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CHAIN CONDITIONS AND CALIBRES IN MOORE SPACES

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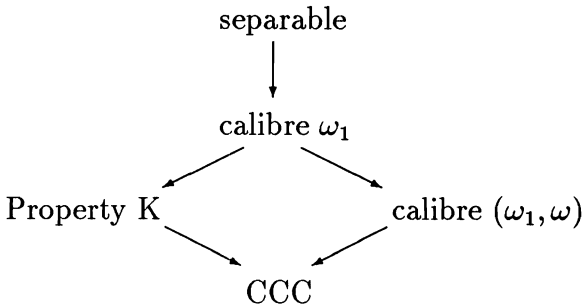
ABSTRACT. The relative strengths of various chain conditions between separability and the countable chain condition (CCC) are considered in the context of Moore spaces. It is shown that the relative strengths are the same as for arbitrary first countable T_3 spaces. In particular, we can deduce from work of F. D. Tall that it is consistent and independent that calibre ω_1 is equivalent to separability for Moore spaces. The space considered in that work is shown to yield an absolute example of a non-separable Moore space with calibre (ω_1, ω). A Moore space due to C. Pixley and P. Roy is shown to have Property K, and hence the CCC, but to lack calibre (ω_1, ω). It is shown to be consistent and independent that either CCC or calibre (ω_1, ω) implies Property K for Moore spaces.

The relationships between separability, the countable chain condition (CCC) and the discrete countable chain condition (DCCC) in Moore spaces have been investigated in [3], [4], [9], [10], [11] and [12]. In this paper, we extend this work to include certain chain conditions, or calibres, which lie between separability and the CCC. This investigation was inspired by a seminar on the comparison of calibres and the CCC in non-first countable spaces given by W. S. Watson at Oxford University in 1987.

A topological space is said to have calibre ω_1 (resp. calibre (ω_1, ω)) if, for every family $\{ U_\alpha : \alpha \in \omega_1 \}$ of non-empty open sets, there is an uncountable (resp. infinite) subset Λ of ω_1 such that $\bigcap \{ U_\alpha : \alpha \in \Lambda \} \neq \emptyset$. It is said to have Property K if, for every family $\{ U_\alpha : \alpha \in \omega_1 \}$ of non-empty open sets, there is an uncountable subset Λ of ω_1 such that, for $\alpha, \beta \in \Lambda$, $U_\alpha \cap U_\beta \neq \emptyset$. Clearly the following implications hold:



The five properties are all equivalent for metric spaces, and for completable Moore spaces [4]. This naturally raises the question of whether any of the above implications is reversible for Moore spaces, and what the relationship between calibre (ω_1, ω) and Property K is for Moore spaces. The first observation is that it is sufficient to study first countable T_3 spaces. This is a result of the following construction, due to G. M. Reed [8,9].

Let $(X, \mathcal{T})$ be a first countable T_3 space, and for each $x \in X$ choose a basis $\{ B_n(x) : n \in \omega \}$ for $\mathcal{T}$ at x such that for each $n \in \omega$, $\overline{B_{n+1}(x)} \subseteq B_n(x)$. For $n \in \omega$, let ${}^n\omega$ denote the set of functions from n to ω, and let ${}^{\omega}\omega = \bigcup_{n \in \omega} {}^n\omega$. If $f \in {}^n\omega$, $g \in {}^m\omega$, define $f^\frown g : n + m \to \omega$ by

$$f^\frown g(t) = \begin{cases} f(t) & \text{if } 0 \leq t < n, \\ g(t-n) & \text{if } n \leq t < n+m. \end{cases}$$

Let $M = X \times {}^{<\omega}\omega$. For each $p = \langle x, f \rangle \in M$ and $n \in \omega$ define

$$G_n(p) = \{p\} \cup \{ \langle y, f^\frown g \rangle \in M :$$
$$(g \neq \emptyset) \wedge (\text{ ran } g \subseteq \omega \setminus n) \wedge (y \in B_{n+g(0)}(x)) \}.$$

Let $\mathcal{S}$ be the topology on M generated by $\{ G_n(p) : n \in \omega, p \in M \}$. Then $(M, \mathcal{S})$ is a Moore space, and has any one of the five conditions given above if and only if $(X, \mathcal{T})$ has the corresponding property. This correspondence for separability, CCC and DCCC was noted in [8]. As an example, we show that CCC is preserved—the other proofs are similar.

Suppose $(X, \mathcal{T})$ has CCC, and $\{ U_\alpha : \alpha \in \omega_1 \}$ is a family of non-empty open sets in M. For each $\alpha \in \omega_1$, we can choose $x_\alpha \in X$, $f_\alpha \in {}^{<\omega}\omega$ and $n_\alpha \in \omega$ such that $G_{n_\alpha}(\langle x_\alpha, f_\alpha \rangle) \subseteq U_\alpha$. There exist $f \in {}^{<\omega}\omega$, $n \in \omega$, and $\Gamma \subseteq \omega_1$ with Γ uncountable, such that for each $\alpha \in \Gamma$, $n_\alpha = n$ and $f_\alpha = f$. Since X has CCC, there exists a point $y \in X$ and $\alpha, \beta \in \omega_1$ with $\alpha \neq \beta$ such that $y \in B_{2n}(x_\alpha) \cap B_{2n}(x_\beta)$. Then $\langle y, g \rangle \in U_\alpha \cap U_\beta$, where $g = f^\frown\{\langle 0, n \rangle\}$

Conversely, suppose $(M, \mathcal{S})$ has CCC, and $\{ V_\alpha : \alpha \in \omega_1 \}$ is a family of non-empty open sets in X. For $\alpha \in \omega_1$, choose $x_\alpha \in X$, $n_\alpha \in \omega$ such that $B_{n_\alpha}(x_\alpha) \subseteq V_\alpha$. Since M has CCC, for some $y \in X$, $f \in {}^{<\omega}\omega$, and $\alpha, \beta \in \omega_1$ with $\alpha \neq \beta$, $\langle y, f \rangle \in G_{n_\alpha}(\langle x_\alpha, \emptyset \rangle) \cap G_{n_\beta}(\langle x_\beta, \emptyset \rangle)$. But then, in particular, $y \in V_\alpha \cap V_\beta$.

In general, if $(X, \mathcal{T})$ is a first countable T_3 space, we will call the space $(M, \mathcal{S})$ constructed above the *Reed space over* $(X, \mathcal{T})$.

As a result of the above, we can extend a result of F. D. Tall in [11]:

Theorem 1. *It is consistent and independent that calibre ω_1 implies separability for Moore spaces.*

Specifically, under CH every first countable T_2 space with calibre ω_1 is separable whereas under MA+$\neg$CH there is a non-separable first countable T_3 space with calibre ω_1, and hence there is also a non-separable Moore space with calibre ω_1.

Theorem 2. *There is a non-separable Moore space with calibre (ω_1, ω).*

The space given in [11] is in fact an absolute example of a first countable T_3 space with calibre (ω_1, ω) which is not separable. Let X be the set of all non-empty compact nowhere dense subsets of the real line. Let $\mathcal{T}$ be the Pixley-Roy topology on X, that is, that topology generated by all sets of the form

$$[a, U] = \{ b \in X : a \subseteq b \subseteq U \},$$

where $a \in X$ and U is an open subset of $\mathbb{R}$ with $a \subseteq U$ (see [1]). It is shown in [2] that $(X, \mathcal{T})$ is a non-separable first countable zero-dimensional T_1 space with the CCC, and that there is a sequence $\{ \mathcal{B}_n : n \in \omega \}$ of families of (closed) subsets of X such that

(1) if $x \in U \in \mathcal{T}$ and $n \in \omega$ then for some $B \in \mathcal{B}_n$, $x \in \operatorname{int}(B)$ and $B \subseteq U$; and
(2) if $\mathcal{A} \subseteq \bigcup_{n\in\omega} \mathcal{B}_n$, $\mathcal{A} \cap \mathcal{B}_n \neq \emptyset$ for each $n \in \omega$, and $\mathcal{A}$ is centred, then $\bigcap \mathcal{A} \neq \emptyset$.

Let $\{ U_\alpha : \alpha \in \omega_1 \}$ be a family of non-empty open subsets of X. Let $\mathcal{C}$ be a maximal disjoint family of non-empty open subsets of X such that each $C \in \mathcal{C}$ meets U_α for only countably many $\alpha \in \omega_1$. Then $\mathcal{C}$ is countable, so for some $\alpha_0 \in \omega_1$, U_{α_0} does not meet any $C \in \mathcal{C}$. By maximality, this implies that any non-empty open subset of U_{α_0} meets U_β for uncountably many $\beta \in \omega_1$. Since $U_{\alpha_0} \neq \emptyset$, we can choose $x_0 \in U_{\alpha_0}$ and $B_0 \in \mathcal{B}_0$ with $x \in \operatorname{int}(B_0)$ and $B_0 \subseteq U_{\alpha_0}$. We can inductively choose sequences $(x_n : n \in \omega)$, $(B_n : n \in \omega)$ and $(\alpha_n : n \in \omega)$ such that, for each $n \in \omega$, $x_n \in X$, $B_n \in \mathcal{B}_n$, $\alpha_n \in \omega_1$, and

(1) $\alpha_n < \alpha_{n+1}$;
(2) $x_n \in \operatorname{int}(B_n)$; and
(3) $B_{n+1} \subseteq U_{\alpha_{n+1}} \cap \operatorname{int}(B_n)$.

Putting $\mathcal{A} = \{ B_n : n \in \omega \}$, we observe that $\mathcal{A}$ is a centred subset of $\bigcup_{n\in\omega} \mathcal{B}_n$ which meets each $\mathcal{B}_n$, and hence $\mathcal{A}$ has non-empty intersection. But then if $\Lambda = \{ \alpha_n : n \in \omega \}$ we see that if $x \in \bigcap \mathcal{A}$ then $x \in \bigcap_{\alpha\in\Lambda} U_\alpha$. Thus $(X, \mathcal{T})$ has calibre (ω_1, ω).

Theorem 3. *Calibre (ω_1, ω) is not implied by CCC, or even by Property K.*

To show this, we consider the space Y of all finite subsets of the real line, again with the Pixley-Roy topology $\mathcal{S}$ (see [7]). If $\mathcal{B}$ is a countable basis for the usual topology on $\mathbb{R}$, and $\mathcal{B}'$ is the set of finite unions of sets in $\mathcal{B}$, then $\mathcal{B}'$ is countable, and for each $a \in U \in \mathcal{S}$ there is a $B \in \mathcal{B}'$ such that $a \subseteq B$ and $[a, B] \subseteq U$. Let $\{ U_\alpha : \alpha \in \omega_1 \}$ be a family of non-empty open subsets of Y. For each $\alpha \in \omega_1$, choose $a_\alpha \in U_\alpha$ and $B_\alpha \in \mathcal{B}'$ such that $a_\alpha \subseteq B_\alpha$ and $[a_\alpha, B_\alpha] \subseteq U_\alpha$. There is some uncountable subset Λ of ω_1 and some $B \in \mathcal{B}'$ such that $B_\alpha = B$ for each $\alpha \in \Lambda$. Then if $\alpha, \beta \in \Lambda$, $a_\alpha \cup a_\beta \in U_\alpha \cap U_\beta$. Thus $(Y, \mathcal{S})$ has Property K (and the CCC). However, $(Y, \mathcal{S})$ does not have calibre (ω_1, ω), for $\{ [a, \mathbb{R}] : a \in Y \}$ is an uncountable point-finite family of non-empty open sets.

Lemma *Let $(X, \mathcal{T})$ be a hereditarily Lindelöf space. Then $(X, \mathcal{T})$ has calibre (ω_1, ω).*

Proof.[1] Let $\mathcal{U}$ be a point-finite collection of non-empty open sets in X. For each $n \in \omega$, let $F_n = \{x \in X : \operatorname{ord}(x, \mathcal{U}) = n + 1\}$, (where $\operatorname{ord}(x, \mathcal{U}) = |\{U \in \mathcal{U} : x \in U\}|$). Let

$$\mathcal{D}_n = \{(\bigcap \mathcal{W}) \cap F_n : \mathcal{W} \subseteq \mathcal{U}, |\mathcal{W}| = n + 1\}.$$

Then $\mathcal{D}_n$ is a disjoint open cover of the (Lindelöf) saubspace F_n of X, and hence countable. Each non-empty set in $\mathcal{D}_n$ is a subset of exactly $n + 1$ many sets in $\mathcal{U}$, and each set in $\mathcal{U}$ contains as a subset some non-empty set in $\mathcal{D}_n$ for some $n \in \omega$. It follows that $\mathcal{U}$ must be countable.

Theorem 4. *It is consistent and independent that either CCC or calibre (ω_1, ω) implies Property K for Moore spaces.*

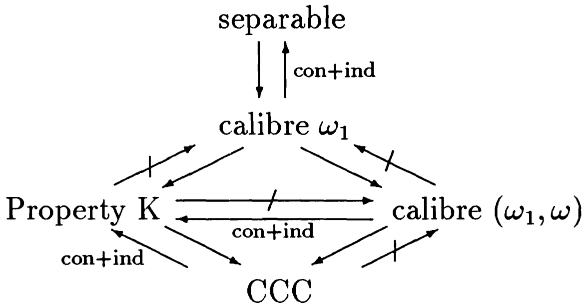
It is well-known (see, for example, [5, p61]) that under MA+¬CH, CCC implies Property K, and that the product of a Property K space with a CCC space has CCC.

A Souslin line is a regular first countable T_3 space with the CCC but without Property K [5,6]. In [3], E. K. van Douwen and G. M. Reed constructed a consistent example of a CCC Moore space whose square is not CCC, and hence a Moore space with CCC but not Property K, by building the Reed space over a Souslin line. By the above lemma, their space also has calibre (ω_1, ω), since any Souslin line is an L-space.

Theorems 1 to 4 together establish the relative strengths of each of the five conditions given, except that the example of a Moore space with calibre (ω_1, ω) but without calibre ω_1 is not an absolute example (it does not exist under MA+¬CH). However, G. M. Reed has recently outlined to the author a modification of the Reed space construction to produce an absolute example of such a space. This space is given in [8]. The

[1]This proof, suggested by the referee, replaces the author's own proof, which required the additional assumption that X is T_1 and first countable.

completed lattice of implications for Moore spaces is given below.



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