PROXIMATELY REFINABLE MAPS AND
θ'_n-CONTINUA

by

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1. INTRODUCTION

A class of functions is of particular interest if many topological properties are preserved by the action of all of the functions. One class of functions that yields such stability for many continuum-theoretic properties is the class of refinable maps, defined by Jo Heath (Ford) and Jack Rogers [2, p. 263]. Proximate refinability, which generalizes refinability, was defined by Grace [4, p. 294] in considering what properties of refinable maps are essential to the preservation of the proximate fixed point property [4, p. 294]. One purpose of this paper is to characterize proximately refinable maps on graphs as those maps that are monotone. Another purpose is to study what properties of continua preserved by refinable maps are or are not preserved by proximately refinable maps where the domain or range is a θ_n-continuum or a θ'_n-continuum.

2. DEFINITIONS AND PRELIMINARIES

A *continuum* is a compact, connected, metric space and, except for Theorem 1, all spaces are continua. A θ_n-*continuum* (defined in [6, p. 261] as a compact, metric θ_n-space, based on the definition of θ_n-space in [1, p. 139]) is a continuum in which the complement of any subcontinuum has at most n components. A θ'_n-*continuum* [7, p. 56] is a θ_n-continuum which admits a unique, monotone, upper semi-continuous decomposition D, called the canonical decomposition, whose elements have void interior and whose quotient space X_D is a graph, i.e., a locally connected θ_n-continuum. There is a

natural function $P : X \rightarrow X_D$, called the *projection func-tion*, that assigns to each element x in X, the element of D that contains it. The elements of D are called *tranches* and the *order* of a tranch is its order considered as an elemen-t of the quotient space. If A is a subset or point of X, then $K(A) = \cap \{Q \mid Q$ is a subcontinuum of X and A is contained in the interior of $Q\}$ [9, Theorem 2, p. 404]. It is easily seen that $K(H) = T(H)$ for each subcontinuum H of a θ_n-continuum X where $T(H) = \{x \in X \mid K(x) \cap H \neq \phi\}$. It also is easily seen that $K(H) \backslash H$ has void interior for any subcontinuum H of a θ'_n-continuum. From this and [6, Theorem 1, p. 263], it follows that a continuum X is a θ'_n-continuum if and only if X is a θ_n-continuum for which $K(H) \backslash H$ has void interior for each subcontinuum H of X.

The notation $f : X \rightarrow\rightarrow Y$ means f is a function from X onto Y. A *map* is a continuous function (but the use of *map* as a verb does not require that the function be continuous). A function g is ϵ-*continuous* if, for each x in the domain of g, there is a neighborhood D of x such that $g[D] \subseteq N_\epsilon(g(x))$, i.e., such that $d(g(x), g(z)) < \epsilon$, for each z in D. A function $f : X \rightarrow\rightarrow Y$ is an ϵ-*function* if diam $(f^{-1}(y)) < \epsilon$, for each y in Y, and a *strong ϵ-function* if, for each y in Y, there is a neighborhood D of y such that diam $(f^{-1}[D]) < \epsilon$. A function $f : X \rightarrow\rightarrow Y$ is *proximately refinable* if, for every positive number ϵ, there is an ϵ-continuous strong ϵ-function $g : X \rightarrow\rightarrow Y$, ϵ-near f, i.e., such that $d(f(x), g(x)) < \epsilon$ for all x in X. If g can be chosen to be continuous also, then f is *refinable*. It is easily seen that a proximately refinable or refinable function is a map. If f is refinable and g in the definition can be chosen to be one to one, then f is a *near homeomorphism*.

Let $f : X \rightarrow\rightarrow Y$ be a proximately refinable map, and, for each positive integer n, let f_n be a strong $\frac{1}{n}$-function that is $\frac{1}{n}$-continuous and $\frac{1}{n}$-near f. Let B be a closed subset of Y. We denote by B' the limit of some convergent subsequence of $\{f_n^{-1}[B]\}$. Observe that B' is not uniquely determined since it depends on the choice of the subsequence. Heath (Ford) and Rogers [2, Theorem 1, p. 264] have proved for refinable maps

that for each subcontinuum B of Y, B' is a subcontinuum of X such that $f[B'] = B$ and $f^{-1}[B^\circ] \subseteq B'$. Their proof [2, Theorem 1, p. 264] also works for proximately refinable maps. We will use both of these properties of B' frequently.

3. PROXIMATELY REFINABLE MAPS ON GRAPHS AND COMPOSITION OF MAPPINGS

The first theorem in this section will be of use later in the section and is also of interest in its own right.

Theorem 1. *Let X, Y, and Z be compacta and $f : X \rightarrow\rightarrow Y$ and $g : Y \rightarrow\rightarrow Z$ be proximately refinable. Then $h = g \circ f$ is proximately refinable, and h is refinable, if f and g are.*

Proof. Let ϵ be a positive number, and let $\delta < \epsilon$ be a positive number such that $d(g(y_1), g(y_2)) < \frac{\epsilon}{2}$, if $d(y_1, y_2) < \delta$. Let F be a strong δ-function that is δ-continuous, and δ-near f. Since strong δ-functions are uniformly strong there is a positive number $\delta' < \frac{\epsilon}{2}$ such that diam $(F^{-1}[D]) < \delta$, if diam $D < \delta'$. Let G be a strong δ'-function that is δ'-continuous and δ'-near g. Let $H = G \circ F$ and let D be any open subset of Z such that diam $(G^{-1}[D]) < \delta'$. Each point of Z is contained in such an open set, since G is a strong δ'-function. So diam $H^{-1}[D] = $ diam $(F^{-1}[G^{-1}[D]]) < \delta < \epsilon$, by the choice of δ'. Hence, H is a strong ϵ-function.

Let $x \in X$. Then $d(f(x), F(x)) < \delta$. Therefore $d(g(f(x)), g(F(x))) < \frac{\epsilon}{2}$, also $d(g(F(x)), G(F(x))) < \delta' < \frac{\epsilon}{2}$. Hence $d(h(x), H(x)) = d(g(f(x)), G(F(x))) \leq d(g(f(x)), g(F(x))) + d(g(F(x)), G(F(x))) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence H is ϵ-near h. Since h is continuous, H is $(2\epsilon + \gamma)$-continuous, for each positive number γ, by [5, Lemma 2, p. 332]. It follows that H is 3ϵ-continuous, and, hence, that h is proximately refinable.

If f and g are refinable, then F and G can be chosen to be continuous (and still be strong δ- and δ'-functions, respectively). Then H is continuous, and hence, h is refinable.

Next we use Theorem 1 to establish a characterization of proximately refinable mappings on graphs.

Theorem 2. *A map defined on a graph is proximately refinable if and only if it is monotone.*

Proof. It has been proved in [5, Lemma 5, p. 335] that any proximately refinable map on a graph is monotone.

To show the converse, let X be a graph and $f : X \twoheadrightarrow Y$ be monotone. Let $\{A_1, \ldots, A_n\}$ be the collection of all arcs and simple closed curves in X having the following properties: (1) each is mapped to a point by f, (2) the end points of the arcs are not of order 2 but all other points are, and (3) all but one point of each simple closed curve is of order 2. Let X_1 be the space obtained from X by identifying A_1 to a point and, in general, for $i = 2, \ldots, n$, let X_i be the factor space obtained from X_{i-1} by identifying A_i in X_{i-1} to a point. Note that an A_i that starts as an arc may become a simple closed curve before it is shrunk to a point. Let $X_0 = X$ and for $i = 1, \ldots, n$, let f_i be the natural map from X_{i-1} to X_i. Then the map $f_{n+1} : X_n \twoheadrightarrow Y$ defined by $f_{n+1} = f \circ [(f_n \circ \cdots \circ f_1)^{-1}]$ is monotone and has the property that $f_{n+1}^{-1}(y)$ contains at most one point of order unequal to 2 and contains no simple closed curve, for all y in Y. Hence by [3, Theorem 2, p. 142], f_{n+1} is proximately refinable (in fact, f_{n+1} is a near homeomorphism). Also f can be taken to be $f_{n+1} \circ f_n \circ \cdots \circ f_1$, and is proximately refinable by Theorem 1, if one can show that f_i is proximately refinable for $i = 1, \ldots, n$. For notational convenience, we only show that $f_1 : X_0 \twoheadrightarrow X_1$ is proximately refinable. Also we show this only for the case where A_1 is an arc with endpoints of order greater than 2. The case where A_1 is an arc with one endpoint of order 1 and the other of order greater than 2 and the case where A_1 is a simple closed curve with all but one point of order 2 are similar.

Let ϵ be a positive number. Let the end points of A_1 be b and c. Let $m + 1$ and $n + 1$ be the order of b and c, respectively, in X_0. For $i = 1, \ldots, m$, and $j = 1, \ldots, n$, let B_i and C_j be arcs in $X_0 \backslash A_1$, of diameter less than $\frac{\epsilon}{2}$, that are mutually disjoint except that all of the B_i's contain b and all of the C_j's contain c, and such that b and c are the only points in their union that

are not of order 2 in X_0. Since $A_1 \cup (\cup_{i=1}^m B_i) \cup (\cup_{j=1}^n C_j)$ is uniquely arcwise connected, no confusion should result from using interval notation for subarcs and half-open subarcs in that subgraph. Let a_1 be the identified point, A_1, as a point in X_1. Let d_0 be the distance in X_0 and d_1 be the distance in X_1. Assume $d_1(x, a_1) = d_0(x, A_1)$ and that $d_1(x, z) \le d_0(x, z)$, for points in $X_1 \backslash \{a_1\} = X_0 \backslash A_1$.

For $i = 1, \dots, m$ and $j = 1, \dots, n$, let b^i and c^j be the end points of order 2 of B_i and C_j, respectively. Let $b_0 = c_0$ be a point of (b, c) and let $b_0, \dots, b_p = b$ and $c_0, \dots, c_q = c$ be partitions of $[b_0, b]$ and $[c_0, c]$, respectively, whose subarcs all have diameter less than $\frac{\epsilon}{2m}$ and $\frac{\epsilon}{2n}$, respectively, and such that p is a multiple of m and q is a multiple of n. Let $u = \frac{p}{m}$ and $v = \frac{q}{n}$.

For $i = 1, \dots, m$, and $j = 1, \dots, n$ let $b = b_0^i, \dots, b_{u+1}^i = b^i$ and $c = c_0^j, \dots c_{v+1}^j = c^j$ be partitions of $[b, b^i]$ and $[c, c^j]$, respectively.

We now wish to show that there is a strong ϵ-function, f_ϵ, that is ϵ-continuous and ϵ-near f_1. Let f_ϵ be a function that maps X_0 onto X_1 in the following way. For $x \in X_0 \backslash [A_1 \cup (\cup_{i=1}^m B_i) \cup (\cup_{j=1}^n C_j)]$, let $f_\epsilon(x) = f_1(x) = x$. For $i = 1, \dots, m$ and $j = 1, \dots, n$, let f_ϵ map $(b, b^i]$ homeomorphically onto $(b_u^i, b^i]$ and map $(c, c^j]$ homeomorphically onto $(c_v^j, c^j]$. Let $f_\epsilon(b_0) = a_1$. For $j = 1, \dots, u$ and $k = 1, \dots, m$, let $B(j, k) = (b_{j-1}^k, b_j^k]$. For $i = 1, \dots, p$, let f_ϵ map $(b_{i-1}, b_i]$ homeomorphically onto the ith member of $\{B(j, k) \mid j = 1, \dots, u$ and $k = 1, \dots, m\}$, in the lexicographical order on the ordered pairs (j, k). Let f_ϵ be defined on $[c_0, c]$ analogously. It is straightforward to check that f_ϵ has the desired properties.

For θ'_n-continua X and Y and a proximately refinable map f from X onto Y, the next theorem uses Theorem 2 to give information about f_D, the induced map on the graph decomposition spaces.

Theorem 3. *If X and Y are θ'_n-continua and $f : X \rightarrow\rightarrow Y$ is*

proximately refinable, then f induces a map f_D from X_D onto Y_D and f_D is proximately refinable.

Proof. Although [8, Theorem 3, p. 233] and [8, Theorem 4, p. 233] are stated for refinable maps, parts of their proofs can be applied to proximately refinable maps. For proximately refinable maps, the last paragraph of the proof of [8, Theorem 3] can be used to establish the existence of the induced map f_D and the first paragraph of the proof of [8, Theorem 4] can be used to show that f_D is monotone. By Theorem 2, then, f_D is proximately refinable.

4. COMPARISONS WITH REFINABLE MAP RESULTS

If X is a θ'_n-continuum and $f : X \twoheadrightarrow Y$ is refinable, it is known that Y is a θ'_n-continuum [8, Theorem 3, p. 233]. However, for proximately refinable maps the most that can be said is given by the following theorem.

Theorem 4. *If $f : X \twoheadrightarrow Y$ is proximately refinable and X is a θ'_n-continuum, then Y is a θ_{2n}-continuum.*

Proof. Let H_0 be a subcontinuum of Y such that $Y \backslash H_0 = H_1 \cup H_2 \cdots \cup H_m$, where $H_1, \cdots, H_m$ are mutually separated. For $i = 1, \ldots, m$, let $K_i = H_0 \cup H_i$. For $i = 1, \ldots,$ let f_i be a strong $\frac{1}{i}$-function that is $\frac{1}{i}$-continuous and $\frac{1}{i}$-near f such that $\{f_i^{-1}[H_0]\}$, $\{f_i^{-1}[K_1]\}, \ldots,$ and $\{f_i^{-1}[K_m]\}$ converge to H'_0, $K'_1, \ldots,$ and K'_m, respectively. For $i = 1, \ldots, m$, K'_i is contained in the component of $f^{-1}[K_i]$ that contains $f^{-1}[H_0^\circ]$. Let H_0^* be the component of $f^{-1}[H_0]$ that contains H'_0. By [6, Lemma 1, p. 262] and the fact that $P[X]$ is a graph, there are mutually disjoint open arcs $A_i \subseteq cl P[K'_i \backslash H_0^*]$, for $i = 1, \ldots, m$, which have one and only one end point in $P[H_0^*]$ and contain only points of order 2. Now $P[X]$ is clearly a θ_n-continuum. Let G be the graph resulting from shrinking $P[H_0^*]$ to a point in $P[X]$. Then G is also a θ_n-continuum and so, by [1, Theorem 49, p. 158], the order of G at $P[H_0^*]$ (as a point of G) is not more than $2n$. But the order of G at $P[H_0^*]$ is at least m, so $m \leq 2n$. Hence Y is a θ_{2n}-continuum.

The following example shows that $2n$ in Theorem 4 cannot be reduced and also shows that Y need not be a θ'_m-continuum for any m.

Example 1. For each natural number n, a proximately refinable map from a θ'_n-continuum X onto a θ_{2n}-continuum Y that is neither a θ'_{2n}-continuum nor a θ_{2n-1}-continuum.

For $n > 1$, the example is gotten by putting together copies of the example for the case $n = 1$.

For $n = 1$, let K be Knaster's indecomposable continuum with two end points [10, Figure 5, p. 205] that is the union of semicircles described using numbers in the unit interval that have a base 5 representation not using "1" or "3." The end points of K are $(0,0)$ and $(1,0)$.

Let C be the Cantor set. X is $C \times K$, considered as a subset of R^3, with the following identifications. $(0,0,0) = (1,0,0)$ and for each gap (a,b) of C, where $b - a = \frac{1}{3^n}$, $(a,0,0) = (b,0,0)$ if n is odd and $(a,1,0) = (b,1,0)$ if n is even. Y is $\{0,1\} \times K$, but with $(0,0,0) = (1,0,0)$. For any point (a,b,c) of X, let $f(a,b,c) = (0,b,c)$, if $0 \le a \le \frac{1}{3}$, and let $f(a,b,c) = (1,b,c)$, if $\frac{2}{3} \le a \le 1$. That is (if we ignore the identifications) f projects each half of X parallel to the x-axis onto the nearest half of Y. We take the distance, d, on X (and Y) to be such that $d((a,b,c),(a',b',c')) \le \sqrt{(a-a')^2 + (b-b')^2 + (c-c')^2}$, with equality where $a = a'$, i.e., where both points are on the same copy of K.

X is easily seen to be a circularly chainable θ'_1-continuum, and Y is the union of two copies of K joined at an end point of each, and, therefore, is neither a θ'_2-continuum nor a θ_1-continuum.

Let X_1 be the left half of X, i.e., let $X_1 = \{(a,b,c) \in X \mid 0 \le a \le \frac{1}{3}\}$, and let $Y_1 = \{0\} \times K$. We wish to define approximating functions for $f \mid X_1$ in such a way that the symmetry of X can be used to extend them to all of X and to see that f is

proximately refinable. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots$ be a sequence of open chains covering $\{0\} \times K$ in such a way that, for $i = 1, 2, \ldots,$ (1) $\mathcal{C}_i$ chains Y_1 from $(0,0,0)$ to $(0,1,0)$, (2) $\mathcal{C}_{i+1}$ refines $\mathcal{C}_i$ and runs straight through $\mathcal{C}_i$ then straight back through $\mathcal{C}_i$ and finally straight through $\mathcal{C}_i$ again, and (3) $\mathcal{C}_i$ is a $\frac{1}{i}$-chain with $(0,0,0)$ only in the first link (in one natural way of defining the chains, C_i would, in fact, be a $1/(4 \cdot 5^{i-1})$-chain).

Let ϵ be a positive number. For some pair i, j of natural numbers with $i < j$, we use chainings of X_1 and Y_1, based on $\mathcal{C}_i$ and $\mathcal{C}_j$ to define a function $f_{i,j}^1$, $X_1 \longrightarrow Y_1$ such that $f_{i,j}^1$ is a strong ϵ-function that is ϵ-continuous and maps each point x of X_1 into a member of C_i that contains $f(x)$. The basic idea in defining $f_{i,j}^1$ is to (1) chain Y_1 from $(0,0,0)$ to $(0,1,0)$ with a certain chain $\mathcal{C}_{i,j}$ that refines $\mathcal{C}_i$ and whose links are, in a certain sense, about as long as the links of $\mathcal{C}_i$ and as wide as the links of $\mathcal{C}_j$, (2) chain X_1 from $(0,0,0)$ to $(\frac{1}{3},1,0)$ with a certain chain $\mathcal{L}_{i,j}$, with the same number of links as $\mathcal{C}_{i,j}$ has, such that each link is the Cartesian product of a "subinterval" of C with the projection onto K of a link in $\mathcal{C}_i$, and (3) map each link of $\mathcal{L}_{i,j}$ onto the corresponding link of $\mathcal{C}_{i,j}$.

Let i be a natural number such that C_i is an $\frac{\epsilon}{4}$-chain. Let j be a natural number that is large enough (e.g., $j > i + |\log_3 \frac{8}{\epsilon}|$) to permit the following selection. Let $\{x_0, x_1, \ldots, x_k\}$ be a partition of $[0, \frac{1}{3}]$, with $k = 3^{j-i}$, where (1) for $p = 1, \ldots, k-1$, the division point x_p is in a component (a, b) of $[0, \frac{1}{3}] \backslash C$ where $(a, 1, 0) = (b, 1, 0)$, if p is odd, and $(a, 0, 0) = (b, 0, 0)$, if p is even, and (2) for $p = 1, \ldots, k$, the diameter of $C \cap [x_{p-1}, x_p]$ is less than $\frac{\epsilon}{8}$. Note that $\mathcal{C}_j$ runs straight through C_i in one direction or the other a total of 3^{j-i} times. Let $\mathcal{C}_{i,j}$ be the chain consisting of all of the sets A such that A is the union of the members of some subchain of C_j that is maximal with respect to being contained in the same link of $\mathcal{C}_i$. Let $\mathcal{L}'_{i,j} = \{(C \cap [x_{p-1}, x_p]) \times A \mid p \in \{1, \ldots, k\} \text{ and } A \in \mathcal{C}_i\}$. Let $\mathcal{L}_{i,j}$ be the chain (in X_1, from $(0,0,0)$ to $(\frac{1}{3},1,0)$) consisting of all of the sets A such that A is the union of the members of some (one- or two-link) subchain of $\mathcal{L}'_{i,j}$ that is maximal

with respect to being mapped into the same member of $\mathcal{C}_i$ by f. Most of the members of $\mathcal{L}_{i,j}$ are members of $\mathcal{L}'_{i,j}$ but, for example, if A is the last member of $\mathcal{C}_i$, then $(C \cap [x_0, x_1]) \times A$ and $(C \cap [x_1, x_2]) \times A$ are replaced by $(C \cap [x_0, x_2]) \times A$.

$\mathcal{L}_{i,j}$ is an open (relative to X_1) $\frac{\epsilon\sqrt{2}}{4}$-chain covering X_1, and the number of links in $\mathcal{L}_{i,j}$ is the same as the number in $\mathcal{C}_{i,j}$ and is $m = (n-1)k + 1 = (n-1)3^{j-i} + 1$, where n is the number of links in $\mathcal{C}_i$.

To define $f^1_{i,j}$, let $\mathcal{L}_{i,j} = \{L_1, \ldots, L_m\}$ and $\mathcal{C}_{i,j} = \{C_1, \ldots, C_m\}$. Let $f^1_{i,j}$ be any function from X_1 onto Y_1 such that $f^1_{i,j}(x) \in C_p$, if and only if $x \in L_p$ for $p = 1, \ldots, m$, and (to facilitate extending $f^1_{i,j}$ to all of X) $f^1_{i,j}$ carries more than one point onto $f^1_{i,j}(\frac{1}{3}, 0, 0)$. Since $\mathcal{C}_{i,j}$ is an $\frac{\epsilon}{4}$-chain, $f^1_{i,j}$ is clearly $\frac{\epsilon}{4}$-continuous. Since $\mathcal{L}_{i,j}$ is an $\frac{\epsilon\sqrt{2}}{4}$-chain, $f^1_{i,j}$ is clearly a strong $\frac{\epsilon}{2}$-function. Since $\mathcal{C}_i$ is an $\frac{\epsilon}{4}$-chain, $f^1_{i,j}$ is clearly $\frac{\epsilon}{4}$-near f.

To extend $f^1_{i,j}$ to $f_{i,j}$, defined on all of X, let $f_{i,j}(a, b, c) = f^1_{i,j}(a, b, c)$ if $(a, b, c) \in X_1$, and let $f_{i,j}(a, b, c)$ be the point in $Y \setminus Y_1$ whose last two coordinates are the same as those of $f^1_{i,j}(1 - a, b, c)$, if $(a, b, c) \in X \setminus X_1$. Now $f_{i,j}$ carries X onto Y, since $f^1_{i,j}$ carries more than one point onto $f^1_{i,j}(\frac{1}{3}, 0, 0)$, even though $f_{i,j}(\frac{2}{3}, 0, 0) \notin Y \setminus Y_1$. From the corresponding properties of $f^1_{(i,j)}$, it is seen that $f_{i,j}$ is an ϵ-continuous, strong ϵ-function that is ϵ-near f. Hence f is proximately refinable.

For $n > 1$, let the domain be the union of n copies of X joined at the points that correspond to $(0, 0, 0)$, let the image be the union of n copies of Y, joined at the points that correspond to $(0, 0, 0)$, and let the proximately refinable function be defined in the obvious way.

It is not known whether the condition that X is a θ'_n-continuum can be weakened in Theorem 4, so we pose the following question.

Question. If X is a θ_n-continuum and $f : X \longrightarrow\!\!\!\rightarrow Y$ is proximately refinable, must Y be a θ_{2n}-continuum?

The reason the situation for proximately refinable maps differs from that for refinable maps, where the image of any θ'_n-continuum is a θ'_n-continuum [8, Theorem 3, p. 233], is as follows. A θ'_n-continuum can be characterized as being a θ_n-continuum in which $K(H)\backslash H$ has void interior for every subcontinuum H. If $f : X \longrightarrow Y$ is refinable, and X has this property (whether X is a θ'_n-continuum or not), then Y has the property also [8, Corollary 2, p. 232]. However, as Example 1 shows, if f is only proximately refinable, then Y may not have this property. But if we add to the hypothesis of Theorem 4 the statement that Y has this property, then we get the following theorem.

Theorem 5. *Suppose $f : X \longrightarrow Y$ is proximately refinable, X is a θ_n-continuum, and Y is a continuum for which $K(H)\backslash H$ has void interior for each subcontinuum H of Y. Then Y is a θ'_n-continuum.*

Proof. Because Y has the property, concerning the set function K, given in the hypothesis, to show that Y is a θ'_n-continuum it suffices to show that Y is a θ_n-continuum. Suppose H_0 is a subcontinuum of Y and $Y\backslash H_0 = H_1 \cup \cdots \cup H_m$, where the H_i's are mutually separated. For $i = 1, \ldots, m$, there is a proper subcontinuum L_i of Y such that $Y\backslash H_i \subseteq L_i^\circ$ since $K(Y\backslash H_i)\backslash(Y\backslash H_i)$ has void interior. Then $M_i = H_0 \cup (L_i \cap H_i)$ is a continuum that does not contain all of H_i or any of H_j for $j \neq i$. Let $M = \cup_{i=1}^m M_i$. Then $H_0 \subseteq M^\circ$ and $H_i\backslash M \neq \phi$, for $i = 1, \ldots, m$. Therefore, $f^{-1}[H_i]\backslash M' \neq \phi$, for $i = 1, \ldots, m$, since $M' \subseteq f^{-1}[M]$. Now $f^{-1}[H_0] \subseteq f^{-1}[M^\circ] \subseteq M'$, so $X = f^{-1}[Y] = f^{-1}((\cup_{i=1}^m H_i) \cup H_0) = [\cup_{i=1}^m f^{-1}(H_i)] \cup f^{-1}[H_0] = [\cup_{i=1}^m f^{-1}[H_i]] \cup M' = (\cup_{i=1}^m [f^{-1}[H_i]\backslash M']) \cup M'$. It follows that M' separates X into the m nonvoid sets $f^{-1}[H_i]\backslash M'$, for $i = 1, \ldots, m$. Therefore $m \leq n$ since X is a θ_n-continuum. This proves that Y is a θ_n-continuum.

Unlike the inverses of refinable maps (see [8, Section 2, pp. 236–238]) the inverses of proximately refinable maps preserve almost nothing. The following theorem can be used to show that.

Theorem 6. *Let $X = V \cup W \cup Z$ and $Y = V \cup W$, where V, X, Y, and Z are continua and W is closed, and (1) V is a chainable continuum with endpoint p, (2) for some neighborhood N of p, $V \cap N \cap (W \cup Z) = \{p\}$, and (3) $Z \cap (V \cup W) = \{p\}$. Then the map $f : X \longrightarrow\longrightarrow Y$ that is the identity on Y and maps Z to p is proximately refinable.*

Proof. Let ϵ be a positive number. First we wish to cover Z with a cover $\mathcal{A} = \{A_1, \ldots, A_n\}$ of sets of cardinal c such that (1) $A_1 \cup A_n$ is contained in the $\frac{\epsilon}{2}$-neighborhood of p, (2) $p \in A_1$, (3) $A_i \cap A_j = \phi$ unless $i = j$, (4) the diameter of $A_i \cup A_{i+1}$ is less than $\frac{\epsilon}{2}$, for $i = 1, \ldots, n-1$, and (5) $n \geq 2$. Let $\mathcal{D}$ be an open (relative to Z) $\frac{\epsilon}{4}$-cover of Z (with subsets of Z) such that (1) no proper subcollection of $\mathcal{D}$ covers Z, (2) p is in only one member of $\mathcal{D}$ and (3) $\mathcal{D}$ has at least 2 members. Let F be a function from some initial segment $\{1, \ldots, n\}$ of the natural numbers onto $\mathcal{D}$, such that (1) $p \in F(1)$, (2) $F(i) \cap F(i+1) \neq \phi$, for $i = 1, \ldots, n-1$, and (3) $F(n) = F(1)$. F can be defined by "walking" over Z using members of $\mathcal{D}$ as "stepping stones" and always "stepping" from one "stone" to an overlapping "stone," starting with the "stone" covering p and returning to that "stone" after "stepping" on all of the "stones." For $i = 1, \ldots, n$, let $F_1(i)$ be the complement in $F(i)$ of the union of all of the other members of $\mathcal{D}$.

We define $\mathcal{A}$ by mathematical induction. Let A_1 be a subset of $F_1(1)$ such that $p \in A_1$ and A_1 and $F_1(1) \backslash A_1$ have cardinal c. Assume A_i is defined for $i = 1, \ldots, j-1$. If $F(j) = F(k)$ for some $k > j$, then let A_j be a subset of $F(j) \backslash \cup_{i=1}^{j-1} A_i$ such that A_j and $F_1(j) \backslash A_j$ have cardinal c. If $F(j) \neq F(k)$, for all $k > j$, then let $A_j = F(j) \backslash \cup_{i=1}^{j-1} A_i$. This defines $\mathcal{A}$ having the desired properties.

Let $\{C_1, \ldots, C_m\}$ be an open (relative to V) $\frac{\epsilon}{2n}$-chain covering V (with subsets of V) such that (1) $m > 2n$, (2) $[\cup_{i=1}^{2n} C_i] \cap (W \cup Z) = \{p\}$, (3) p is only in C_1 and (4) $\overline{C_i} \cap \overline{C_j} \neq \phi$, if and only if $|i - j| \leq 1$. Let $B_1 = \overline{C_1}$ and $B_i = \overline{C_i} \backslash C_{i-1}$, for $i = 2, \ldots, m$. Let g be a one-to-one function from X onto Y such that (1) $g(A_i) = B_i$ and $g(B_{2i-1} \cup B_{2i}) = B_{n+i}$, for

$i = 1, \ldots, n$, and (2) g is the identity on $W \cup \left[\cup_{i=2n+1}^{m} B_i \right]$. Then g is an ϵ-continuous, strong ϵ-function that is ϵ-near f. It follows that f is proximately refinable.

This can be used to show that inverses of proximately refinable maps do not preserve any of the properties considered in [8]. For example, for the property of being a θ_1'-continuum, in the theorem let (1) Z be the cone over the Cantor set, (2) p be the vertex of Z, (3) V and W be arcs, each with end point p and otherwise disjoint from Z, whose union is a simple closed curve, (4) $X = V \cup W \cup Z$, and (5) $Y = V \cup W$. By Theorem 6, the function $f : X \to\to Y$ that is the identity on Y and maps Z to p is proximately refinable. Y is a θ_1'-continuum, but X is not a θ_n-continuum for any n, in fact, $\{p\}$ separates X into c components.

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