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by


SHOU LIN



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**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
**E-mail:** topolog@auburn.edu
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# A DECOMPOSITION THEOREM FOR $\Sigma^*$-SPACES

SHOU LIN*

ABSTRACT. In this note it is shown that if $f$ is a continuous closed mapping from a $T_1, \Sigma^*$-space $X$ onto $Y$, then there is a $\sigma$-closed discrete subspace $Z$ of $Y$ such that $f^{-1}(y)$ is an $\omega_1$−compact subspace of $X$ for each $y \in Y \backslash Z$.

We assume that all spaces are $T_1$, and all mappings are continuous and onto.

In 1985, Y. Tanaka and Y. Yajima [4] obtained a decomposition theorem for $\Sigma$-spaces that every $\Sigma$-space $X$ satisfies the following condition($*$).

($*$) If $f$ is a closed mapping form $X$ onto $Y$, then there is a $\sigma$-closed discrete subspace $Z$ of $Y$ such that $f^{-1}(y)$ is an $\omega_1$-compact subspace of $X$ for each $y \in Y \backslash Z$.

J. Chaber [1] constructed a counterexample to show that $\Sigma^{\#}$-spaces are not always satisfying the above condition ($*$). Since $\Sigma$-spaces are $\Sigma^*$-spaces, and $\Sigma^*$-spaces are $\Sigma^{\#}$-spaces, it is a natural question whether $\Sigma^*$-spaces satisfy($*$). Y. Tanaka and Y. Yajima [4] obtained only a weak form of decomposition theorem for $\Sigma^*$-spaces. The purpose of this note is to prove that $\Sigma^*$-spaces satisfy($*$).

Recall basic definitions concerning $\Sigma^*$-spaces. Suppose that $\mathcal{P}$ is a collection of subsets of a space $X$. $\mathcal{P}$ is called hereditarily closure-preserving(abbre. HCP) if $\{H(P); P \in \mathcal{P}\}$ is closure-preserving for every subset $H(P) \subset P \in \mathcal{P}$. A space $X$ is called a $\Sigma^*$-space (or, strong $\Sigma^*$-space) [3] if there is a covering $\mathcal{K}$ of $X$ by closed countable compact subsets (or, closed compact

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subsets) and a $\sigma$-HCP collection $\mathcal{P}$ of closed subsets of $X$ such that whenever $K \subset U$ with $K \in \mathcal{K}$ and $U$ open in $X$, then $K \subset P \subset U$ for some $P \in \mathcal{P}$. The $\mathcal{P}$ is called a $\sigma$-HCP closed (mod $\mathcal{K}$)-network for $X$.

**Lemma (4, Lemma 1.1)** *If $\mathcal{P}$ is an HCP collection of subsets of $X$, then*

$$\{P_1 \cap P_2 \ldots \cap P_n; P_i \in \mathcal{P}, i \leq n\}$$

*is also HCP in $X$ for each $N \in N$.*

**Theorem** *$\Sigma^*$-spaces satisfy (*).*

*Proof.* Suppose that $f$ is a closed mapping from a $\Sigma^*$-space $X$ onto $Y$. Let $\mathcal{K}$ be a covering of $X$ by closed countable compact subsets, and let $\mathcal{P} = \cup\{\mathcal{P}_n : n \in N\}$ be a $\sigma$-HCP closed (mod $\mathcal{K}$)-network for $X$, where each $\mathcal{P}_n$ is HCP in $X$. Here we can assume that $\mathcal{P}$ is closed under finite intersections by Lemma, and $X \in \mathcal{P}_n \subset \mathcal{P}_{n+1}$. For each $n \in N$, put

$$D_n = \{x \in X : \mathcal{P}_n \text{ is not point-finite at } x\},$$

then

(1) $D_n$ is a $\sigma$-closed discrete subspace of $X$. In fact , for each $m \in N$, put

$$E_m = \{x \in X : \cap\{P \in \mathcal{P}_m : x \in P\} = \{x\}\},$$

then $E_m$ is a closed discrete subspace of $X$ by [3, Lemma 2.5]. It is not difficult to check that $D_n \subset \cup\{E_m : m \in N\}$ (cf.[2]), hence $D_n$ is $\sigma$-closed discrete in $X$. Put

$$\mathcal{Q}_n = \{P \backslash D_n : P \in \mathcal{P}_n\},$$

and

$$\mathcal{Q} = \cup\{\mathcal{Q}_n : n \in N\},$$

then

(2) There are $m \geq n, F \in \mathcal{Q}_m$, and $G \subset D_m$ with $\cap\mathcal{F} = F \cup G$ for each finite $\mathcal{F} \subset \mathcal{Q}$ and $n \in N$.

In fact, let $\mathcal{F} = \{F_i : i \leq k\} \subset \mathcal{Q}$, we might as well grant $\cap\mathcal{F} \neq \emptyset$, and there are $n_i \in N, P_i \in \mathcal{P}_{n_i}$ with $F_i = P_i \backslash D_{n_i}$ and $n_i \leq n_{i+1}$, then $\cap\mathcal{F} = \cap\{P_i : i \leq k\} \backslash D_{n_k}$, and $\cap\{P_i : i \leq$

$k\} = P$ for some $m \geq \max\{n_k, n\}, P \in \mathcal{P}_m$ because $\mathcal{P}$ is closed under finite intersections. Put

$F = P \backslash D_m$, and

$G = P \cap (D_m \backslash D_{n_k})$,

then $F \in \mathcal{Q}_m, G \subset D_m$ and $\cap \mathcal{F} = F \cup G$.

For each $n \in N$, put

$$Z_n = f(D_n) \cup ( \quad \cup\{f(Q) \cap f(Q') : Q, Q' \in \mathcal{Q}_n, \text{ and } F(Q) \cap f(Q') \text{ is finite}\}).$$

Since $\mathcal{Q}_n$ is HCP in $X, \{f(Q) : Q \in \mathcal{Q}_n\}$ is HCP in $Y$. Thus $Z_n$ is $\sigma$-closed discrete in $Y$ by (1) and Lemma. Put

$$Z = \cup\{Z_n : n \in N\},$$

then $Z$ is $\sigma$-closed discrete in $Y$. Take a $y \in Y \backslash Z$, then

(3) $\{Q \in \mathcal{Q}_n : Q \cap f^{-1}(y) \neq \emptyset\}$ is finite.

Assume the contrary, then there is an $m \in N$ and a sequence $\{Q_n\}$ of distinct members of $\mathcal{Q}_m$ such that $Q_n \cap f^{-1}(y) \neq \emptyset$. Pick an $x \in f^{-1}(y)$, then $x \notin X \backslash D_n$ for each $n \in N$ , put

$$R_n = \cap\{Q \in \mathcal{Q}_n : x \in Q\}.$$

Since $\mathcal{Q}_n$ is point-finite on $X$ there are a $k_n \in N, F_n \in \mathcal{Q}_{k_n}$ and $G_n \subset D_{k_n}$ with $m \leq k_n < k_{n+1}$ and $R_n = F_n \cup G_n$ by (2). Put

$F'_n = Q_n \backslash D_{k_n}$,

$G'_n = Q_n \cap D_{k_n}$,

then $F'_n \in \mathcal{Q}_{k_n}, G'_n \subset D_{k_n}$, and $Q_n = F'_n \cup G'_n$. Since $y \in f(R_n) \cap f(Q_n) \backslash Z = f(F_n) \cap f(F'_n) \backslash Z, f(F_n) \cap f(F'_n)$ is an infinite set. So we can choose a sequence $\{y_n\}$ of distinct points in $Y$ such that $y_n \in f(F_n) \cap f(F'_n)$. Pick $p_n \in F_n \cap f^{-1}(y_n)$, and $q_n \in F'_n \cap f^{-1}(y_n)$.

Suppose that the sequence $\{p_n\}$ has not any cluster point in $X$. Take a $K \in \mathcal{K}$ with $x \in K$, then there is an $i \in N$ such that $K \cap \{p_n : n \geq i\} = \emptyset$, thus $x \in K \subset P \subset X \backslash \{p_n : n \geq i\}$ for some $j \geq i, P \in \mathcal{P}_j$. Since $y \in Y \backslash Z, x \notin D_j$, then $x \in P \backslash D_j \in \mathcal{Q}_j$, hence $R_j \subset P \backslash D_j$, so $p_j \in F_j \subset R_j \subset P \backslash D_j \subset X \backslash \{p_n : n \geq i\}$, a contradiction. Consequently, the sequence $\{p_n\}$ has a cluster point in $X$, and the sequence $\{y_n\}$ also has a cluster point in $Y$. On the other hand, since $q_n \in F'_n \subset Q_n \in \mathcal{Q}_m$ for each $n \in N$, the sequence $\{q_n\}$ has not any cluster point in $X$ so the sequence $\{y_n\}$ has not any cluster point in $Y$ either, a contradiction. (3) holds. By

$$y \in Y \backslash Z, f^{-1}(y) \subset X \backslash \cup \{D_n : n \in N\},$$

thus

$$\{f^{-1}(y) \cap P : P \in \mathcal{P}\} = \{f^{-1}(y) \cap Q : Q \in \mathcal{Q}\},$$

therefore it is a countable closed (mod $\mathcal{K}'$)-network from (3), where $\mathcal{K}' = \mathcal{K}_{|f^{-1}(y)}$. Hence $f^{-1}(y)$ is an $\omega_1$-compact subspace of $X$ by [4, Lemma 1.2].

**Corollary** *If $f$ is a closed mapping from a strong $\Sigma^*$-space $X$ onto $Y$, then there is a $\sigma$-closed discrete subspace $Z$ of $Y$ such that $f^{-1}(y)$ is Lindelöf for each $y \in Y \backslash Z$.*

*Proof.* An $\omega_1$-compact strong $\Sigma^*$-space is a space with a countable (mod $\mathcal{K}$)-network with respect to $\mathcal{K}$ by compact subsets. A space with a countable (mod $\mathcal{K}$)-network with respect to $\mathcal{K}$ by compact subsets is Lindelöf.

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Ningde Teachers' College
Ningde, Fujian 352100
P.R. China