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1. Results

If X is a topological space, then a function $\xi : \omega \to X$ is said to be a *sequence converging to a point* $x \in X$ provided that $\operatorname{ran} \xi \setminus U$ is finite for every open neighbourhood U of x, where as usual $\operatorname{ran} \xi = \{\xi(i) : i \in \omega\}$. We say that ξ is *non-trivial* if $\operatorname{ran} \xi$ is infinite. We will often identify a convergent sequence ξ with the set $\operatorname{ran} \xi$.

Recall that a space is *Fréchet-Urysohn* if, whenever a point x is in the closure of a subset A, there is a sequence from A converging to x. Arhangel'skii [2,3] introduced the notion of an α_i-space and showed its importance in determining whether a product of Fréchet-Urysohn spaces is Fréchet-Urysohn (see also [21–24] in this regard).

1.1 Definition. Let X be a space, and let x be a point of X. A *sheaf* at x is a family $\{\xi_n : n \in \omega\}$, where each ξ_n is a sequence converging to x. For $i = 1, 2, 3$ and 4, we call x an *α_i-point* if for each sheaf at x there is a sequence ξ converging to x such that $\operatorname{ran} \xi$ intersects:

(α_1) each $\operatorname{ran} \xi_n$ in a cofinite set;
(α_2) each $\operatorname{ran} \xi_n$ in an infinite set;
(α_3) infinitely many $\operatorname{ran} \xi_n$ in an infinite set;
(α_4) infinitely many $\operatorname{ran} \xi_n$ in a nonempty set.

A space is an *α_i-space* if each point of it is an α_i-point. A *v-space* is a Fréchet-Urysohn α_1-space, and a *w-space* is a Fréchet-Urysohn α_2-space.

The term "w-space" is due to Gruenhage, who originally defined these spaces by means of a natural convergence game

[12]. That this is equivalent to the definition adopted above is essentially due to Sharma [33]. Gruenhage pointed out that a condition equivalent to α_2 is obtained when "infinite" is replaced by "nonempty". (It suffices to split each ξ_n into infinitely many sequences with disjoint ranges; then any ξ whose range meets each of these will meet that of ξ_n in an infinite set.)

First-countable spaces are v-spaces [2,3], and α_i-spaces are α_{i+1}-spaces for $i = 1, 2, 3$. The countable Fréchet-Urysohn fan (the quotient space of the free union of the countable family of non-trivial convergent sequences obtained via identifying the limit points of these sequences to a single point) is an example of a "barely Fréchet-Urysohn space", i.e. Fréchet-Urysohn space which is not even an α_4-space. Nevertheless, countably compact Fréchet-Urysohn spaces are α_4-spaces [2,3]. Simon [35] constructed a compact Fréchet-Urysohn space X whose square $X \times X$ is not Fréchet-Urysohn. Since the product of a countably compact Fréchet-Urysohn α_3-space and a Fréchet-Urysohn space is Fréchet-Urysohn [2,3], X is a compact α_4-space which is not an α_3-space. Reznichenko [29] and somewhat later Gerlits and Nagy [11] and Nyikos [26] independently of each other constructed a compact Fréchet-Urysohn α_3-space which is not an α_2-space. Reznichenko's space X was additionally an image of a first-countable compact space under a continuous map f so that $|f^{-1}(x)| = 1$ for all $x \in X$ except just a single point. Dow [8] showed that in Laver's model for the Borel conjecture (see [18]) each w-space is a v-space. Conversely, Nyikos [28] showed that $b = \omega_1$ implies the existence of a countable w-space which is not a v-space. (The number b is defined to be the least cardinality of an unbounded family in $(\omega^\omega, <^*)$, where $f <^* g$ means that f is eventually less than g, i.e. $f(n) < g(n)$ for all but finitely many n.) Dow and Steprāns constructed a model of ZFC in which every countable v-space is metrizable (announced in [9], but the proof presented here contains a gap; see [10] for a correct proof). On the other hand, every space of character $< b$ is an α_1-space [28], and countable spaces are Fréchet-Urysohn under $MA + \neg CH$ [20], so any countable dense subset of 2^{ω_1} would be a nonmetrizable

v-space under $MA + \neg CH$.

Originally Arhangel'skii added the following line to the list of α_i-properties in Definition 1.1:

(α_5) infinitely many ran ξ_n in a cofinite set.

Nyikos [28] recently showed however that α_5-spaces coincide with α_1-spaces. Nevertheless this concept makes sense if one restricts oneself to considering only sheafs with disjoint ranges.

1.2 Definition. (Nyikos). A sheaf $\{\xi_n : n \in \omega\}$ at x is *disjoint* provided that ran $\xi_n \cap$ ran $\xi_m = \emptyset$ for $m \neq n$. We call x an $\alpha_{1,5}$-*point* if for each disjoint sheaf at x there is a sequence ξ converging to x so that ran ξ intersects infinitely many ran ξ_n in a cofinite set. A space is an $\alpha_{1,5}$-*space* if each point of it is an $\alpha_{1,5}$-point.

Clearly α_1-spaces are $\alpha_{1,5}$-spaces. Nyikos [28] proved that every $\alpha_{1,5}$-space is an α_2-space thus justifying the name for this concept given above. Therefore, as follows from Dow's result cited above, in Laver's model for the Borel conjecture Fréchet-Urysohn $\alpha_{1,5}$-spaces coincide with both w-spaces and v-spaces. On the other hand, Nyikos constructed in [28], under some additional set-theoretic assumptions, examples of countable Fréchet-Urysohn spaces with a single non-isolated point which show that neither of implications $\alpha_1 \Rightarrow \alpha_{1,5} \Rightarrow \alpha_2$ is reversible.

What is the behaviour of α_i-properties in Fréchet-Urysohn topological groups? Nyikos [25] pioneered in this direction. He proved that, unlike general topological spaces, Fréchet-Urysohn groups are α_4-spaces and that a sequential topological group is Fréchet-Urysohn if, and only if, it is an α_4-space. Nyikos asked whether Fréchet-Urysohn topological groups must be w-spaces, i.e α_2-spaces ([25], Problem 5; this question was recently repeated in [26]). He has some partial results which show that most Fréchet-Urysohn groups obtained via popular constructions *must* be w-spaces [26,27]. Since α_2 implies α_3, the next theorem provides a consistent solution of Nyikos' question.

1.3 Main Theorem. *Let H be the countable infinite group all elements of which have order 2 (this group is Abelian and unique up to isomorphism). Add ω_1 Cohen reals to an arbitrary model M of ZFC. Then, in the generic extension $M[G]$, there are Hausdorff group topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ on H of weight ω_1 having the following properties:*

(i) $(H, \mathcal{T}_0)$ is a Fréchet-Urysohn non α_3-space,

(ii) $(H, \mathcal{T}_1)$ is a w-space, but is not an $\alpha_{1,5}$-space (and so is not also a v-space, because α_1 implies $\alpha_{1,5}$).

In particular, the existence of the groups above is consistent with any admissible cardinal arithmetic.

For additional information connected with this theorem see Section 9.

From Dow's result cited above we immediately obtain

1.4 Corollary. *The existence of a (countable) w-group which is not a v-group is consistent with and independent of ZFC.*

Recall that an indexed family $\{A_\alpha : \alpha \in \kappa\}$ of infinite subsets of ω, where κ is a cardinal, is said to be a *tower* provided that $A_\beta \setminus A_\alpha$ is finite whenever $\alpha < \beta < \kappa$, and if $A \subset \omega$ is infinite, then some $A \setminus A_\alpha$ also is infinite. A family $F \subset \omega^\omega$ is said to be *dominating* if for every $g \in \omega^\omega$ there is an $f \in F$ with $g <^* f$. Let t be the least cardinality of a tower and d be the smallest cardinality of a dominating family $F \subset \omega^\omega$.

Under $t = d$ Nyikos constructed in [28, Section 5] a countable w-group which is not a v-group. He pointed out in [28, Section 7] that if $t = c$ is assumed, then he can make the group fail to be an $\alpha_{1,5}$-space. (Equalities $t = d = c$ follow from MA, see for example, the first paragraph in the proof of [7, Theorem 5.1].) Therefore Corollary 1.4 was independently established by Nyikos.

Let $G = \{f \in 2^{\omega_1} : |\{\alpha \in \omega_1 \, f(\alpha) = 1\}| \leq \omega\}$ be the Σ-product in 2^{ω_1}. Then all countable subsets of G are metrizable, so G is a (countably compact) v-group that is not first countable [12]. Is it possible to construct a countable group with the same properties? Assume $MA + \neg CH$ and consider any countable dense subgroup G of the group 2^{ω_1} (which exists since 2^{ω_1} is separable). Since the character of G is $\omega_1 < c$, $MA + \neg CH$

implies that G is Fréchet-Urysohn [20]. Moreover, since $b = c$ under MA, G is an α_1-space [28]. Thus G is a countable v-group that is not first countable. On the other hand, let M be the model in which every countable v-space is metrizable [9,10]. Then, *in M, every separable v-group is metrizable, so the existence of a countable (or separable) nonmetrizable v-group is consistent with and independent of ZFC.*

Our results leave open the following questions.

1.5 Question. Is there a (countable) Fréchet-Urysohn group which is an α_3-space without being an α_2-space?

1.6 Question. Is it consistent with ZFC to have a Fréchet-Urysohn $\alpha_{1,5}$-group which is not a v-group?

1.7 Question. Is there a "real" (= requiring no additional set-theoretic assumptions beyond ZFC) example of a countable nonmtrizable w-group?

1.8 Question. Is there a "real" example of a Fréchet-Urysohn topological group that is not an α_3-space?

It should be noted that it seems to be very hard to answer last two questions positively at present, since to do this one needs at least to construct a "real" example of a countable nonmetrizable Fréchet-Urysohn group the existence of which is the well-known open problem [1,25]. For "real" examples of countable nonmetrizable w-spaces see [11] and [26].

Quite recently Nogura [23] introduced an infinite series of convergence properties generalizing α_3-property.

1.9 Definition [23]. Let X be a space, $x \in X$ and let $\Xi = \{\xi_n : n \in \omega\}$ be a sheaf at x. We call a sheaf $\{\eta_m : m \in \omega\}$ at x a *cross-sheaf* of Ξ provided that $\cup\{\text{ran } \eta_m : m \in \omega\} \subset \cup\{\text{ran } \xi_n : n \in \omega\}$ and each ran η_m meets infinitely many ran ξ_n. A sheaf $\{\eta_m : m \in \omega\}$ is a *subsheaf* of a sheaf $\{\xi_n : n \in \omega\}$ if there exists an injection $j : \omega \to \omega$ so that ran $\eta_m \subset$ ran $\xi_{j(m)}$ for each $m \in \omega$.

For $k \in \omega$ we define by induction what does it mean for Ξ to be k-nice as follows: Ξ is 0-*nice* if $\cup\{\text{ran } \xi_n : n \in \omega\} \subset$ ran η

for some convergent sequence η; and Ξ is $(k+1)$-*nice* if each cross-sheaf of Ξ has a k-nice subsheaf.

For $k \in \omega$ we call x an α^k-*point* if each sheaf at x has a k-nice subsheaf. A space X is an α^k-*space* if each point of it is an α^k-point.

1.10 Definition. We will say that a point x of a space X is an α^∞-*point* if every sheaf at x has a k-nice subsheaf for some $k \in \omega$. If every point of X is an α^∞-point, then X is an α^∞-*space*.

One can easily see that

$$\alpha_3 = \alpha^0 \Rightarrow \alpha^1 \Rightarrow \ldots \Rightarrow \alpha^k \Rightarrow \alpha^{k+1} \Rightarrow \ldots \Rightarrow \alpha^\infty.$$

On the other hand, for every $k \in \omega$ Nogura [23, Example 3.7] constructed, under CH, a countable Fréchet-Urysohn α_4-space X_k which is an α^{k+1}-space but fails to be an α^k-space. Clearly, the disjoint sum $\oplus\{X_k : k \in \omega\}$ of all X_k's is a countable Fréchet-Urysohn α_4-space which is an α^∞-space but is not an α^k-space for every $k \in \omega$. Now we arrive to the natural questions.

1.11 Question. Does convergence properties defined above, $\alpha^0, \alpha^1, \ldots, \alpha^k, \ldots$ and α^∞ coincide for Fréchet-Urysohn topological groups?

1.12 Question. Is every Fréchet-Urysohn group an α^∞-space?

Since α_3 implies α^∞, a counterexample to the last question, if any, would be an improvement of our example constructed in Theorem 1.3(i). At the same time, Question 1.12 can be considered as the "heir" of Nyikos' question cited before Theorem 1.3, and since the property α^∞ looks like to be much more weaker than α_3, it seems to have a good chance for a positive solution in ZFC.

A lot of efforts was spent on distinguishing α_i-properties in compact Fréchet-Urysohn spaces (see [26], [28], [29] and [11]). Since compact groups are dyadic [15,17] and dyadic spaces of countable tightness are metrizable [4], a Fréchet-Urysohn

compact group is metrizable. However, it seems to be worth considering the behaviour of α_i-properties in Fréchet-Urysohn compact-like groups.

1.13 Question. Do some new implications between α_i-properties, $i \in \{1;1,5;2;3;4\}$, and α^k-properties, $k \in \omega \cup \{\infty\}$, appear in Fréchet-Urysohn groups belonging to one of the following classes:

(i) countably compact groups,

(ii) pseudocompact groups,

(iii) precompact groups (= subgroups of compact groups), and

(iv) groups complete in their two-sided uniformity?

Results of this paper were announced in [31].

Our set-theoretic and forcing notations follow [16]. For a function f we use dom f and ran f for denoting the domain and the range of f respectively; the function f itself is considered as a subset of dom $f \times$ ran f. If $X \subset$ dom f, then $f''X = \{f(x) : x \in X\}$. If X is a set and τ is a cardinal, then $[X]^\tau = \{Y \subset X : |Y| = \tau\}$ and $[X]^{<\tau} = \{Y \subset X : |Y| < \tau\}$. We use letters with a dot over the top to denote names in the forcing notion, and usually for typographical reasons we will drop checks over letters when denoting canonical names for objects lying in the ground model. If $G \subset P$ is a generic subset of a forcing notion P and $\dot{X}$ is a P-name, then $val_G \dot{X}$ is the interpretation of $\dot{X}$ via G (see [16, Chapter VII, Section 2] for details).

We fix the symbol H for denoting the countable infinite group all elements of which have order 2 (recall that such a group is Abelian and unique up to isomorphism), and $\mathbf{0}$ stands for the neutral element of H. Since H is Abelian, we adopt $+$ for denoting the group operation of H. If $n \in \omega$ and $K \subset H$, then

$$\mathrm{gr}_n K = \{g_0 + \ldots + g_i : i \leq n,\ g_0, \ldots, g_i \in K\}$$

and $\mathrm{gr}_\omega K = \cup\{\mathrm{gr}_n K : n \in \omega\}$. If $K \subset H$ is finite, then $\mathrm{gr}_{|K|} K$ is the smallest subgroup of H that contains K, so this subgroup is finite. Further, if H' is a subgroup of H and $g \in H \setminus H'$,

then $H' \cup (g + H')$ is the smallest subgroup of H that contains $H' \cup \{g\}$.

The paper is organized as follows. We fix a countable family $\mathcal{F}$ of subsets of H. In Section 2 we define a forcing poset P depending of $\mathcal{F}$ which adds generically a Hausdorff group topology $\mathcal{T}$ on H to be defined in Section 3. All results of Sections 2–6 are independent of the choice of $\mathcal{F}$. In particular, we show that, for any $\mathcal{F}$, $(H, \mathcal{T})$ is Fréchet-Urysohn (Section 5), but is not an $\alpha_{1,5}$-space (Section 6), and P is forcing isomorphic to $Fn(\omega_1, \omega)$, the standard poset adding ω_1 Cohen reals (Section 4). Finally, in Sections 7 and 8 we specify the choice of $\mathcal{F}$ to construct topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ respectively.

We propose the following guidelines during the first reading of the manuscript. First, start with Sections 2 and 3. Then proceed with reading Sections 5–8 returning to statements (but not proofs) of results from Section 4 when necessary. And finally, if, after all these been done, you would be still encoraged to enter into rather messy details of Section 4, you would be free to do this.

2. A FORCING NOTION P

From now on we fix a countable family $\mathcal{F}$ of subsets of H.

2.1 Definition. A condition $p \in P$ is a structure $p = \langle H^p, A^p, B^p, C^p, f^p, \varphi^p \rangle$, where H^p is a finite subgroup of H,

$$A^p \in [\omega_1]^{<\omega}, \ B^p \in [[A^p]^{<\omega} \times \mathcal{F} \times A^p \times \omega]^{\omega}, \ C^p \subset H^p \times A^p \times A^p,$$

and $f^p : A^p \times H^p \to 2$, $\varphi^p : A^p \times H^p \to 2$ are functions satisfying the following properties:

(1_p) $f^p(\alpha, 0) = 1$ for every $\alpha \in A^p$,

(2_p) $(h, \alpha, \beta) \in C^p$ implies that $\alpha < \beta$, $f^p(\alpha, h) = 1$ and if $g_0, g_1 \in H^p$, $f^p(\beta, g_0) = f^p(\beta, g_1) = 1$, then $f^p(\alpha, h + g_0 + g_1) = 1$, and

(3_p) if $(E, F, \alpha, n) \in B^p$, then $\alpha \notin E$ and $\theta^p(\{\alpha\}) \cap \mathrm{gr}_n \theta^p(E) \subset F$, where for each $E \in [A^p]^{\omega}$ we fix the following notation:

$$\theta^p(E) = \{h \in H^p : \varphi^p(\alpha, h) = 1 \text{ for some } \alpha \in E\}.$$

For $p, q \in P$ we define $q \leq p$ iff $H^p \subset H^q$, $A^p \subset A^q$, $B^p \subset B^q$, $C^p \subset C^q$, $f^p \subset f^q$, $\varphi^p \subset \varphi^q$ and

$({}_q\leq_p)$ if $h \in H^q \setminus H^p$, $\alpha, \beta \in A^p$ and $\varphi^q(\alpha, h) = 1$, then $f^q(\beta, h) = 1$.

One can easily verify that $(P, \leq)$ is actually a poset. (Usually we will drop $\leq$ and write simply P.) Clearly, $\langle \{\mathbf{0}\}, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rangle \in P$ is the largest element of P, and in what follows we will use $\Vdash T$ as an abbreviation for $\langle \{\mathbf{0}\}, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rangle \Vdash T$.

2.2 Lemma. *P is c. c. c.*

Proof. Let $Q \in [P]^{\omega_1}$. Since $|\mathcal{F}| \leq \omega$, using standard Δ-system and countability arguments one can find distinct $p, q \in Q$ such that $H^p = H^q = H^*$, $A^p \cap A^q = A^*$, $f^p \restriction_{A^* \times H^*} = f^q \restriction_{A^* \times H^*}$ and $\varphi^p \restriction_{A^* \times H^*} = \varphi^q \restriction_{A^* \times H^*}$. Define $H^r = H^*$, $A^r = A^p \cup A^q$, $B^r = B^p \cup B^q$, $C^r = C^p \cup C^q$, $f^r = f^p \cup f^q$, $\varphi^r = \varphi^p \cup \varphi^q$ and $r = \langle H^r, A^r, B^r, C^r, f^r, \varphi^r \rangle$. One can easily verify that $r \in P$, $r \leq p$ and $r \leq q$.

2.3 Lemma. *For each $h \in H$ the set $Q_h = \{q \in P : h \in H^q\}$ is dense in P.*

Proof. Fix $p \in P \setminus Q_h$ and define

$$H^q = H^p \cup (h + H^p), \; A^q = A^p, \; B^q = B^p, \; C^q = C^p,$$

$$f^q = f^p \cup \{\langle \langle \alpha, g + h \rangle, 0 \rangle : \alpha \in A^p, \; g \in H^p\}$$

and

$$\varphi^q = \varphi^p \cup \{\langle \langle \alpha, g + h \rangle, 0 \rangle : \alpha \in A^p, \; g \in H^p\}.$$

Then $q = \langle H^q, A^q, B^q, C^q, f^q, \varphi^q \rangle \in Q_h$ and $q \leq p$.

2.4 Lemma. *(i) Suppose that $p \in P$ and $\beta \in \omega_1 \setminus A^p$. Then there exists $q \leq p$ such that $\beta \in A^q$ and $f^q(\beta, h) = 0$ whenever $h \in H^p \setminus \{\mathbf{0}\}$.*

(ii) If in addition to the hypothesis of (i) we assume that $\alpha \in A^p$, $h \in H^p$, $\alpha < \beta$ and $f^p(\alpha, h) = 1$, then $q \leq p$ can be chosen to satisfy $(h, \alpha, \beta) \in C^q$.

(iii) For each $\beta \in \omega_1$ the set $Q_\beta = \{q \in P : \beta \in A^q\}$ is dense in P.

Proof. (i) Define $H^q = H^p$, $A^q = A^p \cup \{\beta\}$, $B^q = B^p$, $C^q = C^p$,

$$f^q = f^p \cup \{\langle\langle\beta, \mathbf{0}\rangle, 1\rangle\} \cup \{\langle\langle\beta, h\rangle, 0\rangle : h \in H^q \setminus \{\mathbf{0}\}\}$$

and $\varphi^q = \varphi^p \cup \{\langle\langle\beta, h\rangle, 0\rangle : h \in H^q\}$. Then $q = \langle H^q, A^q, B^q, C^q, f^q, \varphi^q\rangle \in P, q \leq p$ and q satisfies the requirements of (i).

(ii) Consider the q constructed in the proof of (i), and let $H^r = H^q$, $A^r = A^q$, $B^r = B^q$, $C^r = C^q \cup \{(h, \alpha, \beta)\}$, $f^r = f^q$, $\varphi^r = \varphi^q$ and $r = \langle H^r, A^r, B^r, C^r, f^r, \varphi^r\rangle$. Assumptions of (ii) yield that $r \in P$, $r \leq q \leq p$ and moreover, $(h, \alpha, \beta) \in C^r$.

(iii) immediately follows from (i).

2.5 Lemma. *If $p \in P$ and $\mu \in A^p$, then there exist $q \leq p$ and $h^* \in H^q \setminus H^p$ such that $A^q = A^p$, $\varphi^q(\mu, h^*) = 1$, and $\varphi^q(\eta, h) = 0$ whenever $\eta \in A^p \setminus \{\mu\}$ and $h \in H^q \setminus H^p$.*

Proof. Choose $h^* \in H \setminus H^p$ and define

$$H^q = H^p \cup (h^* + H^p),\ A^q = A^p,\ B^q = B^p,\ C^q = C^p,$$

$$(1)\quad f^q = f^p \cup \{\langle\langle\alpha, g + h^*\rangle, f^p(\alpha, g)\rangle : \alpha \in A^p,\ g \in H^p\},$$

$$(2)\quad \varphi^q = \varphi^p \cup \{\langle\langle\mu, h^*\rangle, 1\rangle\} \cup \{\langle\langle\alpha, h^*\rangle, 0\rangle : \alpha \in A^p \setminus \{\mu\}\}$$
$$\cup \{\langle\langle\alpha, g + h^*\rangle, 0\rangle : \alpha \in A^p,\ g \in H^p \setminus \{\mathbf{0}\}\}$$

and $q = \langle H^q, A^q, B^q, C^q, f^q, \varphi^q\rangle$.

2.5.1 Claim. $q \in P$.

Proof. (1_q) trivially follows from (1_p).

(2_q) Assume that $(h, \alpha, \beta) \in C^q = C^p$. Then $h \in H^p$, $\alpha < \beta$ and $f^q(\alpha, h) = f^p(\alpha, h) = 1$ by (2_p). Now suppose that $g_0, g_1 \in H^q$ and $f^q(\beta, g_0) = f^p(\beta, g_1) = 1$. We have to consider three cases.

Case 1. $g_0, g_1 \in H^p$. In this case $f^p(\beta, g_i) = f^q(\beta, g_i) = 1$ for $i \in 2$. Since H^p is a subgroup of H, $h + g_0 + g_1 \in H^p$, and (2_p) yields $f^q(\alpha, h + g_0 + g_1) = f^p(\alpha, h + g_0 + g_1) = 1$.

Case 2. $g_i \in H^p$, $g_{1-i} \in H^q \setminus H^p$, $i \in 2$. In this case $g_{1-i} = g' + h^*$ for some $g' \in H^p$, and since $f^q(\beta, g_{1-i}) = 1$, from (1) we conclude that $f^p(\beta, g') = 1$. Now from (2_p) it

follows that $f^p(\alpha, h+g_i+g') = 1$, and applying (1) once more we obtain

$$f^q(\alpha, h+g_0+g_1) = f^q(\alpha, h+g_i+g'+h^*) = f^p(\alpha, h+g_i+g') = 1.$$

Case 3. Both $g_i \in H^q \setminus H^p$, $i \in 2$. In this case choose $g_i' \in H^p$ with $g_i = g_i' + h^*$, $i \in 2$. Since $f^p(\beta, g_i') = f^q(\beta, g_i) = 1$ for $i \in 2$, $f^p(\alpha, h+g_0'+g_1') = 1$ by (2_p). But

$$h + g_0 + g_1 = h + g_0' + h^* + g_1' + h^* = h + g_0' + g_1',$$

so

$$f^q(\alpha, h+g_0+g_1) = f^q(\alpha, h+g_0'+g_1') = f^p(\alpha, h+g_0'+g_1') = 1.$$

(3_q) Suppose $(E, F, \alpha, n) \in B^q = B^p$. Then $\alpha \notin E$ by (3_p). If $\mu \notin E \cup \{\alpha\}$, then

$$\theta^q(\{\alpha\}) \cap \mathrm{gr}_n\theta^q(E) = \theta^p(\{\alpha\}) \cap \mathrm{gr}_n\theta^p(E) \subset F,$$

because (3_p) holds. If $\mu = \alpha$, then

$$\begin{aligned} \theta^q(\{\alpha\}) \cap \mathrm{gr}_n\theta^q(E) &= (\theta^p(\{\alpha\}) \cup \{h^*\}) \cap \mathrm{gr}_n\theta^p(E) \\ &\subset \theta^p(\{\alpha\}) \cap \mathrm{gr}_n\theta^p(E) \subset F, \end{aligned}$$

because $h^* \notin H^p \supset \mathrm{gr}_n\theta^p(E)$ (we use here the fact that H^p is a subgroup of H and $\alpha \notin E$) and (3_p) holds. Now suppose that $\mu \in E$. Then $\alpha \neq \mu$ and

$$\begin{aligned} &\theta^q(\{\alpha\}) \cap \mathrm{gr}_n\theta^q(E) \\ &\subset \theta^p(\{\alpha\}) \cap (\mathrm{gr}_n\theta^p(E) \cup (h^* + \mathrm{gr}_n\theta^p(E))) \\ &\subset \theta^p(\{\alpha\}) \cap \mathrm{gr}_n\theta^p(E) \subset F, \end{aligned}$$

since H^p is a subgroup of H, $h^* \notin H^p$ and (3_p) holds.

2.5.2 Claim. $q \leq p$.

Proof. All properties beyond $({}_q{\leq}_p)$ are obvious. To verify $({}_q{\leq}_p)$ observe that if $h \in H^q \setminus H^p$, $\alpha, \beta \in A^p$ and $\varphi^q(\alpha, h) = 1$, then $h = h^*$ and $\alpha = \mu$ according to (2); finally, (1) and (1_p) yield $f^q(\beta, h^*) = f^p(\beta, \mathbf{0}) = 1$.

All other properties of q mentioned in Lemma 2.5 follow trivially from the choice of q, so the proof of this lemma is now completed.

3. Introducing generically a group topology $\mathcal{T}$ on H.

In this section we define a Hausdorff group topology $\mathcal{T}$ on H and discuss purposes for including conditions (1_p)–(3_p) and $(_q\leq_p)$ in Definition 2.1. To give the reader an opportunity to understand better the construction of $\mathcal{T}$, let us start from an easy lemma adapting the common way of topologizing groups to the specific group H:

3.1 Lemma. *Let $\mathcal{S} = \{U_\alpha : \alpha \in \omega_1\}$ be a family of subsets of H satisfying the following properties:*

(i) $\cap\{U_\alpha : \alpha \in \omega_1\} = \{\mathbf{0}\}$ and

(ii) if $\alpha \in \omega_1$ and $h \in U_\alpha$, then $h + U_\beta + U_\beta \subset U_\alpha$ for some $\beta \in \omega_1$.

For $h \in H$ and $\Phi \in [\omega_1]^\omega$ set $V_{h,\Phi} = h + \cap\{U_\alpha : \alpha \in \Phi\}$, and let $\mathcal{B} = \{V_{h,\Phi} : h \in H \text{ and } \Phi \in [\omega_1]^\omega\}$. Then $\mathcal{B}$ constitutes a base of some Hausdorff group topology $\mathcal{T}$ on H and $\mathcal{S}$ is a subbase at $\mathbf{0}$ of this topology. Moreover, if $\mathcal{T}'$ is another group topology on H for which $\mathcal{S}$ is a subbase at $\mathbf{0}$, then $\mathcal{T}' = \mathcal{T}$, i.e. $\mathcal{T}$ is uniquely determined by $\mathcal{S}$.

Recall that a family $\mathcal{S}$ is said to be a *subbase at a point x* of a topological space $(X, \mathcal{T})$ if $\mathcal{S} \subset \mathcal{T}$ and, whenever $x \in V \in \mathcal{T}$, there is a finite subfamily $\mathcal{S}_V \subset \mathcal{S}$ with $x \in \cap\mathcal{S}_V \subset V$.

Proof. Since all elements of H have order 2,

(iii) $-U_\alpha = U_\alpha$ for every $\alpha \in \omega_1$.

Since H is Abelian, we trivially have

(iv) $g + U_\alpha - g = U_\alpha$ for all $\alpha \in \omega_1$.

Now the conclusion of our lemma is a well-known consequence of (i)–(iv) (see, for example, [14, Chapter 2, Theorems 4.3 and 4.5]).

3.2 Definition. For $\alpha \in \omega_1$ set

$$\dot{U}_\alpha = \{\langle \check{h}, p\rangle : p \in P,\ h \in H^p \text{ and } f^p(\alpha, h) = 1\}$$

and

$$\dot{S}_\alpha = \{\langle \check{h}, p\rangle : p \in P,\ h \in H^p \text{ and } \varphi^p(\alpha, h) = 1\}.$$

Obviously, $\dot{U}_\alpha$ and $\dot{S}_\alpha$ are canonical P-names and

$$\Vdash \text{“}\dot{U}_\alpha \subset H \text{ and } \dot{S}_\alpha \subset H \text{ for every } \alpha \in \omega_1\text{”}.$$

Our main aim in this section will be to show that, in every generic extension $M[G]$ via a generic filter $G \subset P$,

($\star$) the family $\mathcal{S} = \{val_G \dot{U}_\alpha : \alpha \in \omega_1\}$ satisfies the assumption of Lemma 3.1, and so, in accordance with the conclusion of this lemma, it generates the Hausdorff group topology $\mathcal{T}$ on H,

($\star\star$) each interpretation $val_G \dot{S}_\beta$ is a non-trivial sequence converging to $\mathbf{0}$ in the space $(H, \mathcal{T})$.

In the process of verifying this we will demonstrate the impact of different conditions from Definition 2.1 on the behaviour of $\dot{U}_\alpha$'s and $\dot{S}_\alpha$'s. We start with the simplest condition (1_p).

3.3 Lemma. $\Vdash$ “$\cap\{\dot{U}_\alpha : \alpha \in \omega_1\} = \{\mathbf{0}\}$”.

Proof. For $\alpha \in \omega_1$ and $p \in P$ fixed, use 2.4(iii) to choose $q \leq p$ with $\alpha \in A^q$ and note that $q \Vdash$ “$\mathbf{0} \in \dot{U}_\alpha$”, since $f^q(\alpha, \mathbf{0}) = 1$ by (1_q). Now a standard density argument yields $\Vdash$ “$\mathbf{0} \in \cap\{\dot{U}_\alpha : \alpha \in \omega_1\}$”. On the contrary, from 2.3 and 2.4(i) it follows that for each $h \in H \setminus \{\mathbf{0}\}$ the set $\{q \in P : \exists \beta \in A^q\ (f^q(\beta, h) = 0)\}$ is dense in P, so $\Vdash$ “$\cap\{\dot{U}_\alpha : \alpha \in \omega_1\} \subset \{\mathbf{0}\}$”.

To prevent the reader from treating really simply-sounding condition (1_p) as being something not very much serious, it should be specially emphasized that we have already used this condition in a non-trivial way in the proof of Claim 2.5.2. In its turn, Lemma 2.5 incorporating this claim will be applied in the proof of important Lemma 3.9.

The proof of our next lemma displays what reason the property (2_p) was designed for.

3.4 Lemma. *If $q \in P$ and $(h, \alpha, \beta) \in C^q$, then $q \Vdash$ “$h + \dot{U}_\beta + \dot{U}_\beta \subset \dot{U}_\alpha$”.*

Proof. Let $g_0, g_1 \in H$, $r \leq q$ and $r \Vdash$ " $g_i \in \dot{U}_\beta$ for $i \in 2$". Use 2.3 to pick $p \leq r$ so that $g_0, g_1 \in H^p$. Since $p \leq r \leq q$, $(h, \alpha, \beta) \in C^q \subset C^p$. From $p \leq r \Vdash$ " $g_0, g_1 \in \dot{U}_\beta$" it follows that $f^p(\beta, g_0) = f^p(\beta, g_1) = 1$. Now the condition (2_p) enters into the game: $f^p(\alpha, h + g_0 + g_1) = 1$, and so $p \Vdash$ " $h + g_0 + g_1 \in \dot{U}_\alpha$". Finally, a standard density argument finishes the proof.

3.5 Lemma. $\Vdash$ "$\forall \alpha \in \omega_1 \, \forall h \in \dot{U}_\alpha \, \exists \beta \in \omega_1 \;\; (h + \dot{U}_\beta + \dot{U}_\beta \subset \dot{U}_\alpha)$".

Proof. Assume that $\alpha \in \omega_1$, $h \in H$, $r \in P$ and $r \Vdash$ " $h \in \dot{U}_\alpha$". According to a standard density argument, to prove our lemma it suffices to find $q \leq r$ and $\beta \in \omega_1$ such that $q \Vdash$ " $h + \dot{U}_\beta + \dot{U}_\beta \subset \dot{U}_\alpha$". So use 2.3 and 2.4(iii) to pick $p \leq r$ such that $h \in H^p$ and $\alpha \in A^p$. Since $p \leq r \Vdash$ " $h \in \dot{U}_\alpha$", we obviously have $f^p(\alpha, h) = 1$. Then apply 2.4(ii) to choose $\beta \in \omega_1$ and $q \leq p$ with $(h, \alpha, \beta) \in C^q$ and note that, by Lemma 3.4, q is as required.

3.6 Lemma. $\Vdash$ "*the family* $\dot{\mathcal{S}} = \{\dot{U}_\alpha : \alpha \in \omega_1\}$ *can be taken as a subbase at* 0 *of some (uniquely determined by* $\dot{\mathcal{S}}$*) Hausdorff group topology* $\mathcal{T}$ *on* H".

Proof. By 3.3 and 3.5, $\Vdash$ " $\dot{\mathcal{S}}$ satisfies the assumption of Lemma 3.1", and the result follows.

3.7 Definition $\dot{\mathcal{T}}$ is a P-name satisfying the following property: $\Vdash$ "$(H, \dot{\mathcal{T}})$ is the Hausdorff topological group for which $\dot{\mathcal{S}}$ is a subbase at 0". (Such a P-name exists by 3.6 and the maximal principle (see, for example, [16], Ch. VII, Theorem 8.2).

Now we turn to handling $\dot{S}_\beta$'s. For the reader convenience we state explicitly a simple topological lemma wich will be helpful in verifying $(\star\star)$.

3.8 Lemma. *Let* $(X, \mathcal{T})$ *be a Hausdorff space,* $x \in X$ *and* $\mathcal{S} = \{U_\alpha : \alpha \in \omega_1\}$ *be a subbase at* x. *Assume also that* S *is an infinite countable subset of* X *and* $S \setminus U_\alpha$ *is finite for each* $\alpha \in \omega_1$. *Then (after any one-to-one enumeration)* S *is a non-trivial sequence* $\mathcal{T}$*-converging to* x.

Proof. We have to show that $S \setminus V$ is finite whenever $x \in V \in \mathcal{T}$. Indeed, let $x \in V \in \mathcal{T}$. Since $\mathcal{S}$ is a subbase at x, there is a finite set $\alpha_0, \ldots, \alpha_n \in \omega_1$ such that $x \in \cap\{U_{\alpha_i} : i \in n+1\} \subset V$. Then

$$S \setminus V \subset S \setminus \cap\{U_{\alpha_i} : i \in n+1\} = \cup\{S \setminus U_{\alpha_i} : i \in n+1\},$$

and the last set is finite by our assumption.

3.9 Lemma *For each $\beta \in \omega_1$, $\Vdash$ "$\dot{S}_\beta$ is infinite".*

Proof. For $\beta \in \omega_1$ fixed, from 2.3, 2.4(iii) and 2.5 it follows that for every $K \in [H]^{<\omega}$ the set $\{p \in P : \varphi^p(\beta, h) = 1$ for some $h \in H^p \setminus K\}$ is dense in P, so $\Vdash$ "$\dot{S}_\beta$ is infinite".

The proof of our next lemma displays how property $({}_q\leq_p)$ works:

3.10 Lemma. $\Vdash$ "$\dot{S}_\beta \setminus \dot{U}_\alpha$ *is finite whenever* $\alpha, \beta \in \omega_1$".

Proof. Let $\alpha, \beta \in \omega_1$. For $p \in P$ fixed, use 2.4(iii) to pick $q \leq p$ with $\alpha, \beta \in A^q$ and then observe that $q \Vdash$ "$\dot{S}_\beta \setminus \dot{U}_\alpha \subset H^q$", because $({}_r\leq_q)$ holds for each $r \leq q$. Since H^q is finite, the result follows from a standard density argument.

3.11 Lemma. *For each $\beta \in \omega_1$, $\Vdash$ "$\dot{S}_\beta$ is a non-trivial sequence converging to* $\mathbf{0}$ *in* $(H, \dot{T})$".

Proof. Combine 3.6, 3.9. 3.10 and apply 3.8.

3.12 Lemma. *If $p \in P$ and $(E, F, \alpha, n) \in B^p$, then $p \Vdash$ "$\dot{S}_\alpha \cap gr_n \cup \{\dot{S}_\beta : \beta \in E\} \subset F$".*

Proof. If $q \leq p$, then $(E, F, \alpha, n) \in B^p \subset B^q$, and so $\theta^q(\{\alpha\}) \cap \mathrm{gr}_n\theta^q(E) \subset F$ by (3_q); now it remains only to remember the definition of $\dot{S}_\alpha$.

Now it is time to summarize the results of this section. Let $G \subset P$ be a generic filter and $M[G]$ the generic extension of M. In $M[G]$ define $f = \cup\{f^p : p \in G\}$ and $\varphi = \cup\{\varphi^p : p \in G\}$. From 2.3, 2.4(iii) and easy density arguments it follows that f and φ are functions with $\mathrm{dom}\, f = \mathrm{dom}\, \varphi = \omega_1 \times H$ and $\mathrm{ran}\ f = \mathrm{ran}\ \varphi = 2$. For $\alpha \in \omega_1$ set

$$U_\alpha = \{h \in H : f(\alpha, h) = 1\} \text{ and}$$

$S_\alpha = \{h \in H : \varphi(\alpha, h) = 1\}$.

It is routine to check that $U_\alpha = val_G \dot{U}_\alpha$ and $S_\alpha = val_G \dot{S}_\alpha$ for all $\alpha \in \omega_1$, so the following lemma is an immediate consequence of 3.6, 3.7 and 3.11:

3.13 Lemma. *(i) $\mathcal{S} = \{U_\alpha : \alpha \in \omega_1\}$ is a subbase at $\mathbf{0}$ of the Hausdorff group topology $\mathcal{T} = val_G \mathcal{T}$ on H, and*

(ii) $\{S_\alpha : \alpha \in \omega_1\}$ is a family consisting of non-trivial sequences $\mathcal{T}$-converging to $\mathbf{0}$.

It is properties (1_p) and (2_p) that are responsible for 3.13(i), and the property $({}_q{\leq}_p)$ was designed to ensure 3.13(ii). The various f^p and φ^p could be regarded as carrying the finite piece of information on what p forces to be in future U_α's and S_α's, respectively. And the interplay between f^p and φ^p incorporated into the condition (3_p) will then determine, by means of 3.12, which of the α_i-properties would eventually hold in $M[G]$. Finally, the sequences S_α will play a key role in verifying both the Fréchet-Urysohn property and α_i-properties of $(H, \mathcal{T})$.

4. Technical Lemmas

4.1 Definition. Let $p \in P$ and $\gamma \in \omega_1$. We define a condition $p|\gamma \in P$, a *restriction of p to γ*, by the following rules:

$$H^{p|\gamma} = H^p,\ A^{p|\gamma} = A^p \cap \gamma,\ B^{p|\gamma} = B^p \cap ([A^{p|\gamma}]^{<\omega} \times \mathcal{F} \times A^{p|\gamma} \times \omega),$$

$$C^{p|\gamma} = C^p \cap (H^{p|\gamma} \times A^{p|\gamma} \times A^{p|\gamma}),$$

$$f^{p|\gamma} = f^p \restriction_{A^{p|\gamma} \times H^{p|\gamma}},\ \ \varphi^{p|\gamma} = \varphi^p \restriction_{A^{p|\gamma} \times H^{p|\gamma}}$$

and $p \mid \gamma = \langle H^{p|\gamma}, A^{p|\gamma}, B^{p|\gamma}, C^{p|\gamma}, f^{p|\gamma}, \varphi^{p|\gamma} \rangle$.

The following facts can be verified trivially.

4.2 Lemma. *$p|\gamma \in P$ and $p \leq p|\gamma$ whenever $p \in P$ and $\gamma \in \omega_1$.*

4.3 Lemma. *If $q \leq p$ and $\gamma \in \omega_1$, then $q|\gamma \leq p|\gamma$.*

4.4 Lemma. *For each $p \in P$ there exists $\delta \in \omega_1$ such that $p|\gamma = p$ for all $\gamma \in \omega_1 \setminus \delta$.*

4.5 Definition. Let $p \in P$ and $\gamma \in \omega_1$. We say that p is *γ-good* if there exists $\sigma : A^p \to A^p \cap \gamma$ (witnessing that p is γ-good) such that:

(i) $\sigma \restriction_{A^p \setminus \gamma}$ is an injection,

(ii) $\sigma(\beta) = \beta$ for every $\beta \in A^p \cap \gamma$,

(iii) if $(E, F, \alpha, n) \in B^p$, then $(\sigma''E, F, \sigma(\alpha), n) \in B^p$,

(iv) if $(h, \alpha, \beta) \in C^p$, then $(h, \sigma(\alpha), \sigma(\beta)) \in C^p$; in particular, $\sigma(\alpha) < \sigma(\beta)$,

(v) $f^p(\alpha, h) = f^p(\sigma(\alpha), h)$ for $\alpha \in A^p$ and $h \in H^p$,

(vi) $\varphi^p(\alpha, h) = \varphi^p(\sigma(\alpha), h)$ provided that $\alpha \in A^p$ and $h \in H^p$.

For $A, B \subset \omega_1$ we write $A \prec B$ if $\alpha < \beta$ whenever $\alpha \in A$ and $\beta \in B$.

4.6 Lemma. *Suppose that $p \in P$ and $\gamma \in \omega_1$ is a limit ordinal. Then there is a γ-good $q \leq p$ so that $A^p \cap \gamma \prec A^q \setminus A^p \prec A^q \setminus \gamma$.*

Proof. If $A^p \subset \gamma$, then $q = p$ does the job, since the identity function $\sigma : A^p \to A^p$ witnesses γ-goodness of p. So assume that $A^p \setminus \gamma \neq \emptyset$. Since A^p is finite and γ is a limit ordinal, one can find $K \subset \gamma \setminus A^p$ with $|K| = |A^p \setminus \gamma|$ and $A^p \cap \gamma \prec K \prec A^p \setminus \gamma$. In particular, there is a map $\sigma : A^p \cup K \to A^p \cup K$ such that $\sigma(\beta) = \beta$ for $\beta \in (A^p \cap \gamma) \cup K$, and $\sigma \restriction_{A^p \setminus \gamma}$ is a bijection from $A^p \setminus \gamma$ to K preserving order, i.e. so that $\alpha, \beta \in A^p \setminus \gamma$ and $\alpha < \beta$ imply $\sigma(\alpha) < \sigma(\beta)$. Define $H^q = H^p$, $A^q = A^p \cup K$,

$$B^q = B^p \cup \{(\sigma''E, F, \sigma(\alpha), n) : (E, F, \alpha, n) \in B^p\},$$

$$C^q = C^p \cup \{(h, \sigma(\alpha), \sigma(\beta)) : (h, \alpha, \beta) \in C^p\},$$

$$f^q = f^p \cup \{\langle\langle \sigma(\alpha), h\rangle,\ f^p(\alpha, h)\rangle : \alpha \in A^p \setminus \gamma,\ h \in H^p\},$$

$$\varphi^q = \varphi^p \cup \{\langle\langle \sigma(\alpha), h\rangle,\ \varphi^p(\alpha, h)\rangle : \alpha \in A^p \setminus \gamma,\ h \in H^p\}$$

and $q = \langle H^q, A^q, B^q, C^q, f^q, \varphi^q\rangle$. An easy verification that $q \in P$ and $q \leq p$ is left to the reader. On the other hand, by our construction σ witnesses that q is γ-good.

4.7 Definition. $P' = \{p \in P : p$ is γ-good for each limit $\gamma \in \omega_1\}$ and $P'_\alpha = \{p \in P' : p|\alpha = p\}$ for $\alpha \in \omega_1$.

4.8 Lemma. *P' is dense in P.*

Proof. For $\mu \in \omega_1$ let $\bar{\mu}$ denote the smallest limit ordinal greater than μ. Fix $r \in P$ and choose $p \leq r$ with $\omega \in A^p$ (Lemma 2.4(iii)). Let $\{\overline{\mu} : \mu \in A^p\} = \{\mu_1, \dots, \mu_n\}$, where $\mu_1 > \mu_2 > \dots > \mu_n$. Starting from $p_0 = p$ use 4.6 to choose, by induction on $i \in (n+1) \setminus \{0\}$, $p_i \in P$ such that $p_0 \geq p_1 \geq \dots \geq p_n$, p_i is μ_i-good and

$$A^{p_{i-1}} \cap \mu_i \prec A^{p_i} \setminus A^{p_{i-1}} \prec A^{p_i} \setminus \mu_i \text{ for } i \in (n+1) \setminus \{0\}.$$

Then observe that $p_n \in P'$.

4.9 Lemma. *Suppose that $\gamma \in \omega_1$, $p, q \in P$, $q|\gamma = q$, $q \leq p|\gamma$ and p is γ-good. Then p and q are compatible in P.*

Proof. Fix σ witnessing that p is γ-good. Define

$$H^s = H^q,\ A^s = A^p \cup A^q,\ B^s = B^p \cup B^q,\ C^s = C^p \cup C^q,$$

$$(3)\ f^s = f^q \cup \{\langle\langle \alpha, h\rangle, f^q(\sigma(\alpha), h)\rangle : \alpha \in A^p \setminus \gamma,\ h \in H^s\},$$

$$(4)\ \varphi^s = \varphi^p \cup \varphi^q \cup \{\langle\langle \alpha, h\rangle, 0\rangle : \alpha \in A^p \setminus \gamma,\ h \in H^s \setminus H^p\}$$

and $s = \langle H^s, A^s, B^s, C^s, f^s, \varphi^s\rangle$.

4.9.1. Claim $s \in P$.

Proof. (1_s) follows from (1_p) and (1_q).

(2_s) Suppose that $(h, \alpha, \beta) \in C^s$. If $(h, \alpha, \beta) \in C^q$, then (2_s) for these h, α, β follows from 4.5(ii) for σ, (2_q) for h, α, β and (3). Otherwise $(h, \alpha, \beta) \in C^p$, and so $\alpha < \beta$ by (2_p). Moreover, 4.5(iv) for σ yields $(h, \sigma(\alpha), \sigma(\beta)) \in C^p$. Since $\sigma(\alpha), \sigma(\beta) \in \gamma$ and $q \leq p|\gamma$, we conclude that $(h, \sigma(\alpha), \sigma(\beta)) \in C^{p|\gamma} \subset C^q$. Therefore $f^s(\alpha, h) = f^q(\sigma(\alpha), h) = 1$. Now suppose that $g_i \in H^s = H^q$ and $f^s(\beta, g_i) = f^q(\sigma(\beta), g_i) = 1$ for each $i \in 2$. Applying (2_q) for $h, \sigma(\alpha), \sigma(\beta)$ and (3) we obtain $f^s(\alpha, h + g_0 + g_1) = f^q(\sigma(\alpha), h + g_0 + g_1) = 1$.

(3_s) First of all let us show that

$$(5) \qquad \theta^s(\{\alpha\}) \subset \theta^q(\{\sigma(\alpha)\}) \text{ whenever } \alpha \in A^p.$$

Indeed, since $\sigma(\alpha) \in A^p \cap \gamma = A^{p|\gamma}$ and $q \leq p|\gamma$, $\varphi^{p|\gamma} \subset \varphi^q$ and so

$$(6) \qquad \theta^p(\{\sigma(\alpha)\}) = \theta^{p|\gamma}(\{\sigma(\alpha)\}) \subset \theta^q(\{\sigma(\alpha)\}).$$

If $\alpha \in A^p \setminus \gamma$, then from (4) and 4.5(vi) it follows that $\theta^s(\{\alpha\}) = \theta^p(\{\alpha\}) = \theta^p(\{\sigma(\alpha)\})$; this, together with (6), immediately yields (5). Finally, for $\alpha \in A^p \cap \gamma$ we have $\sigma(\alpha) = \alpha$ by 4.5(ii), and so $\theta^s(\{\alpha\}) = \theta^s(\{\sigma(\alpha)\}) = \theta^q(\{\sigma(\alpha)\})$ according to (4).

Secondly, as a consequence of (5), for each $E \in [A^p]^{<\omega}$ we have

$$\begin{aligned}(7)\qquad \theta^s(E) &= \cup\{\theta^s(\{\alpha\}) : \alpha \in E\} \\ &\subset \cup\{\theta^q(\{\sigma(\alpha)\}) : \alpha \in E\} \\ &= \cup\{\theta^q(\{\beta\}) : \beta \in \sigma''E\} = \theta^q(\sigma''E).\end{aligned}$$

Now assume that $(E, F, \alpha, n) \in B^s$. If $(E, F, \alpha, n) \in B^q$, then $\alpha \notin E$ and

$$\theta^s(\{\alpha\}) \cap \mathrm{gr}_n\theta^s(E) = \theta^q(\{\alpha\}) \cap \mathrm{gr}_n\theta^q(E) \subset F$$

by (3_q). Otherwise $(E, F, \alpha, n) \in B^p$ and $\alpha \notin E$ by (3_p). Further, $(\sigma''E, F, \sigma(\alpha), n) \in B^p$ by 4.5(iii) for σ. On the other hand, $\sigma''E \subset \gamma$, $\sigma(\alpha) \in \gamma$ and $q \leq p|\gamma$, so $(\sigma''E, F, \sigma(\alpha), n) \in B^{p|\gamma} \subset B^q$. Finally from (5), (7) and (3_q) we obtain

$$\theta^s(\{\alpha\}) \cap \mathrm{gr}_n\theta^s(E) \subset \theta^q(\{\sigma(\alpha)\}) \cap \mathrm{gr}_n\theta^q(\sigma''E) \subset F.$$

4.9.2 Claim. $s \leq p$ and $s \leq q$.

Proof. Inequality $s \leq q$ immediately follows from definition of s. By the choice of s,

$$H^p \subset H^q = H^s,\ A^p \subset A^s,\ B^p \subset B^s,\ C^p \subset C^s \text{ and } \varphi^p \subset \varphi^s.$$

Further, from $q \leq p|\gamma$, 4.5(v) and (3) it follows that $f^p \subset f^s$. Hence to show that $s \leq p$ it remains only to verify $({}_s{\leq}_p)$. So fix $h \in H^s \setminus H^p$ and $\alpha, \beta \in A^p$ with $\varphi^s(\alpha, h) = 1$. Then (4) implies that $\alpha \in A^p \cap \gamma = A^{p|\gamma}$ and $\varphi^q(\alpha, h) = \varphi^s(\alpha, h) = 1$. Since $h \in H^s \setminus H^p = H^q \setminus H^{p|\gamma}$, $\sigma(\beta) \in A^p \cap \gamma = A^{p|\gamma}$ and $q \leq p \mid \gamma$, from $({}_q{\leq}_{p|\gamma})$ we conclude that $f^q(\sigma(\beta), h) = 1$. Now if $\beta \in A^p \cap \gamma$, then $\beta = \sigma(\beta) \in A^p \cap \gamma \subset A^q$ by 4.5(ii) and $q \leq p|\gamma$, so $f^s(\beta, h) = f^q(\beta, h) = f^q(\sigma(\beta), h) = 1$. Otherwise $\beta \in A^p \setminus \gamma$ and $f^s(\beta, h) = f^q(\sigma(\beta), h) = 1$ by (3).

4.10 Lemma. *In addition to hypothesis of Lemma 4.9 assume also that*

(i) $\mu \in A^p \setminus \gamma$,

(ii) $h^* \in H^q \setminus H^p$,

(iii) $f^q(\alpha, h^*) = 1$ *for each* $\alpha \in A^p \cap \gamma$, *and*

(iv) $h^* \notin \mathrm{gr}_n\theta^q(E) \cup (\theta^q(\{\alpha\}) + \mathrm{gr}_n\theta^q(E))$ *provided that* $(E, F, \alpha, n) \in B^{p|\gamma}$.

Then there exists $r \in P$ *such that* $r \leq p$, $r \leq q$ *and* $\varphi^r(\mu, h^*) = 1$.

Proof. Consider the condition s constructed in the proof of Lemma 4.9 and σ from the same proof. Define

$$H^r = H^s,\ A^r = A^s,\ B^r = B^s,\ C^r = C^s,\ f^r = f^s$$

and $\varphi^r = \varphi^s \cup \{\langle\langle \mu, h^*\rangle, 1\rangle\} \setminus \{\langle\langle \mu, h^*\rangle, 0\rangle\}$.

4.10.1 Claim. $r = \langle H^r, A^r, B^r, C^r, f^r, \varphi^r\rangle \in P$.

Proof. First of all note that $\mu \notin A^q$. Indeed, $A^q \subset \gamma$ because $q = q|\gamma$, and $\mu \in A^p \setminus \gamma$ by (i).

(1_r) follows from (1_s).

(2_r) follows from (2_s) since $f^r = f^s$.

(3_r) Suppose $(E, F, \alpha, n) \in B^r = B^s$.Then $\alpha \notin E$ by (3_r).

1°. If $\mu \notin E \cup \{\alpha\}$, then (3_r) for this (E, F, α, n) follows from (3_s).

2°. If $\mu = \alpha$, then, since $\mu \notin A^q$, we conclude that $(E, F, \alpha, n) \in B^p$ in this case. Therefore $(\sigma''E, F, \sigma(\alpha), n) \in B^{p|\gamma} \subset B^q$, because $q \leq p|\gamma$ and 4.5(iii) for σ holds. Moreover (compare the proof of 4.9.1, item (3_s)),

$$\theta^r(\{\alpha\}) = \theta^s(\{\alpha\}) \cup \{h^*\} \subset \theta^q(\{\sigma(\alpha)\}) \cup \{h^*\}$$

and $\mathrm{gr}_n\theta^r(E) = \mathrm{gr}_n\theta^s(E) \subset \mathrm{gr}_n\theta^q(\sigma''E)$, because $\mu \notin E$. Since $(\sigma''E, F, \sigma(\alpha), n) \in B^{p|\gamma}$, (iv) yields $h^* \notin \mathrm{gr}_n\theta^q(\sigma''E)$, and so

$$\begin{aligned}\theta^r(\{\alpha\}) \cap \mathrm{gr}_n\theta^r(E) &\subset (\theta^q(\{\sigma(\alpha)\}) \cup \{h^*\}) \cap \mathrm{gr}_n\theta^q(\sigma''E)\\ &\subset \theta^q(\{\sigma(\alpha)\}) \cap \mathrm{gr}_n\theta^q(\sigma''E) \subset F\end{aligned}$$

by (3_q).

3°. Now consider the last case $\mu \in E$. Since $\mu \notin A^q$, we again have $(E, F, \alpha, n) \in B^p$ and $(\sigma''E, F, \sigma(\alpha), n) \in B^{p|\gamma} \subset B^q$. But now $\alpha \neq \mu$, and so $\theta^r(\{\alpha\}) = \theta^s(\{\alpha\}) \subset \theta^q(\{\sigma(\alpha)\})$. On the other hand,

$$\mathrm{gr}_n\theta^r(E) \subset \mathrm{gr}_n\theta^q(\{\sigma''E\}) \cup (h^* + \mathrm{gr}_n\theta^q(\sigma''E)).$$

Since $(\sigma''E, F, \sigma(\alpha), n) \in B^{p|\gamma}$, applying (iv) once more we conclude that $h^* \notin \theta^q(\{\sigma(\alpha)\}) + \mathrm{gr}_n\theta^q(\sigma''E)$, and hence $\theta^q(\{\sigma(\alpha)\}) \cap (h^* + \mathrm{gr}_n\theta^q(\sigma''E)) = \emptyset$. Now from (3_q) it follows that

$$\begin{aligned} &\theta^r(\{\alpha\}) \cap \mathrm{gr}_n\theta^r(E) \\ &\subset (\theta^q(\{\sigma(\alpha)\}) \cap (\mathrm{gr}_n\theta^q(\sigma''E) \cup (\{h^*\} + \mathrm{gr}_n\theta^q(\sigma''E))) \\ &\subset \theta^q(\{\sigma(\alpha)\}) \cap \mathrm{gr}_n\theta^q(\sigma''E) \subset F. \end{aligned}$$

4.10.2 Claim. $r \leq p$.

Proof. Since $s \leq p$ and $h^* \notin H^p$ by (ii), we have only to check property $({}_r\leq_p)$. So let $h \in H^r \setminus H^p = H^s \setminus H^p$, $\alpha, \beta \in A^p$ and $\varphi^r(\alpha, h) = 1$. If $h \neq h^*$, then $\varphi^s(\alpha, h) = \varphi^r(\alpha, h) = 1$ according to the choice of φ^r, and then $f^r(\beta, h) = f^s(\beta, h) = 1$ by $({}_s\leq_p)$. On the other hand, since $\sigma(\beta) \in A^p \cap \gamma$ in accordance with the choice of σ, $f^r(\beta, h^*) = f^s(\beta, h^*) = f^q(\sigma(\beta), h^*) = 1$ by (3), 4.5(ii) and condition (iii) from the assumption of our lemma.

4.11 Lemma. *Suppose that $\gamma \in \omega_1$, $p, q \in P$, $q|\gamma = q$, $q \leq p|\gamma$ and p is γ-good. Assume also that $\mu \in A^p \setminus \gamma$, $h^* \in H^q$ and*

$$h^* \notin H^p \cup (H^p + \mathrm{gr}_{2|A^p|}\theta^q(A^p \cap \gamma)). \tag{8}$$

Then there is $r \in P$ so that $r \leq p$, $r \leq q$ and $f^r(\mu, h^*) = 0$.

Proof. Let σ witnes that p is γ-good, $A^p \setminus A^q = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ and $\lambda_0 > \lambda_1 \cdots > \lambda_n$. By induction on $i \in n+1$ we will define Z_i as follows. Set

$$Z_0 = \{h \in H^p : f^p(\lambda_0, h) = 1\} \cup (\theta^q(A^p \cap \gamma) \setminus H^p). \tag{9}$$

If $0 < i \leq n$ and $Z_0, \ldots, Z_{i-1}$ have already been defined, then we let

$$\begin{aligned} Z_i = \{h \in H^p : f^p(\lambda_i, h) = 1\} &\cup (\theta^q(A^p \cap \gamma) \setminus H^p) \\ &\cup \{h + Z_j + Z_j : (h, \lambda_i, \lambda_j) \in C^p\}. \end{aligned} \tag{10}$$

(Note that $(h, \lambda_i, \lambda_j) \in C^p$ implies $\lambda_i < \lambda_j$ by (2_p), so all Z_j's mentioned in the third subset of Z_i in (10) were already

defined.) Now define

(11)
$$H^r = H^q,\ A^r = A^p \cup A^q,\ B^r = B^p \cup B^q,\ C^r = C^p \cup C^q,$$

(12)
$$f^r = f^q \cup \{\langle\langle\lambda_j, h\rangle, 1\rangle : j \in n+1,\ h \in Z_j\} \cup \{\langle\langle\lambda_j, h\rangle, 0\rangle : j \in n+1,\ h \in H^q \setminus Z_j\}$$

and

(13)
$$\varphi^r = \varphi^p \cup \varphi^q \cup \{\langle\langle\alpha, h\rangle, 0\rangle : \alpha \in A^p \setminus A^q,\ h \in H^q \setminus H^p\}.$$

It is clear that f^r and φ^r are functions. We claim that $r = \langle H^r, A^R, B^r, C^r, f^r, \varphi^r\rangle$ is as required. The proof of this will be divided into a sequence of claims.

4.11.1 Claim $f^q(\sigma(\lambda_i), h) = 1$ for each $i \in n+1$ and any $h \in Z_i$.

Proof. We will use induction on $i \in n+1$. Let $i = 0$ and $h \in Z_0$. If $h \in H^p$, then $f^p(\lambda_0, h) = 1$ by (9), and since $q \leq p|\gamma$,

$$f^q(\sigma(\lambda_0), h) = f^p(\sigma(\lambda_0), h) = f^p(\lambda_0, h) = 1$$

by 4.5(v). Otherwise $h \in H^q \setminus H^p$ and $\varphi^q(\alpha, h) = 1$ for some $\alpha \in A^p \cap \gamma$ (see (9)). Since $\alpha, \sigma(\lambda_0) \in A^p \cap \gamma = A^{p|\gamma}$ and $q \leq p|\gamma$, from $({}_q\leq_{p|\gamma})$ it follows that $f^q(\sigma(\lambda_0), h) = 1$.

Now suppose that $0 < i \leq n$ and that for $Z_0, Z_1, \ldots, Z_{i-1}$ our claim was verified. Let $h \in Z_i$. If h belongs either to the first or the second subset of Z_i in (10), then the same argument as above works to show that $f^q(\sigma(\lambda_i), h) = 1$. So assume that h belongs to the third subset of Z_i in (10), i.e. there exist $j \in n+1$, $h' \in H^p$ and $z_0, z_1 \in Z_j$ so that $(h', \lambda_i, \lambda_j) \in C^p$ and $h = h' + z_0 + z_1$. Observe that $j < i$ (see remark after (10)), so $f^q(\sigma(\lambda_j), z_0) = f^q(\sigma(\lambda_j), z_1) = 1$ by our inductive hypothesis. Since $q \leq p|\gamma$, from 4.5(iv) it follows that $(h', \sigma(\lambda_i), \sigma(\lambda_j)) \in C^{p|\gamma} \subset C^q$. Thus $f^q(\sigma(\lambda_i), h') = 1$ by (2_q). Applying (2_q) once more, we conclude that

$$f^q(\sigma(\lambda_i), h) = f^q(\sigma(\lambda_i), h' + z_0 + z_1) = 1.$$

4.11.2 Claim. $f^p \subset f^r$.

Proof. Since $q \leq p|\gamma$, (12) yields that $f^{p|\gamma} \subset f^q \subset f^r$. Since $A^p \setminus \gamma = A^p \setminus A^q = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$, it remains only to show that $f^p(\lambda_j, h) = f^r(\lambda_j, h)$ for every $j \in n+1$ and any $h \in H^p$. So fix arbitrarily $j \in n+1$ and $h \in H^p$. If $f^p(\lambda_j, h) = 1$, then $h \in Z_j$ by (9) and (10), and so $f^r(\lambda_j, h) = 1$ by (12). If, on the contrary, $f^p(\lambda_j, h) = 0$, then from 4.5(v) and $q \leq p|\gamma$ it follows that

$$f^q(\sigma(\lambda_j), h) = f^p(\sigma(\lambda_j), h) = f^p(\lambda_j, h) = 0,$$

so 4.11.1 yields that $h \notin Z_j$; therefore, $f^r(\lambda_j, h) = 0$ by (12).

4.11.3 Claim. $r \in P$.

Proof. (1_r) follows from (1_p), (1_q), 4.11.2 and (12).

(2_r) Let $(h, \alpha, \beta) \in C^r$. If $(h, \alpha, \beta) \in C^q$, then (2_r) for these h, α, β follows from (2_q) and (12). So assume that $(h, \alpha, \beta) \in C^p \setminus C^q$. From (2_p) and 4.11.2 it follows that $\alpha < \beta$ and $f^r(\alpha, h) = f^p(\alpha, h) = 1$. In particular, $(h, \alpha, \beta) \in C^p \setminus C^q$ implies that $\beta \in A^p \setminus A^q$, i.e. $\beta = \lambda_j$ for some $j \in n+1$. Further, suppose additionally that $g_0, g_1 \in H^r$ and $f^r(\beta, g_0) = f^r(\beta, g_1) = 1$. Then $g_0, g_1 \in Z_j$ and $f^q(\sigma(\beta), g_0) = f^q(\sigma(\beta), g_1) = 1$ by 4.11.1. Now we have to consider two cases.

1° $\alpha \in A^q$. Since $q|\gamma = q \leq p|\gamma$, from 4.5(ii) and 4.5(iv) it follows that $(h, \alpha, \sigma(\beta)) \in C^q$. Since $f^q(\alpha, h) = f^p(\alpha, h) = 1$, (2_q) yields $f^q(\alpha, h + g_0 + g_1) = 1$, and so $f^r(\alpha, h + g_0 + g_1) = 1$ by (12).

2° $\alpha \in A^p \setminus A^q$. In this case $\alpha = \lambda_i$ for some $i \in n+1$. Since $(h, \lambda_i, \lambda_j) = (h, \alpha, \beta) \in C^p$, (10) implies that $h + g_0 + g_1 \in Z_i$, and so $f^r(\alpha, h + g_0 + g_1) = 1$ by (12).

(3_r) could be verified via the same arguments as in the proof of 4.9.1, item (3_s).

4.11.4 Claim. $r \leq p$.

Proof. In view of (11), 4.11.2 and (13) it suffices only to check property $({}_r \leq_p)$. So let $h \in H^r \setminus H^p = H^q \setminus H^{p|\gamma}$, $\alpha, \beta \in A^p$ and $\varphi^r(\alpha, h) = 1$. From the last equality, (13) and $q|\gamma = q \leq p|\gamma$ we conclude that $\alpha \in A^p \cap A^q = A^p \cap \gamma = A^{p|\gamma}$. If $\beta \in A^p \cap A^q$, then

from (12) and $({}_q \leq_{p|\gamma})$ it follows that $f^r(\beta, h) = f^q(\beta, h) = 1$. Otherwise $\beta \in A^p \setminus A^q$ and so $\beta = \lambda_i$ for some $i \in n+1$. Now notice that $\alpha \in A^p \cap \gamma$, $h \in H^q \setminus H^p$ and $\varphi^q(\alpha, h) = \varphi^r(\alpha, h) = 1$, so $h \in \theta^q(A^p \cap \gamma) \setminus H^p \subset Z_i$ by (9) or (10); finally (12) yields $f^r(\beta, h) = f^r(\lambda_i, h) = 1$.

4.11.5 Claim. $r \leq q$.

Proof. Since $H^r = H^q$, $({}_r \leq_q)$ holds trivially. All other properties immediately follow from (11)–(13).

4.11.6 Claim. $f^r(\mu, h^*) = 0$.

Proof. Using (9), (10) and induction on $i \in n+1$ it can be easily verified that

(14) $Z_i \subset H^p \cup (H^p + \mathrm{gr}_{2^i}\theta^q(A^p \cap \gamma))$ for each $i \in n+1$.

Since $\mu \in A^p \setminus \gamma$, $\mu = \lambda_s$ for some $s \in n+1$. Since $s < n+1 = |A^p \setminus A^q| \leq |A^p|$, from (8) and (14) it follows that $h^* \notin Z_s$. Therefore $f^r(\mu, h^*) = 0$ by (12) as required.

4.12 Definition. Let T be a sentence in the forcing notion P and $\delta \in \omega_1$. We say that T is *δ-definable* provided that, for each $\gamma \in \omega_1 \setminus \delta$ and any γ-good condition $p \in P$, $p \Vdash T$ implies $p|\gamma \Vdash T$.

Note that if T is δ-definable, then T is also δ'-definable for any $\delta' \in \omega_1 \setminus \delta$.

4.13 Lemma. *Each sentence T in the forcing notion P is δ-definable for some $\delta \in \omega_1$.*

Proof. Let $R \subset P$ be a maximal antichain consisting of conditions which force $\neg T$. Since P is *c. c. c.*, $|R| \leq \omega$, and so there exists $\delta \in \omega_1$ such that $r|\gamma = r$ whenever $r \in R$ and $\gamma \in \omega_1 \setminus \delta$ (Lemma 4.4). We will prove that this δ is as required. Indeed, fix $\gamma \in \omega_1 \setminus \delta$ and a γ-good condition $p \in P$ with $p \Vdash T$. To show that $p|\gamma \Vdash T$ it suffices to check that no $t \leq p|\gamma$ forces $\neg T$. If this is not the case, then $t \Vdash \neg T$ for some $t \leq p|\gamma$. Since R is a maximal antichain in $\{r \in P : r \Vdash \neg T\}$, one can

find $s \in P$ and $r \in R$ so that $s \leq t$ and $s \leq r$. For $q = s|\gamma$ we would have $q = s|\gamma \leq t|\gamma \leq p|\gamma$ (see 4.3). Moreover, $q|\gamma = q$ and p is γ-good, so by 4.9 there is $r^* \in P$ such that $r^* \leq p$ and $r^* \leq q$. Now in view of 4.3 and the choice of δ we have $r^* \leq q = s|\gamma \leq r|\gamma = r \in R$, thus $r^* \Vdash \neg T$. On the other hand, $r^* \leq p$ implies $r^* \Vdash T$, a contradiction.

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. A map $j : P \to Q$ is a *dense embedding* provided that j is an injection, $j''P$ is dense in $(Q, \leq_Q)$, and $p \leq_P p'$ is equivalent to $j(p) \leq_Q j(p')$ for all $p, p' \in P$. Any poset P can be densely embedded into the complete Boolean algebra $\tilde{P}$ which is unique up to isomorphism. Posets P and Q are *forcing isomorphic* iff Boolean algebras $\tilde{P}$ and $\tilde{Q}$ are isomorphic. If $j : P \to Q$ is a dense embedding, then P and Q are forcing isomorphic (see [16], Chapter VII, Section 7).

4.14 Lemma. *P is forcing isomorphic to $Fn(\omega_1, \omega)$, the standard poset adding ω_1 Cohen reals. In particular, P and $Fn(\omega_1, \omega)$ provide the same generic extensions.*

Proof. For $p \in P$ and $\gamma \in \omega_1$ define

$$p \parallel \gamma = \langle H^p, A^p \setminus \gamma, B^p \setminus B^{p|\gamma}, C^p \setminus C^{p|\gamma}, f^p \setminus f^{p|\gamma}, \varphi^p \setminus \varphi^{p|\gamma} \rangle.$$

Let $\{\gamma(\alpha) : \alpha \in \omega_1 \setminus \{0\}\}$ be the increasing enumeration of all limit ordinals $\lambda \in \omega_1$. By induction on $\alpha \in \omega_1$ we will define a finite support iteration

$$Q = \langle \langle Q_\alpha : \alpha \in \omega_1 \rangle, \langle \dot{R}_\alpha : \alpha \in \omega_1 \rangle \rangle$$

and dense embeddings $j_\alpha : P'_{\gamma(\alpha)} \to Q_\alpha$, $\alpha \in \omega_1 \setminus \{0\}$, so that $j_\alpha \restriction P'_{\gamma(\beta)} = i_{\beta\alpha} \circ j_\beta$ for $\beta < \alpha$, where $i_{\beta\alpha} : Q_\beta \to Q_\alpha$ is the canonical complete embedding. Set $Q_0 = \{0\}$, $Q_1 = P'_\omega$ and let $j_1 : P'_\omega \to Q_1$ be the identity map. For α limit, let Q_α be the finite support iteration $\langle \langle Q_\beta : \beta \in \alpha \rangle, \langle \dot{R}_\beta : \beta \in \alpha \rangle \rangle$ and $j_\alpha = \cup \{i_{\beta\alpha} \circ j_\beta : \beta \in \alpha\}$. For $\alpha = \beta + 1$ define Q_β-names

$$\dot{R}_\beta = \{\langle (p \parallel \gamma(\beta))^\vee, j_\beta(p|\gamma(\beta)) \rangle : p \in P'_{\gamma(\alpha)}\}$$

and

$$\dot{\leq}_\beta = \{\langle \langle \langle p' \parallel \gamma(\beta), p \parallel \gamma(\beta) \rangle \rangle^\vee, 1_{Q_\beta} \rangle : p, p' \in P'_{\gamma(\alpha)} \text{ and } p' \leq p\}.$$

It is clear that

$$1_{Q_\beta} \Vdash_{Q_\beta} \text{“}(\dot{R}_\beta, \dot{\leq}_\beta) \text{ is a poset with the largest element } 1_\beta = \langle\{0\}, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\rangle\text{”},$$

so (omitting $\dot{\leq}_\beta$) let $Q_\alpha = Q_\beta * \dot{R}_\beta$. Finally, let $j_\alpha(p) = (j_\beta(p|\gamma(\beta)),\ p \parallel \gamma(\beta))$ for all $p \in P'_{\gamma(\alpha)}$.

Let us verify by induction on $\alpha \in \omega_1$ that j_α's are as required. For α limit this is clear, so let $\alpha = \beta + 1$. If $\beta = 0$, then $j_\alpha = j_1$ is trivially a dense embedding, so we will assume that $\beta > 0$. Since j_β is an injection, so is j_α. We leave to the reader to check that $p \leq q$ is equivalent to $j_\alpha(p) \leq j_\alpha(q)$. To show that $j_\alpha{}''P'_{\gamma(\alpha)}$ is dense in Q_α fix $(q, \dot{r}) \in Q_\alpha$. Observe that $1_{Q_\beta} \Vdash_{Q_\beta}$ “$\dot{R}_\beta \subset \{p \parallel \gamma(\beta) : p \in P'_{\gamma(\alpha)}\}$”, so we can find $t \in Q_\beta$ and $\tilde{p} \in P'_{\gamma(\alpha)}$ with $t \leq q$ and $t \Vdash_{Q_\beta}$ “$\tilde{p} \parallel \gamma(\beta) = \dot{r} \in \dot{R}_\beta$”. Since, by inductive hypothesis, $j_\beta{}''P'_{\gamma(\beta)}$ is dense in Q_β, $j_\beta(s) \leq t$ for some $s \in P'_{\gamma(\beta)}$. Now $j_\beta(s) \Vdash_{Q_\beta}$ “$\tilde{p} \parallel \gamma(\beta) = \dot{r} \in \dot{R}_\beta$” implies the existence of $p \in P'_{\gamma(\alpha)}$ such that $j_\beta(p \mid \gamma(\beta))$ and $j_\beta(s)$ are compatible in Q_β, and $p \parallel \gamma(\beta) = \tilde{p} \parallel \gamma(\beta)$. By inductive hypothesis, j_β is a dense embedding, so $p|\gamma(\beta)$ and s are compatible in $P'_{\gamma(\beta)}$. Now 4.9, 4.8 and 4.3 yield that there is $p^* \in P'_{\gamma(\alpha)}$ with $p^* \leq p$ and $p^* \leq s$. Then $j_\alpha(p^*) \leq (q, \dot{r})$. Finally, note that $p \in P'_{\gamma(\beta)}$ implies $p \parallel \gamma(\beta) = 1_\beta$, so $j_\alpha(p) = i_{\beta\alpha} \circ j_\beta(p)$.

Recall that a poset P is *non-atomic* if for every $p \in P$ there exist incompatible $q, r \in P$ with $q \leq p$ and $r \leq p$. The following claim seems to be folklore, but we include its proof for the sake of completeness.

Claim. Suppose that $Q = \langle\langle Q_\alpha : \alpha \in \kappa\rangle, \langle \dot{R}_\alpha : \alpha \in \kappa\rangle\rangle$ is a finite support iteration such that, $1_{Q_\beta} \Vdash_{Q_\beta}$ “$\dot{R}_\beta$ is a countable non-atomic poset” for every $\beta \in \kappa$. Then there is a dense embedding of $Fn(\kappa, \omega)$ into Q.

Proof. By induction on $\alpha \in \kappa$ we will define dense embeddings $j_\alpha : Fn(\alpha \times \omega, \omega) \to Q_\alpha$ such that $j_\alpha \restriction_{Fn(\beta \times \omega, \omega)} = i_{\beta\alpha} \circ j_\beta$ for $\beta < \alpha < \kappa$, where $i_{\beta\alpha} : Q_\beta \to Q_\alpha$ is the canonical complete embedding. For the basis of induction we set $Q_0 = \{0\}$, and

since $Fn(0 \times \omega, \omega) = \{\emptyset\}$, the map $j_0 : Fn(0 \times \omega, \omega) \to Q_0$ sending $\emptyset$ to 0 is obviously a dense embedding. If α is a limit ordinal, then from inductive hypothesis it follows that

$$j_\alpha = \cup\{i_{\beta\alpha} \circ j_\beta : \beta \in \alpha\} : Fn(\alpha \times \omega, \omega) \to Q_\alpha$$

is a dense embedding. So assume that $\alpha = \beta + 1$. Since

$$1_{Q_\beta} \Vdash_{Q_\beta} \text{“}\dot{R}_\beta \text{ is a countable non-atomic poset”},$$

from the proof of [34, Ch.II, Theorem 5.6] we conclude that

$$1_{Q_\beta} \Vdash_{Q_\beta} \text{“there is a dense embedding } J_\beta : Fn(\omega, \omega) \to \dot{R}_\beta \text{ such that } J_\beta(\emptyset) = 1_{\dot{R}_\beta}\text{”},$$

where $Fn(\omega, \omega)$ is the standard poset adding a single Cohen real. By the maximal principle [16, Ch. VII, Theorem 8.2], there exists a Q_β-name $\dot{J}_\beta$ such that

$$1_{Q_\beta} \Vdash_{Q_\beta} \text{“}\dot{J}_\beta : Fn(\omega, \omega) \to \dot{R}_\beta \text{ is a dense embedding such that } \dot{J}_\beta(\emptyset) = 1_{\dot{R}_\beta}\text{”}.$$

For $f \in Fn(\alpha \times \omega, \omega)$ define $f' = f \restriction_{\beta \times \omega}$, and let $f'' \in Fn(\omega, \omega)$ be the function defined by $f''(n) = f(\beta, n)$ for every $n \in \omega$. For every $g \in Fn(\omega, \omega)$ use the maximal principle again to fix a Q_β-name $\dot{r}_g$ such that $1_{Q_\beta} \Vdash_{Q_\beta} \text{“}\dot{J}_\beta(g) = \dot{r}_g\text{”}$. Now one can check that the map $j_\alpha : Fn(\alpha \times \omega, \omega) \to Q_\alpha = Q_\beta * \dot{R}_\beta$ defined by

$$\begin{aligned} j_\alpha &= \{\langle f, \langle j_\beta(f'), \dot{r}_{f''}\rangle\rangle : f \in Fn(\alpha \times \omega, \omega)\} \\ &\subset Fn(\alpha \times \omega, \omega) \times (Q_\beta * \dot{R}_\beta) \end{aligned}$$

is a dense enbedding and $j_\alpha \restriction_{Fn(\beta \times \omega, \omega)} = i_{\beta\alpha} \circ j_\beta$. Since $Fn(\kappa, \omega)$ is isomorphic to $Fn(\kappa \times \omega, \omega)$, the result follows.

One could easily check that $1_{Q_\alpha} \Vdash_{Q_\alpha} \text{“}\dot{R}_\alpha$ is a countable non-atomic poset” for all $\alpha \in \omega_1$, so $Fn(\omega_1, \omega)$ and Q are forcing isomorphic by our Claim. If $i_\alpha : Q_\alpha \to Q$ denotes the canonical complete embedding, then $j : P' \to Q$ defined by $j = \cup\{i_\alpha \circ j_\alpha : \alpha \in \omega_1\}$ is a dense embedding. Now observe that P and P' are forcing isomorphic by 4.8.

5. $(H,\mathcal{T})$ is Fréchet-Urysohn

In what follows we will use $[X]$ for denoting the closure of $X \subset H$ in $(H,\mathcal{T})$.

5.1 Lemma. *Suppose that $s \in P$, $\dot{X}$ is a P-name and $s \Vdash \dot{X} \subset H$ and $\mathbf{0} \in [\dot{X}]$. Then there exist $t \leq s$, $E \in [\omega_1]^{<\omega}$ and $m \in \omega$ so that*

$$t \Vdash \text{“}\mathbf{0} \in [\dot{X} \cap \mathrm{gr}_m \cup \{\dot{S}_\alpha : \alpha \in E\}]\text{”}.$$

Proof. Suppose the contrary, then

(15)
$$s \Vdash \text{“}\dot{X} \subset H \text{ and } \mathbf{0} \in [\dot{X}] \setminus [\dot{X} \cap \mathrm{gr}_m \cup \{\dot{S}_\alpha : \alpha \in E\}] \text{ for each } m \in \omega \text{ and any } E \in [\omega_1]^{<\omega}\text{”}.$$

We are going to find $r \leq s$ which forces the contradiction. For each $h \in H$ Lemma 4.13 allows us to find $\delta_h \in \omega_1$ so that the sentence “$h \in \dot{X}$” is δ_h-definable. Since $|H| \leq \omega$, we can choose a limit ordinal $\gamma \in \omega_1$ with $\gamma > \sup\{\delta_h : h \in H\}$. Then each sentence “$h \in \dot{X}$” is γ-definable. Pick $\mu \in \omega_1 \setminus \gamma$. Since $s \Vdash \text{“}\mathbf{0} \notin [\dot{X} \cap \dot{S}_\mu]\text{”}$, we can use 3.11, 2.3, 2.4(iii) and 4.8 to find a γ-good $p \leq s$ such that $\mu \in A^p$ and

(16)
$$p \Vdash \text{“}\dot{X} \cap \dot{S}_\mu \subset H^p\text{”}.$$

Set

(17)
$$n^* = \max\,(\{n : (E,F,\alpha,n) \in B^p\} \cup \{0\}) + 1.$$

Since $A^p \cap \gamma$ is finite, from (15), 3.6, 3.7, 2.3 and 4.8 it follows that there exist a γ-good $t \leq p$ and $h^* \in H^t \setminus H^p$ with

$$t \Vdash \text{“}h^* \in \cap\{\dot{U}_\alpha : \alpha \in A^p \cap \gamma\} \cap \dot{X} \setminus \mathrm{gr}_{n^*} \cup \{\dot{S}_\alpha : \alpha \in A^p \cap \gamma\}\text{”}.$$

In particular, for $q = t \mid \gamma$ we would have $f^q(\alpha,h^*) = 1$ whenever $\alpha \in A^p \cap \gamma$, and moreover $h^* \notin \mathrm{gr}_{n^*}\theta^q(A^p \cap \gamma)$. Applying (17), we conclude that

$$h^* \notin \mathrm{gr}_n\theta^q(E) \cup (\theta^q(\{\alpha\}) + \mathrm{gr}_n\theta^q(E))$$

for each $(E,F,\alpha,n) \in B^{p\restriction\gamma}$. On the other hand, since the sentence “$h^* \in \dot{X}$” is γ-definable and t is γ-good, $q \Vdash \text{“}h^* \in \dot{X}\text{”}$. Now all the requirements of 4.10 are satisfied, so there

is $r \in P$ such that $r \leq p$, $r \leq q$ and $\varphi^r(\mu, h^*) = 1$. Since $r \leq q \Vdash$ " $h^* \in \dot{X}$", $r \Vdash$ " $h^* \in X \cap \dot{S}_\mu$". Since $r \leq p$, (16) yields $r \Vdash$ " $h^* \in H^p$", a contradiction with $h^* \notin H^p$.

5.2 Lemma. *$M[G] \models$ "$(H, \mathcal{T})$ is Fréchet-Urysohn".*

Proof. We argue in $M[G]$. Since $(H, \mathcal{T})$ is a topological group (see 3.13(i)), it suffices to check the Fréchet-Urysohn property of $(H, \mathcal{T})$ at $\mathbf{0}$. So let $X \subset H$ and $\mathbf{0} \in [X]$. From 5.1 it follows that $\mathbf{0} \in [X \cap \mathrm{gr}_m \cup \{S_\alpha : \alpha \in E\}]$ for some $E \in [\omega_1]^{<\omega}$ and $m \in \omega$. Since each S_α is a convergent sequence (Lemma 3.13(ii)), $Z = (\cup\{S_\alpha : \alpha \in E\})^m$ is a countable compact subspace of $(H, \mathcal{T})^m$. Since the m-fold multiplication $\phi_m : (H, \mathcal{T})^m \to (H, \mathcal{T})$ is continuous, $\mathrm{gr}_m \cup \{S_\alpha : \alpha \in E\} = \phi_m(Z)$ is also a countable compact (hence metrizable) subspace of $(H, \mathcal{T})$. Now $\mathbf{0} \in [X \cap \mathrm{gr}_m \cup \{S_\alpha : \alpha \in E\}]$ yields that there is a sequence of points of X converging to $\mathbf{0}$ in $(H, \mathcal{T})$.

6. $(H, \mathcal{T})$ IS NOT AN $\alpha_{1,5}$-SPACE.

6.1 Lemma. *Suppose that $s \in P$, $\dot{\xi}$ and $\dot{\Omega}$ are P-names and*

(18)

$s \Vdash$ "$\dot{\xi} : \omega \to H$ *is a function,* $\dot{\Omega} \subset \omega$ *is infinite and*

$\dot{S}_m \setminus (\cup\{\dot{S}_n : n < m\} \cup \mathit{ran}\, \dot{\xi})$ *is finite for all* $m \in \dot{\Omega}$".

Then $s \Vdash$ "$\dot{\xi}$ is not a sequence $\dot{\mathcal{T}}$-converging to $\mathbf{0}$".

Proof. If not, then there is $s' \leq s$ so that

(19) $\quad s' \Vdash$ "$\dot{\xi}$ is a sequence $\dot{\mathcal{T}}$-converging to $\mathbf{0}$".

For $i, m \in \omega$ and $K \in [H]^{<\omega}$ use 4.13 to choose $\delta_{i,m,K} \in \omega_1$ so that the sentence

$$\text{"}(\dot{S}_m \setminus (\cup\{\dot{S}_n : n < m\} \cup \mathrm{ran}\, \dot{\xi})) \cup \{\dot{\xi}(j) : j \leq i\} \subset K\text{"}$$

is $\delta_{i,m,K}$-definable. Fix a limit ordinal $\gamma \in \omega_1$ with

$$\gamma > \{\delta_{i,m,K} : i, m \in \omega,\ K \in [H]^{<\omega}\}.$$

Pick $\mu \in \omega_1 \setminus \gamma$. Then use (19), 3.6, 3.7 and 4.8 to find a γ-good $p \leq s'$ and $i \in \omega$ such that

$$p \Vdash \text{“}\dot{\xi}(j) \in \dot{U}_\mu \text{ for all } j > i\text{”}. \tag{20}$$

Since $p \leq s \Vdash$ “$\dot{\Omega} \subset \omega$ is infinite”, we can find $m \in \omega \setminus A^p$ and $p' \leq p$ so that $p' \Vdash$ “$m \in \dot{\Omega}$”. Now from $p' \leq s$, (18), 2.3, 2.4(iii) and 4.8 it follows that there exists a γ-good $t \leq p'$ such that $\{0, 1, \dots, m\} \subset A^t$ and

$$t \Vdash \text{“}(\dot{S}_m \setminus (\cup\{\dot{S}_n : n < m\} \cup \operatorname{ran} \dot{\xi})) \cup \{\dot{\xi}(j) : j \leq i\} \subset H^t\text{”}.$$

Since the last sentence is γ-definable and t is γ-good,

$$t|\gamma \Vdash \text{“}(\dot{S}_m \setminus (\cup\{\dot{S}_n : n < m\} \cup \operatorname{ran} \dot{\xi})) \cup \{\dot{\xi}(j) : j \leq i\} \subset H^t\text{”}. \tag{21}$$

Now use 2.5 to find $q \leq t|\gamma$ and $h^* \in H^q \setminus H^t$ such that $A^q = A^{t|\gamma}$, $\varphi^q(m, h^*) = 1$, and $\varphi^q(\alpha, h) = 0$ whenever $\alpha \in A^q \setminus \{m\}$ and $h \in H^q \setminus H^t$. Since $m \notin A^p \cap \gamma \subset A^q$, H^t is a subgroup of H and $H^p \subset H^t$, we conclude that

$$h^* \notin H^t \supset \operatorname{gr}_\omega(H^p \cup \theta^q(A^p \cap \gamma)). \tag{22}$$

In particular

$$h^* \notin H^t \supset H^p \cup (H^p + \operatorname{gr}_{2^{|A^p|}} \theta^q(A^p \cap \gamma)).$$

Note that $q \mid \gamma = q$, because $A^q = A^{t|\gamma}$. So 4.11 can be applied to find $r \in P$ with $r \leq p$, $r \leq q$ and $f^r(\mu, h^*) = 0$. Since $r \leq q$ and $\{0, 1, \dots, m-1\} \subset A^{t|\gamma} = A^q$, $\varphi^r(m, h^*) = \varphi^q(m, h^*) = 1$ and $\varphi^r(n, h^*) = \varphi^q(n, h^*) = 0$ for $n < m$, i.e. $r \Vdash$ “$h^* \in \dot{S}_m \setminus \cup\{\dot{S}_n : n < m\}$”. Now $r \leq q \leq t \mid \gamma$, $h^* \notin H^t$ and (21) imply that $r \Vdash$ “$h^* = \dot{\xi}(j)$ for some $j > i$”. Since $r \leq p$ and (20) holds, this yields $r \Vdash$ “$h^* \in \dot{U}_\mu$”, a contradiction with $f^r(\mu, h^*) = 0$.

6.2 Lemma. *$M[G] \models$ “$(H, \mathcal{T})$ is not an $\alpha_{1,5}$-space”.*

Proof. We argue in $M[G]$. From 2.4(iii) and 2.5 it follows that $T_m = S_m \setminus \cup\{S_n : n < m\}$ is infinite for every $m \in \omega$. Since each S_m is a sequence converging to $\mathbf{0}$ in $(H, \mathcal{T})$ (see 3.13(ii)), we conclude that $\{T_m : m \in \omega\}$ is a disjoint sheaf at $\mathbf{0}$. Finally,

Lemma 6.1 implies that the set $\{m \in \omega : |T_m \setminus \operatorname{ran} \xi| < \omega\}$ is finite for each sequence ξ that converges to $\mathbf{0}$ in $(H, \mathcal{T})$.

7. Construction of $\mathcal{T}_0$

In this section we will show that for $\mathcal{F} = [H]^{<\omega}$ the topology $\mathcal{T}$ constructed in Section 3 can be taken as $\mathcal{T}_0$ from Main Theorem.

7.1 Lemma. *Let $p \in P$ and $\mu \in \omega_1 \setminus A^p$. Then there exists $r \in P$ so that $r \leq p$, $A^r = A^p \cup \{\mu\}$, $H^r = H^p$ and $\varphi^r(\mu, h) = 1$ for each $h \in H^r$.*

Proof. Define $H^r = H^p$, $A^r = A^p \cup \{\mu\}$, $B^r = B^p$, $C^r = C^p$,

$$f^r = f^p \cup \{\langle\langle \mu, \mathbf{0}\rangle, 1\rangle\} \cup \{\langle\langle \mu, h\rangle, 0\rangle : h \in H^p \setminus \{\mathbf{0}\}\},$$

$$\varphi^r = \varphi^p \cup \{\langle\langle \mu, h\rangle, 1\rangle : h \in H^p\}.$$

We leave to the reader an easy verification that $r = \langle H^r, A^r, B^r, C^r, f^r, \varphi^r\rangle$ is as required.

7.2 Lemma. *Let $\mathcal{F} = [H]^{<\omega}$, $K \in [H]^{<\omega}$, $J \in [\omega_1]^{<\omega}$, $\beta \in \omega_1 \setminus J$ and $k \in \omega$. Then the set*

$$\{q \in P : q \Vdash \text{“}\dot{S}_\beta \cap (K + \operatorname{gr}_k \cup \{\dot{S}_\alpha : \alpha \in J\}) \text{ is finite”}\}$$

is dense in P.

Proof. Fix $p \in P$. In view of 2.3 and 2.4(iii) we can assume, without loss of generality, that $K \subset H^p$, $J \subset A^p$ and $\beta \in A^p$. Choose $\mu \in \omega_1 \setminus A^p$, and let $r \leq p$ be the condition constructed in 7.1. Define

$$H^q = H^r,\ A^q = A^r,\ B^q = B^r \cup \{(J \cup \{\mu\}, H^r, \beta, k+1)\},$$

$$C^q = C^r,\ f^q = f^r,\ \varphi^q = \varphi^r \text{ and } q = \langle H^q, A^q, B^q, C^q, f^q, \varphi^q\rangle.$$

It is clear that $q \in P$ and $q \leq r \leq p$. Furthermore, from 3.12 it follows that

$$q \Vdash \text{“}\dot{S}_\beta \cap \operatorname{gr}_{k+1} \cup \{\dot{S}_\alpha : \alpha \in J \cup \{\mu\}\} \subset H^r\text{”}.$$

Since $K \subset H^r$, and $\varphi^r(\mu, h) = 1$ for each $h \in H^r$, we conclude that

$$q \Vdash \text{“}\dot{S}_\beta \cap (K + \operatorname{gr}_k \cup \{\dot{S}_\alpha : \alpha \in J\}) \subset H^r\text{”}.$$

7.3 Lemma. *Let $\mathcal{F} = [H]^{<\omega}$. Then $\Vdash$ " the set $\{n \in \omega :$ ran $\xi \cap \dot{S}_n$ is infinite$\}$ is finite for every sequence ξ which converges to $\mathbf{0}$ in $(H, \dot{T})$ ".*

Proof. Assume the contrary, and fix $s \in P$ and a P-name $\dot{\xi}$ so that

(23)

$$s \Vdash \text{"}\dot{\xi} : \omega \to H \text{ is a sequence } \dot{T}\text{- converging to } \mathbf{0}, \text{ and } \{n \in \omega : \operatorname{ran} \dot{\xi} \cap \dot{S}_n \text{ is infinite}\} \text{ is infinite"}.$$

For each $j \in \omega$ and every $h \in H$ fix $\delta_{j,h} \in \omega_1$ so that the sentence "$\dot{\xi}(j) = h$" is $\delta_{j,h}$-definable (Lemma 4.13). Let $\gamma \in \omega_1$ be a limit ordinal such that $\gamma > \sup\{\delta_{j,h} : j \in \omega,\ h \in H\}$. Pick $\mu \in \omega_1 \setminus \gamma$. Since

$$s \Vdash \text{"}\dot{\xi} \text{ is a sequence converging to } \mathbf{0} \text{ in } (H, \dot{T})\text{"},$$

we can apply 3.6, 3.7, 2.4(iii) and 4.8 to find $i \in \omega$ and a γ-good $p \in P$ such that $p \leq s$, $\mu \in A^p$ and

$$p \Vdash \text{"}\dot{\xi}(j) \in \dot{U}_\mu \text{ for all } j > i\text{"}. \tag{24}$$

Now use (23) to choose $n \in \omega \setminus A^p$ and $p' \leq p$ so that $p' \Vdash$ " ran $\dot{\xi} \cap \dot{S}_n$ is infinite". From 7.2, 2.3 and 4.8 it follows that there exist a γ-good $t \in P$, $h^* \in H^t$ and $j > i$ such that $t \leq p'$ and

(25)

$$t \Vdash \text{"}\dot{\xi}(j) = h^* \notin H^p \cup (H^p + \mathrm{gr}_{2|A^p|} \cup \{\dot{S}_\alpha : \alpha \in A^p \cap \gamma\})\text{"}.$$

Define $q = t|\gamma$. From (25) it follows that

$$h^* \notin H^p \cup (H^p + \mathrm{gr}_{2|A^p|}\theta^q(A^p \cap \gamma)).$$

Since $t \leq p$, 4.3 implies that $q \leq p|\gamma$. Hence p, q, γ, μ and h^* satisfy all the requirements of 4.11 which yields the existence of $r \in P$ with $r \leq p$, $r \leq q$ and $f^r(\mu, h^*) = 0$, i.e. $r \Vdash$ " $h^* \notin \dot{U}_\mu$". Since t is γ-good and the sentence "$\dot{\xi}(j) = h^*$" is γ-definable, (25) implies that $q \Vdash$ " $\dot{\xi}(j) = h^*$". Now from $r \leq q$ it follows that $r \Vdash$ " $\dot{\xi}(j) \notin \dot{U}_\mu$", a contradiction with $j > i$, $r \leq p$ and (24).

7.4 Lemma. *If $\mathcal{F} = [H]^{<\omega}$, then $M[G] \models$ "$(H,\mathcal{T})$ is not an α_3-space".*

Proof. From 3.13(ii) and 7.3 it follows that, in $M[G]$, $\{S_n : n \in \omega\}$ is a sheaf at $\mathbf{0}$ so that the set

$$\{n \in \omega : S_n \cap \operatorname{ran} \xi \text{ is infinite}\}$$

is finite for each sequence ξ which converges to $\mathbf{0}$ in $(H,\mathcal{T})$.

Proof of Main Theorem, item (i). Set $\mathcal{F} = [H]^{<\omega}$ and combine 3.13, 4.14, 5.2 and 7.4.

8. Construction of $\mathcal{T}_1$

Here we will show that for $\mathcal{F} = \emptyset$ the topology $\mathcal{T}$ constructed in Section 3 can be taken as $\mathcal{T}_1$ from Main Theorem.

8.1 Lemma. *Suppose that $\mathcal{F} = \emptyset$, $s \in P$, $\dot{X}$ is a P-name, $s \Vdash$ "$\dot{X} \subset H$ and $\mathbf{0} \in [\dot{X}] \setminus \dot{X}$", $\gamma \in \omega_1$ is a limit ordinal, and for all $h \in H$ the sentence "$h \in \dot{X}$" is γ-definable. Then $s \Vdash$ "$\dot{S}_\mu \cap \dot{X}$ is infinite" for any $\mu \in \omega_1 \setminus \gamma$.*

Proof. Pick $\mu \in \omega_1 \setminus \gamma$. To verify that $s \Vdash$ "$\dot{S}_\mu \cap \dot{X}$ is infinite" it suffices to find, for $t \leq s$ and $K \in [H]^{<\omega}$ fixed, $r \leq t$ and $h^* \in H \setminus K$ with $r \Vdash$ "$h^* \in \dot{S}_\mu \cap \dot{X}$". So pick a γ-good $p \leq t$ so that $\mu \in A^p$ (see 2.4(iii) and 4.8). Since $p \Vdash$ "$\mathbf{0} \in [\dot{X}] \setminus \dot{X}$", we can use 3.6, 3.7, 2.3 and 4.8 to find a γ-good $q' \leq p$ and $h^* \in H^{q'} \setminus (H^p \cup K)$ with

$$q' \Vdash \text{"}h^* \in \cap\{\dot{U}_\alpha : \alpha \in A^p \cap \gamma\} \cap \dot{X}\text{"}.$$

Set $q = q'|\gamma$. Then $h^* \in H^q \setminus H^p$ and $f^q(\alpha, h^*) = 1$ for $\alpha \in A^p \cap \gamma$. Since $\mathcal{F} = \emptyset$ implies $B^{p|\gamma} = \emptyset$, we can apply 4.10 to obtain $r \in P$ such that $r \leq p$, $r \leq q$ and $\varphi^r(\mu, h^*) = 1$. Since $q' \Vdash$ "$h^* \in \dot{X}$", q' is γ-good and the sentence "$h^* \in \dot{X}$" is γ-definable, $q \Vdash$ "$h^* \in \dot{X}$", and so $r \Vdash$ "$h^* \in \dot{S}_\mu \cap \dot{X}$".

8.2 Lemma. *Assume that $\mathcal{F} = \emptyset$, $s \in P$, $\dot{X}$ is a P-name and*

$$s \Vdash \text{"}\dot{X} \subset H \times \omega \text{ and } \mathbf{0} \in [\dot{X}_n] \setminus \dot{X}_n \text{ for all } n \in \omega\text{"},$$

where $\dot{X}_n = \{\langle \check{h}, p\rangle : p \Vdash$ "$(h,n) \in \dot{X}$"$\}$. Then there is $\mu \in \omega_1$ such that

$$s \Vdash \text{"}\dot{S}_\mu \cap \dot{X}_n \text{ is infinite for all } n \in \omega\text{"}.$$

Proof. For all $h \in H$ and $n \in \omega$ use 4.13 to find $\delta_{h,n} \in \omega_1$ for which the sentence "$h \in \dot{X}_n$" is $\delta_{h,n}$-definable. Choose a limit ordinal $\gamma > \sup\{\delta_{h,n} : h \in H,\ n \in \omega\}$ and $\mu \in \omega_1 \setminus \gamma$. Now 8.1 finishes the proof.

8.3 Corollary *If $\mathcal{F} = \emptyset$, then $M[G] \models$ "$(H,\mathcal{T})$ is an α_2-space".*

Proof of Main Theorem, item (ii). Set $\mathcal{F} = \emptyset$ and combine 3.13, 4.14, 5.2 and 8.3.

9. Addendum to Main Theorem

One can easily see that in case $\mathcal{F} = \emptyset$ the condition (3_p) from Definition 3.1 is superfluous, so in this case poset P can be simplified by dropping B^p and condition (3_p) from $p \in P$. The only reason for involving so complicated poset in solution of "$\alpha_2 \Rightarrow \alpha_1$ problem" is author's intention to present a unified approach to both problems and to provide their solutions by complete proofs, and also the question of size of the paper. In fact, to construct the topology $\mathcal{T}_1$ with the properties described in Main Theorem one can use the following poset P^* which is much more simple than P.

9.1 Definition. A condition $p \in P^*$ is a structure $p = \langle H^p, A^p, f^p, \varphi^p, \rangle$, where H^p is a finite subgroup of H, $A^p \in [\omega_1]^{<\omega}$ and $f^p : A^p \times H^p \to 2$, $\varphi^p : A^p \times H^p \to 2$ are functions satisfying the following property:

$(\bullet_p)$ For every $\alpha \in A^p$ the set $H^p_\alpha = \{h \in H^p : f^p(\alpha, h) = 1\}$ is a subgroup of H.

For $p, q \in P^*$ we define $q \leq p$ iff $H^p \subset H^q$, $A^p \subset A^q$, $f^p \subset f^q$, $\varphi^p \subset \varphi^q$ and

$({}_q\leq_p)$ if $h \in H^q \setminus H^p$, $\alpha, \beta \in A^p$ and $\varphi^q(\alpha, h) = 1$, then $f^q(\beta, h) = 1$.

One may easily check that all results of Sections 2-4 except for 4.11 remain valid for P^* instead of P — simply replace P by P^*, cut B^p's and C^p's, drop verifications of (2_p) and (3_p) in the proofs and delete items (iii) and (iv) from Definition 4.5. (Observe that 3.5 now trivially satisfied by letting $\beta = \alpha$.) As

for 4.11, its analogue, say 4.11*, also holds if we substitute P by P^* and replace condition (8) by the stronger condition

$$h^* \notin \mathrm{gr}_\omega(H^p \cup \theta^q(A^p \cap \gamma)). \tag{26}$$

To see this, define $r = \langle H^r, A^r, f^r, \varphi^r\rangle$ by the same rules as in the proof of 4.11, but for each $i \in n+1$ replace old Z_i by

$$Z_i = \mathrm{gr}_\omega(\{h \in H^p : f^p(\lambda_i, h) = 1\} \cup (\theta^q(A^p \cap \gamma) \setminus H^p)). \tag{27}$$

Since each Z_i is a subgroup of $H^r = H^q$, $r \in P^*$. Further, $f^{p|\gamma} \subset f^q \subset f^r$ and $Z_i \cap H^p = \{h \in H^p : f^p(\lambda_i, h) = 1\}$ for every $i \in n+1$, so $f^p \subset f^r$. From (13), the equality $H^r = H^q$ and $q \leq p|\gamma$ we conclude that $({}_r \leq_p)$ holds, so $r \leq p$; and since $H^r = H^q$, the property $({}_r \leq_q)$ holds trivially, which immediately yields $r \leq q$. As was noted in the proof of 4.11.6, $\mu = \lambda_s$ for some $s \in n+1$. Finally, from (26) and (27) we conclude that $h^* \notin Z_s$, and so $f^r(\mu, h^*) = 0$. This finishes the proof of 4.11*.

Let $\mathcal{T}_1^*$ be the topology constructed via P^* the same way as $\mathcal{T}_1$ was constructed from P. The argument from Section 6 works to show that $(H, \mathcal{T}_1^*)$ fails to be an $\alpha_{1,5}$-space (note that condition (26) coincides with (22), so 4.11* can be applied!). We will show in Lemma 9.2 below that $(H, \mathcal{T}_1^*)$ is Fréchet-Urysohn, i.e. 5.2*, the analogue of 5.2 for P^*, holds. Since in the proofs in Section 8 only results from Sections 2–4 and 5.2* are applied, $(H, \mathcal{T}_1^*)$ is an α_2-space. So summarizing what was said above, we see that $\mathcal{T}_1^*$ has the same properties as $\mathcal{T}_1$ from our Main Theorem. Moreover, P^* is forcing isomorphic to $Fn(\omega_1, \omega)$. The first advantage of $\mathcal{T}_1^*$ with respect to $\mathcal{T}_1$ is that, in view of $(\bullet_p)$ for all $p \in P^*$, $\mathcal{T}_1^*$ has a base of open neighbourhoods of $\mathbf{0}$ consisting of subgroups of H. To display the second one, we need the following

9.2 Lemma. *If G is P^*-generic over M, then $M[G] \models$ "$(H, \mathcal{T}_1^*)^k$ is Fréchet-Urysohn for all $k \in \omega$".*

Proof. Fix $k \in \omega$, and let $[X]_k$ denote the closure of $X \subset H^k$ in $(H, \mathcal{T}_1^*)^k$ and $\mathbf{0}_k = (0, \ldots, 0) \in H^k$. Since in $M[G]$ each S_μ,

$\mu \in \omega_1$, is a sequence $\mathcal{T}$-converging to $\mathbf{0}$ (analogue of 3.13(ii) for $\mathcal{T}_1^*$), to check our lemma it suffices to show that

$$M[G] \models \text{“if } X \subset H^k \text{ and } \mathbf{0}_k \in [X]_k, \text{ then } \mathbf{0}_k \in [X \cap (S_\mu)^k]_k \text{ for some } \mu \in \omega_1\text{”}.$$

In its turn to prove this it suffices to verify that if $s \in P^*$, $\dot{X}$ is a P^*-name and

$$s \Vdash \dot{X} \subset H^k \text{ and } \mathbf{0}_k \in [\dot{X}]_k, \tag{28}$$

then there is $\mu \in \omega_1$ such that, whenever $t \leq s$ and $K \in [H]^{<\omega}$, $r \Vdash$ “ $(\dot{S}_\mu \setminus K)^k \cap \dot{X} \neq \emptyset$” for some $r \leq t$. So fix $s \in P^*$ and a P^*-name $\dot{X}$ satisfying (28). Choose a limit ordinal $\gamma \in \omega_1$ such that for any $(h_0, \dots, h_{k-1}) \in H^k$ the sentence “$(h_0, \dots, h_{k-1}) \in \dot{X}$” is γ-definable. Pick $\mu \in \omega_1 \setminus \gamma$, and fix $t \leq s$ and $K \in [H]^{<\omega}$. Choose a γ-good $p \leq t$ with $\mu \in A^p$ and $K \subset H^p$. Then use (28) to find a γ-good $q' \leq t$ and $h_0^*, \dots, h_{k-1}^* \in H^{q'}$ (not necessary distinct) so that

$$q' \Vdash \text{“}(h_0^*, \dots, h_{k-1}^*) \in (\cap\{\dot{U}_\alpha \setminus H^p : \alpha \in A^p \cap \gamma\})^k \cap \dot{X}\text{”}.$$

Let $q = q'|\gamma$. Then $f^q(\alpha, h_j^*) = 1$ and $h_j^* \notin H^p$ whenever $\alpha \in A^p \cap \gamma$ and $j \in k$. Further, $q \Vdash$ “ $(h_0^*, \dots, h_{k-1}^*) \in \dot{X}$”, because q' is γ-good and the sentence “$(h_0^*, \dots, h_{k-1}^*) \in \dot{X}$” is γ-definable. Now let σ witnes that p is γ-good, and define $H^r = H^q$, $A^r = A^p \cup A^q$,

$$f^r = f^p \cup f^q \cup \{\langle (\alpha, h), f^q(\sigma(\alpha), h)\rangle : \alpha \in A^q \setminus A^p,\ h \in H^q \setminus H^p\}$$

and

$$\begin{aligned} \varphi^r \ = \ & \varphi^p \cup \varphi^q \cup \{\langle (\mu, h_j^*), 1\rangle : j \in k\} \\ & \cup \{\langle (\mu, h), 0\rangle : h \in H^q \setminus (H^p \cup \{h_j^* : j \in k\})\} \\ & \cup \{\langle (\alpha, h), 0\rangle : \alpha \in A^p \setminus (A^q \cup \{\mu\}),\ h \in H^q \setminus H^p\}. \end{aligned}$$

One can easily check that $r = \langle H^r, A^r, f^r, \varphi^r\rangle \in P^*$, $r \leq p$ and $r \leq q$. Finally note that

$$r \Vdash \text{“}(h_0^*, \dots, h_{k-1}^*) \in (\dot{S}_\mu \setminus K)^k \cap \dot{X}\text{”}.$$

Now, in $M[G]$, $(H, \mathcal{T}_1^*)$ is an α_2-space (analogue of 8.3 for $\mathcal{T}_1^*$), and $(H, \mathcal{T}_1^*)^k$ is Fréchet-Urysohn for all $k \in \omega$ (Lemma

9.2), so $(H, \mathcal{T}_1^*)^\omega$ is a w-space ([22], Corollary 3.8). Thus we have verified the following

9.3 Addendum to Main Theorem *In addition to the conclusion of Main Theorem we can assume that $(H, \mathcal{T}_1^*)^\omega$ is a w-space and that $\mathcal{T}_1$ has a base of open neighbourhoods of* **0** *consisting of subgroups.*

In connection with this Addendum it might be interesting to know whether the group $(H, \mathcal{T}_0)$ from Main Theorem can be chosen to have a base of open neighbourhoods of **0** consisting of subgroups. I am inclined to an opinion that this is impossible. Moreover, I could propose the following

9.4 Conjecture Each Fréchet-Urysohn group having a base of open neighbourhoods of its neutral element consisting of subgroups is a w-space.

Historical remarks. The trick of using a special function σ for preserving necessary information in the process of cutting a condition, employed in the definition of a γ-good condition, was implicitly introduced by the author in [30] (observe that the property (10) from item 4 of this paper can be restated in terms of the existence of a special function $\sigma : B^p \to B^p \cap A_0$; see also the property (14) in [32, Definition 2.1.2] for detailed exposition). Such a function usually garantees that some subposets of a poset are completely embedded in it. In our case Lemma 4.9 actually states that P_γ is completely embedded in P for every limit ordinal $\gamma \in \omega_1$ (and $p \mid \gamma$ is a reduction of $p \in P$ to P_γ in the sense of [16, Chapter VII, Definition 7.1]).

The notion of a δ-definable sentence is the precise representation of a rather common idea that one could cut a condition, or more generally, change it in a somewhat more complicated way, without losing some piece of information this condition forces. This idea, based essentially on the concept of a minimally forcing set of conditions introduced, by the way, by Paul Cohen (see definition before [6, Chapter IV, Lemma 8.7]), was exploited in a non-trivial manner by Hechler [13], Bell [5], Malyhin [19] and probably by many others. In particular, based

upon this fruitful idea, Malyhin [19, Section 5] discovered an ingenious way of getting Fréchet-Urysohn non-metrizable groups in forcing extensions. The author would like to emphasize that it is Malyhin's ideas from [19, Section 5] which were the starting (but afterwards become far off) point of his constructions.[1]

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[1]In fairness it should be mentioned that, although Malyhin's ideas from [19, Section 5] got the author an opportunity to understand better how one might construct Fréchet-Urysohn group topologies in forcing extensions, the author was unable to follow the arguments from the proof of [19, Proposition 5.4] (referred to afterwards simply as "the Proof"). Indeed, imagine that $\langle\{\delta,\mu\},i\rangle \in \mathcal{E}^s$ for some $\mu < \theta$ and $i \in \omega$ (we will use the original notations from the Proof). Suppose also that the following property holds:

(*) Whenever $q \in \mathcal{P}$, $q \leq s$, $m \in \Omega$ and $q \Vdash$ "$\check{m} \in \underline{B}$", then there is $n \in (\operatorname{dom} d^q) \setminus (i+1)$ so that $d^q(n) = \langle m, m'\rangle$ and $\mathcal{A}^q_\mu(m') = 1$.

Now if this situation somehow happens (and the author could not extract from the existing proof any indication how to avoid this), then the argument presented in the Proof unfortunately does not work. Moreover, in this case the quadruple $r = \langle \mathcal{A}^r, d^r, \mathcal{E}^r, T^r\rangle$ constructed in the Proof *is not* even a condition, i.e. $r \notin \mathcal{P}$ in contrast with what is claimed there. To see this observe that $q \in \mathcal{P}$ and $m \in \Omega$ chosen in the Proof satisfy the assumption of (*), so we can pick n in accordance with the conclusion of (*). Since $\mathcal{A}^r_\delta(m) = \mathcal{A}^r_\mu(m') = 1$ and $\langle\{\delta,\mu\},i\rangle \in \mathcal{E}^s \subset \mathcal{E}^r$ by the choice of r, if r really *were* a condition, then the property 3c) for this r would yield $n \leq i$, a contradiction with $n \in \omega \setminus (i+1)$.

However, if in the Proof someone drops items 2) and 3) from the definition of a condition and items 3) and 4) from the definition of the comparison relation between conditions, then it can be easily seen that the resulting group would be Fréchet-Urysohn and non-metrizable (but, of course, now nobody could state that its square is not Fréchet-Urysohn). Therefore the following question seems to remain open:

Question. Is there a countable Fréchet-Urysohn group G such that $G \times G$ is not Frechét-Urysohn?

REFERENCES

[1] A.V. Arhangel'skii, *Classes of topological groups*, Uspekhi Matem. Nauk **36**(3) (1981), 127–146 (in Russian, English trasl. in: Russian Math. Surveys **36**(3) (1981), 151–174).

[2] A.V. Arhangel'skii, *The frequency spectrum of a topological space and the classification of spaces*, Doklady Akad. Nauk SSSR **206** (1972), 265–268 (in Russian, English trasl. in: Soviet Math. Doklady **13** (1972), 265–268).

[3] A.V. Arhangel'skii, *The frequency spectrum of a topological space and the product operation*, Trudy Mosk. Mat. Obšč. **40** (1979), 171–206 (in Russian, English trasl. in: Trans. Moscow Math. Soc. 1981 (Issue 2), 163–200).

[4] A.V. Arhangel'skii and V.I. Ponomarev, *On dyadic spaces*, Doklady Akad. Nauk SSSR **182** (1968), 993–996 (in Russian, English transl. in: Soviet Math. Doklady 9 (1968), 1220–1224).

[5] M.G. Bell, *The space of complete subgraphs of a graph*, Comment. Math. Univ. Carolinae **23** (1982), 525–536.

[6] P.J. Cohen, *Set theory and the Continuum hypothesis*, Benjamin, New–York–Amsterdam, 1966.

[7] E.K. van Douwen, *The integers and topology*, in: Handbook of set-theoretic topology (K. Kunen and J.E. Vaughan, Eds.), North-Holland, Amsterdam, 1984, p. 111–165.

[8] A. Dow, *Two classes of Fréchet-Urysohn spaces*, Proc. Amer. Math. Soc. **108** (1990), 241–247.

[9] A. Dow and J. Steprāns, *A model in which countable Fréchet α_1-spaces are first countable*, Math. Proc. Cambridge Philosophical Soc. **105** (1989),

[10] A. Dow and J. Steprāns, *Countable Fréchet α_1-spaces may be first countable*, in: Dept. of Math. of York Univ., Report no. 90–20.

[11] J. Gerlits and Zs. Nagy, *On Fréchet spaces*, Rend. Circ. Matem. Palermo, Ser. II, **18** (1988), 51–71.

[12] G. Gruenhage, *Infinite games and generalizations of first countable spaces*, Gen. Topol. Appl. **6** (1976), 339–352.

[13] S.H. Hechler, *Short complete nested sequences in $\beta\mathbb{N} \setminus \mathbb{N}$ and small maximal almost disjoint families*, Gen. Topol. and Appl. **2** (1972), 139–149.

[14] E. Hewitt and K.A. Ross, Abstract harmonic analysis, vol. 1. Topological groups. Integration theory. Group representations, Springer–Verlag, Berlin–Göttingen–Heidelberg, 1963.

[15] L.N. Ivanovskii, *On a hypothesis of P.S. Alexandroff*, Doklady Akad. Nauk SSSR N.S. **123** (1958), 785–786 (in Russian).

[16] K. Kunen, Set theory. An introduction to independence proofs, North-Holland, Amsterdam, 1980.

[17] V. Kuz'minov, *On a hypothesis of P.S. Alexandroff in the theory of topological groups*, Doklady Akad. Nauk SSSR N.S. **125** (1959), 727–729 (in Russian).

[18] R. Laver, *On the consistency of Borel's Conjecture*, Acta Math. **137** (1976), 151–169.

[19] V.I. Malyhin, *Results on consistency in topology*, Trudy Mosk. Mat. Obšč. **49** (1986), 141–166 (in Russian, English trasl. in: Trans. Moscow Math. Soc. 1987, 145–171).

[20] V.I. Malyhin and B.È. Šapirovskii, *Martin's Axiom and properties of topological spaces*, Doklady Akad. Nauk SSSR **213** (1974), 532–535 (in Russian, English trasl. in: Soviet Math. Doklady 14 (1974), 1746–1751).

[21] T. Nogura, *A counterexample for a problem of Arhangel'skii concerning the product of Fréchet spaces*, Topol. Appl. **25** (1987), 75–80.

[22] T. Nogura, *Fréchetness of inverse limits and products*, Topol. Appl. **20** (1985), 59–66.

[23] T. Nogura, *Products of sequential convergence properties*, Czechoslov. Math. J. **39** (114) (1989), 262–279.

[24] T. Nogura, *The product of $\langle\alpha_i\rangle$-spaces*, Topol. Appl. **21** (1985), 251–259.

[25] P. Nyikos, *Metrizability and the Fréchet-Urysohn property in topological groups*, Proc. Amer. Math. Soc. **83** (1981), 793–801.

[26] P. Nyikos, *The Cantor tree and the Fréchet-Urysohn property*, in: Papers on general topology and related category theory and topological algebra (Ann. New York Acad. Sci. 552), New York Acad. Sci., New York, 1989, pp. 109–123.

[27] P. Nyikos, *Topological groups and vector spaces defined with the help of a single filter*, preprint

[28] P. Nyikos, *${}^{\omega}\omega$ and the Fréchet-Urysohn and α_i-properties*, preprint

[29] E.A. Reznichenko, *On the number of countable Fréchet-Urysohn spaces*, in: Continuous functions on topological spaces, Latv. Gos. Univ., Riga, 1986, pp. 147–154 (in Russian, English summary in: Math. Reviews 88h: 54016).

[30] D.B. Shakhmatov, *A regular symmetrizable L-space*, C. R. Acad. Bulgare Sci. **40**(11) (1987), 5–8.

[31] D.B. Shakhmatov, *Convergence properties in topological groups*, in: Abstracts of Fifth Intern. Conf. on Topology and its Appl. held at Dubrovnik, June 18-23, 1990 (Zagreb, Union of Math., Phys. and Astron. of Yugoslavia, 1990), p. 74.

[32] D.B. Shakhmatov, *Lindelöf property and separability in regular symmetrizable spaces*, Trudy sem. I.G. Petrovskogo **15** (1991), 196–220 (in Russian; English transl. will appear in J. Soviet Math.)

[33] P.L. Sharma, *Some characterizations of W-spaces and w-spaces*, Gen. Topol. Appl. **9** (1978), 289–293.

[34] S. Shelah, *Proper forcing* (Lecture Notes in Math. 940) (Springer-Verlag, Berlin–Heidelberg–New York, 1982).
[35] P. Simon, *A compact Fréchet space whose product is not Fréchet*, Comment. Math. Univ. Carolinae **21** (1980), 749–753.

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