Slot, *Properties which are closely related to compactness*, Indag. Math. **29** (1967), 524-529.

[20] T. Jech, *Set Theory*, Academic Press, New York, 1978.

[21] A. Kanamori, *Finest partitions for ultrafilters*, J. Symbolic Logic **51** (1986), 327-332.

[22] M. Katětov, *Characters and types of point sets*, Fund. Math. **50** (1961/62), 369-380 (Russian).

[23] ———, *Products of filters*, Comment. Math. Univ. Carolinae **9** (1968), 173-189.

[24] H. J. Keisler, *On the cardinalities of ultraproducts*, Bull. Amer. Math. Soc. **70** (1964), 644-647.

[25] J. F. Kennison, *Reflective functors in general topology and elsewhere*, Trans. Amer. Math. Soc. **118** (1965), 303-315.

[26] K. Kunen, Set Theory, *An Introduction to Independence Proofs*, Vol 102 of Studies of Logic and Foundations of Mathematics, North-Holland, 1980.

[27] P. J. Nyikos and J. E. Vaughan, *Sequentially compact Franklin-Rajagopalan spaces*, Proc. Amer. Math. Soc. **101** (1987), 149-155.

[28] L. O'Callaghan, *Topological Endohomeomorphisms and Compactness Properties of Products and Generalized Σ-products*, Doctoral Dissertation, Wesleyan University, 1975.

[29] W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. **23** (1956), 409-419.

[30] V. Saks, *Countably Compact Topological Groups*, Doctoral Dissertation, Wesleyan University, 1972.

[31] ———, *Ultrafilter invariants in topological spaces*, Trans. Amer. Math. Soc. **241** (1978), 79-97.

[32] V. Saks and R. M. Stephenson Jr., *Products of $\mathcal{M}$-compact spaces*, Proc. Amer. Math. Soc. **28** (1971), 279-288.

[33] J. Van der Slot, *Universal Topological Properties*, ZW 1966-011. Math. Centrum, Amsterdam, 1966.

[34] Y. M. Smirnov, *On topological spaces, compact in a given interval of powers*, Izv. Akad. Nauk. SSSR Ser. Mat. **14** (1950), 155-178.

[35] R. M. Stephenson Jr., *Initially κ-compact and related spaces*, in: K.Kunen and J.E. Vaughan, ed. Handbook of Set-Theoretic Topology, North-Holland, 1984.

[36] R. M. Stephenson Jr. and J.E. Vaughan, *Products of initially $\mathcal{M}$-compact spaces*, Trans. Amer. Math. Soc. **196** (1974), 177-189.

[37] J. E. Vaughan, *Powers of spaces of non-stationary ultrafilters*, Fund. Math. **117** (1983), 5-14.

[38] R. G. Woods, *Topological extension properties*, Trans. Amer. Math. Soc. **210** (1975), 365-385.

Instituto de Matemáticas
Ciudad Universitaria (UNAM)
México, D.F. 04510
México