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ABSTRACT. Let 2^κ denote the "Cantor Cube" of weight κ (the Tychonoff product of κ many 2-point spaces), and let X denote the discrete union of 2^κ with a singleton space. We show that X^κ is homeomorphic to 2^κ, answering a question asked recently by Murray Bell.

Let κ be an infinite cardinal, and let 2^κ denote the product of κ many discrete 2-point spaces with the usual product topology. Let $p \notin 2^\kappa$, and let $X = 2^\kappa \oplus \{p\}$. If $\kappa = \omega$ then X^κ is a compact, 0-dimensional metric space with no isolated points, so X^κ is homeomorphic to the Cantor set. For $\kappa \geq \omega_1$, we will define a homeomorphism $h : X^k \to 2^k$, proving the assertion in the abstract.

We will treat members of product spaces as functions. Interval notation will be used in the obvious way for sets of ordinals. We will consider 0 to be a limit ordinal.

1. DESCRIPTION OF THE HOMEOMORPHISM.

Fix $\kappa \geq \omega_1$. Choose sets S_α for each $\alpha \in \kappa$ which satisfy all of the following:

(A1.) $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$.

(A2.) Each $|S_\alpha| = \kappa$.

(A3.) Each $S_\alpha \subset [\alpha + w, \kappa)$.

(A4.) $\bigcup_{\alpha<\kappa} S_\alpha$ contains no limit ordinals (including 0).

(A5.) For each limit $\gamma \in \kappa$, $[\gamma, \gamma + \omega) - \bigcup_{\alpha\in\kappa} S_\alpha$ is infinite.

It is easily verified that such S_α's can be chosen—start with a partition of the "odd" ordinals into κ sets each of size κ, then remove initial segments if necessary to get condition A3. For each $\alpha < \kappa$, fix a 1:1 correspondence $\eta_\alpha : \kappa \to S_\alpha$ between κ and S_α.

Now fix an $x \in X^\kappa$. So, for each α, either $x(\alpha) = p$ or $x(\alpha) \in 2^\kappa$. Let

$$\Gamma_x = \kappa - \bigcup\{S_\alpha : x(\alpha) \in 2^\kappa\}.$$

Note that $|\Gamma_x| = \kappa$. In fact, by A5, each $\Gamma_x \cap [\gamma, \gamma + \omega)$ is infinite. Let $\imath_x : \kappa \to \Gamma_x$ be the (unique) order preserving 1:1 correspondence from κ onto Γ_x. Now define the coordinates of $h(x) = y$ as follows. If $\alpha \in \Gamma_x$, let

$$y(\alpha) = \begin{cases} 0 & \text{if } x(\imath_x^{-1}(\alpha)) = p, \\ 1 & \text{if } x(\imath_x^{-1}(\alpha)) \in 2^\kappa. \end{cases}$$

If $\alpha \in \kappa - \Gamma_x$, then $\alpha \in S_\beta$ for some β with $x(\beta) \in 2^\kappa$. For each such α (if there are any), let $y(\alpha)$ be defined by

$$y(\alpha) = x(\beta)(\eta_\beta^{-1}(\alpha)) \text{ where } \alpha \in S_\beta.$$

It is useful to think of the coordinates of y as coding information about x. For each α, $\imath_x(\alpha)$ indicates which coordinate of y will be used to code whether or not $x(\alpha)$ is equal to p. For each α with $x(\alpha) \in 2^\kappa$, the function η_α is used to distribute $x(\alpha)$ onto the coordinates of y which are in S_α. The values of $y = h(x)$ are completely specified by the above two equations, so we have described a well defined function $h : X^\kappa \to 2^\kappa$.

2. Proof that h is a homeomorphism.

Lemma 1. *Fix $x \in X^\kappa$. If γ is a limit ordinal, then*
(a) $\imath_x(\gamma) = \gamma$,
(b) $\imath_x(\gamma + n) \in [\gamma + n, \gamma + \omega)$ for each $n \in \omega$.

Proof. Recall that $\imath_x : \kappa \to \Gamma_x$ is the unique order preserving 1:1 correspondence. This means that for each α,

$$\imath_x(\alpha) = \min\left(\Gamma_x - \{\imath_x(\beta) : \beta < \alpha\}\right).$$

Clearly, $\imath_x(0) = 0$. Since $\Gamma_x \cap [0, \omega)$ is infinite, $\imath_x(n) \in [0, \omega)$ for all $n \in \omega$. But $\imath_x$ is increasing, so we in fact have that each $\imath_x(n) \in [n, \omega)$. Now fix a limit $\gamma > 0$, and suppose that conditions (a) and (b) hold for all limits $\delta < \gamma$. Condition (b) implies that $\gamma \notin \{\imath_x(\beta) : \beta < \gamma\}$, so $\imath_x(\gamma) \leq \gamma$. Suppose that $\imath_x(\gamma) < \gamma$. Then $\imath_x(\gamma) = \delta + n$ for some limit $\delta < \gamma$ and some $n \in \omega$. But then $\imath_x(\delta + n + 1) > \imath_x(\gamma)$, contradicting the fact that $\imath_x$ is increasing. Thus, $\imath_x(\gamma) = \gamma$. And condition (b) is once again true because $\imath_x$ is increasing and each $\Gamma_x \cap [\gamma, \gamma+\omega)$ is infinite.

Lemma 2. *h is 1:1.*

Proof. Let x and x' be distinct members of X^κ. Let $\alpha < \kappa$ be minimal such that $x(\alpha) \neq x'(\alpha)$. If both $x(\alpha)$ and $x'(\alpha)$ are in 2^κ, then there is some $\beta \in S_\alpha$ such that $x(\alpha)(\eta_\alpha^{-1}(\beta)) \neq x'(\alpha)(\eta_\alpha^{-1}(\beta))$, and thus $h(x) \neq h(x')$. Now suppose that one of $x(\alpha)$ or $x'(\alpha)$ is equal to p. Note that condition A3 implies that $\Gamma_x \cap [0, \alpha + \omega) = \Gamma_{x'} \cap [0, \alpha + \omega)$. But $\imath_x(\alpha)$ and $\imath_{x'}(\alpha)$ lie in $[0, \alpha + \omega)$, and $\imath_x(\beta) = \imath_{x'}(\beta)$ for all $\beta < \alpha$. It follows that $\imath_x(\alpha) = \imath_{x'}(\alpha)$. But $h(x)(\imath_x(\alpha)) \neq h(x')(\imath_{x'}(\alpha))$, so $h(x) \neq h(x')$ for this case also.

Lemma 3. *h is onto.*

Proof. Fix $y \in 2^\kappa$. Inductively construct an increasing function $\imath : \kappa \to \kappa$ as follows. Let $\imath(0) = 0$. For $\alpha > 0$, let

$$\imath(\alpha) = \min(\kappa - \{\imath(\beta) : \beta < \alpha\} \cup \bigcup\{S_\beta : \beta < \alpha \text{ and } y(\imath(\beta)) = 1\}).$$

Now define $x \in X^\kappa$ by:

$x(\alpha) = p$ iff $y(\imath(\alpha)) = 0$,

$x(\alpha)(\beta) = y(\eta_\alpha(\beta))$ for each α such that $y(\imath(\alpha)) = 1$.

A straightforward induction shows that $\imath_x = \imath$ (condition A3 is again important), from which it easily follows that $h(x) = y$.

Lemma 4. *h is continuous.*

Proof. Fix $x \in X^\kappa$, and let $y = h(x)$. Let $\mathcal{V}$ be an open set containing y; we need to find an open set $\mathcal{U}$ containing x which maps into $\mathcal{V}$. Choose a finite set $F \subset \kappa$ such that if $y' \in 2^\kappa$ and $y'(\alpha) = y(\alpha)$ for all $\alpha \in F$, then $y' \in \mathcal{V}$. Assume, wlog, that F has the form

$$F = \bigcup_{i<n} [\gamma_i, \gamma_i + m_i)$$

where $n \in \omega$, each γ_i is a limit ordinal, and each $m_i \in \omega$.

Let $\mathcal{U}$ be the open subset of X^κ which contains exactly the points x' which satisfy all of the following conditions:

(B1.) $x'(\alpha) = p$ if $S_\alpha \cap F \neq \emptyset$ and $x(\alpha) = p$.

(B2.) $x'(\alpha) \in 2^\kappa$ if $S_\alpha \cap F \neq \emptyset$ and $x(\alpha) \in 2^\kappa$.

(B3.) $x'(\alpha)(\beta) = x(\alpha)(\beta)$ if $S_\alpha \cap F \neq \emptyset$ and $x(\alpha) \in 2^\kappa$ and $\eta_\alpha(\beta) \in F$.

(B4.) $x'(\alpha) = p$ if $\imath_x(\alpha) \in F$ and $x(\alpha) = p$.

(B5.) $x'(\alpha) \in 2^\kappa$ if $\imath_x(\alpha) \in F$ and $x(\alpha) \in 2^\kappa$.

Note that U is open since only finitely many S_α intersect F and only finitely many α map into F under $\imath_x$. Note also that $x \in \mathcal{U}$. Now choose an arbitrary $x' \in \mathcal{U}$ and let $y' = h(x')$. All that remains is to show that $y'(\alpha) = y(\alpha)$ for all $\alpha \in F$. By B1 and B2, we have that $\Gamma_{x'} \cap F = \Gamma_x \cap F$. Thus, B3 implies that $y'(\alpha) = y(\alpha)$ for all $\alpha \in F - \Gamma_x$. So let $\alpha \in F \cap \Gamma_x$, and let $\beta = \imath_x^{-1}(\alpha)$. By B4 and B5 it suffices to show that $\imath_{x'}(\beta) = \alpha$. Now $\alpha = \gamma + n$ for some limit ordinal γ and some $n \in \omega$. Recall that $\imath_x(\gamma) = \imath_{x'}(\gamma) = \gamma$ (so we are done if $n = 0$). Note that $[\gamma, \gamma + n] \subset F$ (by the way F was chosen) and that $\Gamma_x \cap [\gamma, \gamma + n] = \Gamma_{x'} \cap [\gamma, \gamma + n]$. Thus, we have that $\imath_{x'}(\gamma + m) = \imath_x(\gamma + m)$ for all $1 \leq m < \omega$ such that $\imath_x(\gamma + m) \leq \gamma + n$. Thus, $\imath_{x'}(\beta) = \imath_x(\beta)$, which completes the proof.

We have shown that h is continuous, 1:1, and onto. Since 2^κ and X^κ are compact, h is a homeomorphism.

3. Remarks.

I would like to thank Amer Bešlagić for suggesting condition A3. While a homeomorphism can be constructed using S_α's satisfying only the other conditions, the proof becomes rather murky.

It is easy to see that the above proof can be extended to include a product of κ spaces where each factor is of the form $2^\beta \oplus \{p\}$ for some $\beta \leq \kappa$. In fact, it seems reasonable to conjecture that $\prod_{\alpha<\kappa} X_\alpha \cong 2^\kappa$ iff each X_α is a continuous, open, non-degenerate image of 2^κ (the only if part is immediate because projection maps are open).

The fact that $X^\kappa \cong 2^\kappa$ also follows from theorem 9 of [S1], which states (essentially) that any retract of 2^κ which has uniform local character κ is homeomorphic to 2^κ. The proof in [S1] is much more complex than the above construction, and it also contains several technical errors (they can be corrected, although this takes a bit of work). In light of this theorem, it is natural to rewrite the above conjecture on product spaces in more general terms.

Conjecture *If X is a continuous, open image of 2^κ and the local character at each point of X is equal to κ, then X is homeomorphic to 2^κ.*

This really asks whether or not "retract" can be replaced with "open image" in [S1]. Theorem 5 of [S2] appears to imply that this can be done for the special case $\kappa = \omega_1$.

Added in proof: Another example of Schepin's is a counterexample for the case $\kappa = \omega_2$.

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