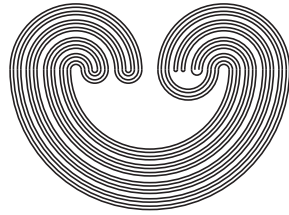

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PRODUCTS OF PARACOMPACT SPACES AND LAŠNEV SPACES

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Dedicated to Professor Yukihiro Kodama on his 60th birthday

ABSTRACT. We shall prove that for a Lašnev space Y the product $X \times Y$ is paracompact for any paracompact space X iff Y is σ -locally compact. This extends K. Morita's theorem for the metrizable case.

1. INTRODUCTION.

A space which is the closed image of a metric space was characterized by Lašnev [12], and it is called a Lašnev space. The normality of a product space $X \times Y$ with a Lašnev space Y has been studied by Chiba [4], Bešlagić and Chiba [3] and the author [7], [8]. Their results generalize or closely relate to those obtained so far for the case Y metrizable.

Spaces Y with the property that $X \times Y$ is paracompact for any paracompact space X have been long considered by many authors (for surveys, see Przymusiński [16], Atsugi [2]), but they are not yet characterized explicitly (cf. Katuta [10]). Following Telgárski [18] let us denote by Π the class of such Y 's. In case Y is metrizable Morita [14] proved a theorem that $Y \in \Pi$ iff Y is σ -locally compact (= a countable union of closed locally compact subspaces); the "if" part is always true for an arbitrary paracompact space Y (Morita [15]). This theorem of Morita is also valid for a paracompact M -space Y (Morita[15]), but is not valid when Y is the closed image of a paracompact M -space; indeed, in [9] Ishii constructed a space which is the closed image of a locally compact paracompact

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space, and hence belong to Π (see[18]), but is not σ -locally compact. In view of these facts it is natural to ask for what spaces Y the theorem of Morita above is true, especially for Lašnev spaces. In this paper we will show it is true for Lašnev spaces; that is, we establish a theorem that a Lašnev space Y belongs to Π iff it is σ -locally compact.

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2. PROOF OF THE MAIN THEOREM

All spaces considered in this paper are assumed to be Hausdorff and closed maps continuous onto. N denotes the set of positive integers.

The following lemma is fundamental in our discussions.

Lemma 2.1. *Let Y be a Lašnev space. Then there exist a metric space T and a one-to-one onto continuous map $f : Y \rightarrow T$ such that $T - f(D_0)$ is σ -discrete (= a countable union of closed discrete subsets of T), where*

$$D_0 = \{y \in Y \mid \{f^{-1}(G) \mid G \text{ is an open nbd of } f(y)\} \text{ is a nbd base of } y \text{ in } Y\}.$$

Proof. This is essentially proved in Arhangel'skii [1]. Indeed, let Z be a metric space and $g : Z \rightarrow Y$ a closed map. Then by Lašnev's decomposition theorem (Lašnev [11]) $Y = Y_0 \cup \cup_{i \in N} Y_i$, where for each $y \in Y_0$ $g^{-1}(y)$ is compact, each Y_i , $i \in N$ is closed discrete, and $Y_0 \cap \cup_{i \in N} Y_i = \emptyset$. Then in the proof of his theorem Arhangel'skii [1] constructed a metric space T and a one-to-one onto continuous map $f : Y \rightarrow T$ such that each point $y \in Y_0$ has $\{f^{-1}(G) \mid G \text{ is an open nbd of } f(y) \text{ in } T\}$ as its nbd base in the whole space Y and each $f(Y_i)$, $i \in N$ is closed discrete in T . Hence, if we define D_0 as above, we have $Y_0 \subset D_0$ and $T - f(D_0) \subset \cup_{i \in N} f(Y_i)$. Hence $T - f(D_0)$ is σ -discrete.

In what will follow whenever we mention a Lašnev space Y , T , $f : Y \rightarrow T$ and D_0 are defined as in Lemma 2.1, and let

$T_0 = f(D_0)$, $T - T_0 = \cup_{i \in N} T_i$, where T_i , $i \in N$ is closed discrete in T , and $D_i = f^{-1}(T_i)$, $i \in N$.

We shall prove some preliminary results. First we show

Theorem 2.2. *Let X be a countably paracompact normal space and Y a Lašnev space. If $X \times Y$ is normal, then $X \times T$ is normal.*

Proof. Let E and F be disjoint closed subsets of $X \times T$. Let $i \geq 1$ and let t be any point of T_i . Since $X \times \{t\}$ is normal, there exists an open set G_t of X such that

$$E \cap (X \times \{t\}) \subset G_t \times \{t\} \text{ and } (\overline{G_t} \times \{t\}) \cap F = \emptyset.$$

Then we have $((\overline{G_t} \times T) \cap F) \cap (X \times \{t\}) = \emptyset$. Since X is countably paracompact normal and t has a countable nbd base, it is easy to see that there exists an open set H_t of $X \times T$ such that

$$X \times \{t\} \subset H_t \text{ and } \overline{H_t} \cap ((\overline{G_t} \times T) \cap F) = \emptyset.$$

Since T_i is closed discrete and T is metrizable, there exists an open nbd W_t of t for each $t \in T_i$ such that $\{W_t \mid t \in T_i\}$ is discrete. Let

$$U_i = \cup \{(G_t \times Y) \cap H_t \cap (X \times W_t) \mid t \in T_i\}$$

Then U_i is an open set of $X \times T$, $E \cap (X \times T_i) \subset U_i$ and $\overline{U_i} \cap F = \emptyset$. Let $E_0 = E - \cup_{i \in N} U_i$. Then E_0 is closed and contained in $X \times T_0$. Let $1_X \times f : X \times Y \rightarrow X \times T$ be the product map, where 1_X is the identity on X . Since $X \times Y$ is normal, there exists an open set G of $X \times Y$ such that

$$(1_X \times f)^{-1}(E_0) \subset G \text{ and } \overline{G} \cap (1_X \times f)^{-1}(F) = \emptyset.$$

Now let $\mathcal{B} = \cup_{i \in N} \mathcal{B}_i$ be a σ -locally finite base for T , where \mathcal{B}_i is locally finite. For each $B \in \mathcal{B}_i$ define an open set G_B of X by

$$G_B = \cup \{P \mid P \text{ is open in } X \text{ such that } P \times f^{-1}(\overline{B}) \subset G\}.$$

Since $(1_X \times f)^{-1}(E_0) \subset X \times D_0$, and by the definition of D_0 , we have

$$(1_X \times f)^{-1}(E_0) \subset \cup\{G_B \times f^{-1}(\overline{B}) \mid B \in \mathcal{B}\}.$$

Moreover, since $G_B \times f^{-1}(\overline{B}) \subset G$ and $\overline{G} \cap (1_X \times f)^{-1}(F) = \emptyset$, we have

$$(\overline{G}_B \times \overline{B}) \cap F = \emptyset.$$

Let $V_i = \cup\{G_B \times B \mid B \in \mathcal{B}_i\}$. Then $\overline{V}_i \cap F = \emptyset$ since \mathcal{B}_i is locally finite. Since we have $E_0 \subset \cup_{i \in N} V_i$, the above shows that U_i and V_i , $i \in N$ are open sets of $X \times T$ which satisfy

$$E \subset \cup_{i \in N} U_i \cup \cup_{i \in N} V_i, \overline{U}_i \cap F = \overline{V}_i \cap F = \emptyset, i \in N.$$

Similarly we can find open sets U'_i and V'_i of $X \times T$ so that $F \cap (X \times T_i) \subset U'_i$, $\overline{U}'_i \cap E = \emptyset$ for each $i \in N$ and $F - \cup_{i \in N} U'_i \subset \cup_{i \in N} V'_i$, $\overline{V}'_i \cap E = \emptyset$ for each $i \in N$. That is,

$$F - \cup_{i \in N} U'_i \cup \cup_{i \in N} V'_i, \overline{U}'_i \cap E = \overline{V}'_i \cap E = \emptyset, i \in N.$$

This shows, as is well-known, E and F are separated by open sets. Hence $X \times T$ is normal

Remark. The converse of this theorem does not hold. For let X be the Michael line. Let Y be the space obtained from the real line R by making each rational number isolated (Michael [13]). Then Y is metrizable. Clearly the identity map $f : Y \rightarrow R$ satisfies the properties in Lemma 2.1. $X \times R$ is paracompact, but $X \times Y$ is not normal.

Corollary 2.3. *Let Y be a Lašnev space. If Y belongs to Π , then T is σ -locally compact.*

Proof. Let X be a paracompact space. Then $X \times Y$ is normal. Hence by Theorem 2.2 $X \times T$ is normal. Therefore, by Rudin-Starbird [17] and Morita [14] $X \times T$ is paracompact. That is, T belongs to Π . Hence by Morita [14], T is σ -locally compact.

Let $\beta f : \beta Y \rightarrow \beta T$ be the Stone extension of $f : Y \rightarrow T$.

Lemma 2.4. $\beta f^{-1}(f(y)) \cap Y = \{y\}$ for any $y \in \cup_{i \in N} D_i$. $\beta f^{-1}(f(y)) = \{y\}$ for any $y \in D_0$. Hence $\beta f^{-1}(T) = \cup_{i \in N} \beta f^{-1}(T_i) \cup D_0$.

Proof. Since f is one-to-one, the first statement is obvious. To prove the second, let $y \in D_0$. Suppose there is $y' \in \beta f^{-1}(f(y))$ with $y' \neq y$. Since $y \in D_0$, there exist an open nbd G of $f(y)$ and a nbd H of y' in βY such that $f^{-1}(G) \cap H = \emptyset$. Since $\beta f(y') = f(y)$, we may assume further $\beta f(H) \cap T \subset G$. Take $y'' \in H \cap Y$. Then $y'' \in f^{-1}(G) \cap H$, a contradiction.

Lemma 2.5. For each $y \in \cup_{i \in N} D_i$ there exists an open nbd V_y of y such that $f^{-1}(G) - \overline{V}_y \neq \emptyset$ for any open nbd G of $f(y)$ in T .

Proof. The lemma follows since Y is regular and $y \notin D_0$.

Let $y \in \cup_{i \in N} D_i$ and V_y be as in Lemma 2.5. Define

$$C_y = \beta f^{-1}(f(y)) - V_y^*,$$

where $V_y^* = \beta Y - \overline{Y - V}_y$. Then using Lemma 2.5 and the fact that βf is a closed map, it is easy to see that C_y is non-empty. Pick a point $p_y \in C_y$ for each $y \in \cup_{i \in N} D_i$. Define a subspace Y_c of βY by

$$Y_c = \{p_y \mid y \in \cup_{i \in N} D_i\} \cup D_0.$$

We shall prove the following lemma.

Lemma 2.6. Y_c is a hereditarily paracompact space.

Proof. We shall only prove Y_c is paracompact. The heredity can be proved similarly. Clearly Y_c is a regular space. Let $g = \beta f \mid Y_c : Y_c \rightarrow T$. Let \mathcal{B} be a σ -locally finite base of T . Let \mathcal{U} be an open cover of Y_c . Define $\mathcal{V}_0 = \{g^{-1}(B) \mid B \in \mathcal{B}, g^{-1}(B) \text{ is contained in some member of } \mathcal{U}\}$.

\mathcal{V}_0 is a σ -locally finite collection of open sets of Y_c and it is easy to see $\cup \mathcal{V}_0 \supset D_0$. Let $i \in N$. Since $f(D_i) = T_i$ and T_i is closed discrete in T , there exists an open nbd $W_{f(y)}$ of $f(y)$ for each $y \in D_i$ such that $\{W_{f(y)} \mid y \in D_i\}$ is discrete in T .

Since $g(p_y) = f(y)$, we can take an open nbd V_{p_y} of p_y in Y_c such that $g(V_{p_y}) \subset W_{f(y)}$ and V_{p_y} contained in some member of \mathcal{U} . Define $\mathcal{V}_i = \{V_{p_y} \mid y \in D_i\}$. Now the above show $\cup_{0 \leq i} \mathcal{V}_i$ is a σ -locally finite open cover of Y_c which refines \mathcal{U} . Hence Y_c is paracompact.

Remark. Y_c is, in addition, a σ -space. For,

$$\{\{p_y\} \mid y \in D_i, i \in N\} \cup \{g^{-1}(B) \mid B \in \mathcal{B}\}$$

is a σ -locally finite net for Y_c .

Let us define a space M_Y . Let M_Y be the space obtained from the set Y_c with the topology generated by $\{U \mid U \text{ open in } Y_c \text{ or } U \subset D_0\}$, hence points in D_0 are isolated.

Since by Lemma 2.6 Y_c is hereditarily paracompact, M_Y is also hereditarily paracompact (see [5]).

Modifying Michael's proof that Michael line has a non-normal product with the irrationals [13], we show:

Lemma 2.7. *Let Y be a Lašnev space. Assume that the metric space T is Čech-complete. If Y belongs to Π then either D_0 or $\cup_{i \in N} D_i$ is not dense in Y .*

Proof. Suppose D_0 is dense in Y . We show $\cup_{i \in N} D_i$ is not dense in Y . Let Y_c and M_Y be spaces defined above. Let $E = \{p_y \mid y \in \cup_{i \in N} D_i\}$. Consider the product $M_Y \times Y$. Define

$$A = E \times Y \text{ and } B = \{(y, y) \mid y \in D_0\}.$$

Then $A \cap B = \emptyset$. Clearly A is closed in $M_Y \times Y$. Since $M_Y \cap Y = D_0$, B is closed in $M_Y \times Y$. Since $M_Y \times Y$ is normal, there exist open sets K and L in $M_Y \times Y$ such that

$$A \subset K, B \subset L \text{ and } K \cap L = \emptyset.$$

Consider the subspace $M_Y \times D_0$, and let

$$A' = E \times D_0, K' = K \cap (M_Y \times D_0), L' = L \cap (M_Y \times D_0).$$

Then $A' \subset K', B \subset L'$ and $K' \cap L' = \emptyset$.

Let d be a metric on T which defines the topology of T . Let $g = \beta f \mid Y_c : Y_c \rightarrow T$. Since D_0 is a subspace of Y_c , by

Lemma 2.4 for each $y \in D_0$ there exist an $n \in N$, such that $\{y\} \times (g^{-1}(B(g(y), \frac{1}{n})) \cap D_0) \subset L'$, where $B(*, \epsilon)$ is the usual ϵ -ball. For $n \in N$, define

$$P_n = \{y \in D_0 \mid \{y\} \times (g^{-1}(B(g(y), \frac{1}{n})) \cap D_0) \subset L'\}.$$

Then $D_0 = \cup_{n \in N} P_n$. We shall show $D_0 = \cup_{n \in N} \overline{P_n}^{Y_c}$, where $\overline{P_n}^{Y_c}$ is the closure of P_n in Y_c . Suppose not. Then for some $n \in N$, $E \cap \overline{P_n}^{Y_c}$ contains a point p_y . Take $0 < \epsilon < \frac{1}{2^n}$ and choose $y' \in g^{-1}(B(g(p_y), \epsilon)) \cap P_n$. Since $(p_y, y') \in A' \subset K'$, there exists a nbd U of p_y in M_Y such that $U \times \{y'\} \subset K$. Since U is a nbd of p_y also in Y_c , we may assume $U \subset g^{-1}(B(g(p_y), \epsilon))$. Since $p_y \in \overline{P_n}^{Y_c}$, $U \cap P_n \neq \emptyset$. Choose $y'' \in U \cap P_n$. Now the above shows that $d(g(y''), g(y')) < \frac{1}{n}$ and $\{y''\} \times (g^{-1}(B(g(y''), \frac{1}{n})) \cap D_0) \subset L'$. Hence $(y'', y') \in L'$, whereas $(y'', y') \in U \times \{y'\} \subset K'$, which contradicts $K' \cap L' = \emptyset$. Thus $D_0 = \cup_{n \in N} \overline{P_n}^{Y_c}$. Observe that D_0 is Čech-complete since D_0 is homeomorphic to $f(D_0) = T_0$ and $T_0 = T - \cup_{i \in N} T_i$ is a G_δ -subset of Čech-complete T . And note that D_0 is dense in Y_c since D_0 is dense in Y . Thus, by Baire's theorem for some $n \in N$ we have

$$\emptyset \neq \text{Int}_{Y_c} \overline{P_n}^{Y_c} \subset \overline{P_n}^{Y_c} \subset D_0.$$

This shows E is not dense in Y_c . By definition of D_0 this implies that $\cup_{i \in N} D_i$ is not dense in Y .

We shall now prove our main theorem.

Theorem 2.8. *Let Y be a Lašnev space. Then $X \times Y$ is paracompact for any paracompact space X iff Y is σ -locally compact.*

Proof. We only prove the “only if” part. Assume that $X \times Y$ is paracompact for any paracompact space X , that is, Y belongs to Π . Then, by Corollary 2.3 T is σ -locally compact, or equivalently, T has a σ -locally finite cover $\{C_\lambda\}$ of compact subsets. Let $Y_\lambda = f^{-1}(C_\lambda)$ and $f_\lambda = f \mid Y_\lambda : Y_\lambda \rightarrow C_\lambda$. Then Y_λ is a Lašnev space and closed in Y , and hence $Y_\lambda \in \Pi$. Therefore, if we could prove that Y_λ is σ -locally compact, or equivalently, Y_λ has a σ -locally finite cover of compact subsets,

then, since $\{Y_\lambda\}$ is a σ -locally finite cover of closed subsets, Y would have a σ -locally finite cover of compact subsets, this is Y would be σ -locally compact. Note that Y_λ, C_λ and $f_\lambda : Y_\lambda \rightarrow C_\lambda$ have the same properties as those for $Y, T, f : Y \rightarrow T$ described in Lemma 2.1. Thus, without loss of generality, we may and shall assume that T is compact.

First we note that Y is hereditarily Lindelöf. For, D_0 is homeomorphic to T_0 which is separable metrizable, and each $T_i (i \geq 1)$ is a finite set since it is closed discrete in T .

For a subspace A of Y we define

$$A_0 = \{y \in A \mid \{f^{-1}(G) \cap A \mid G \text{ is an open nbd of } f(y)\} \\ \text{is a nbd base of } y \text{ in } A\},$$

and $A_1 = A - A_0$. Hence Y_0 and Y_1 are D_0 and $\cup_{i \in \mathbb{N}} D_i$ used above, respectively. Note that $Y_0 \cap A \subset A_0$ and $|A_1| \leq \omega$.

We shall prove a claim

Claim. *Let F be a closed subset of Y . Then $f(\overline{F_0} \cap \overline{F_1})$ is G_σ in $f(F)$.*

Proof of Claim. Observe that $f(\overline{F_1})$ is closed in $f(F)$, and it is G_δ in $f(F)$. Since $f(F) - f(\overline{F_0}) = f(F - \overline{F_0}) \subset f(F_1)$ which is countable, $f(\overline{F_0})$ is G_δ in $f(F)$. Hence $f(\overline{F_0} \cap \overline{F_1}) = f(\overline{F_0}) \cap f(\overline{F_1})$ is G_δ in $f(F)$.

Now we define $Y^{(\alpha)}$ for each ordinal number α . Define $Y^{(0)} = Y$. Assume that $Y^{(\beta)}$ has been defined for $\beta < \alpha$. We define $Y^{(\alpha)}$: if α is a successor, say $\alpha = \beta + 1$, define

$$Y^{(\alpha)} = \overline{Y_0^{(\beta)}} \cap \overline{Y_1^{(\beta)}},$$

and if α is a limit, define $Y^{(\alpha)} = \bigcap_{\beta < \alpha} Y^{(\beta)}$.

Then $Y^{(\alpha)}$ is closed in Y and $\alpha < \beta$ implies $Y^{(\alpha)} \supset Y^{(\beta)}$. Therefore, since Y is hereditarily Lindelöf, it is easy to see that for some $\alpha^* < \omega_1$, $Y^{(\alpha^*)} = Y^{(\omega_1)}$.

We show $Y^{(\alpha^*)} = \emptyset$. Suppose $Y^{(\alpha^*)} \neq \emptyset$. Since $\alpha^* < \omega_1$, by Claim $f(Y^{(\alpha^*)})$ is G_δ in T . Hence $f(Y^{(\alpha^*)})$ is Čech-complete.

Since $Y^{(\alpha^*)}$ is closed and it belongs to Π , by Lemma 2.7 we must have

$$Y^{(\alpha^*)} \neq \overline{Y_0^{(\alpha^*)}} \cap \overline{Y_1^{(\alpha^*)}} = Y^{(\alpha^*+1)},$$

which is a contradiction. Thus, $Y^{(\alpha^*)} = \emptyset$. Therefore $Y = \bigcup_{\alpha < \alpha^*} (Y^{(\alpha)} - Y^{(\alpha+1)})$. Since $Y^{(\alpha)} - \overline{Y_1^{(\alpha)}} \subset Y_0^{(\alpha)}$ and $Y_0^{(\alpha)}$ is homeomorphic to $f(Y_0^{(\alpha)})$, $Y^{(\alpha)} - \overline{Y_1^{(\alpha)}}$ is a countable union of closed metrizable subspaces of $Y^{(\alpha)}$. So the same holds for $Y^{(\alpha)} - Y^{(\alpha+1)} = (Y^{(\alpha)} - \overline{Y_0^{(\alpha)}}) \cup (Y^{(\alpha)} - \overline{Y_1^{(\alpha)}})$ since $Y^{(\alpha)} - \overline{Y_0^{(\alpha)}} \subset Y_1^{(\alpha)}$. Therefore, since each $Y^{(\alpha)}$ is closed, Y is also a countable union of closed metrizable subspaces. Let $Y = \bigcup_{n \in \mathbb{N}} E_n$, where E_n is closed metrizable. Then $E_n \in \Pi$, and hence E_n is σ -locally compact. Thus, Y is σ -locally compact. This completes the proof of Theorem 2.8.

There have been defined several notions of generalized metric spaces (see, Gruenhage ([6]). The author does not know whether Theorem 2.8 is true for the classes of paracompact σ -spaces, stratifiable spaces or M_1 -spaces. If a space Y is perfectly normal paracompact and admits T and $f : Y \rightarrow T$ described in Lemma 2.1, then the same proof shows that Theorem 2.8 holds for Y . But easy examples show that Lemma 2.1 does not hold even for M_1 -spaces.

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