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A STUDY OF D-SPACES

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ABSTRACT. The key results herewith are: (1) Semistratifiable spaces are D-spaces. (2) Perfect inverse images and closed images of D-spaces are D-spaces. (3) Paracompact p-spaces are D-spaces. (4) Monotonically normal D-spaces are paracompact. Several applications demonstrate the usefulness of D-spaces. Several interesting questions remain open.

A *neighborhood assignment* for a topological space (X, τ) is a function $\varphi : X \to \tau$ such that each $x \in \varphi(x)$. X is a *D-space* if, for each neighborhood assignment φ, there exist a *closed discrete* subset D_φ of X such that $\{\varphi(d) \mid d \in D_\varphi\}$ covers X.

The first published results on D-spaces appear in [8], where it is proved that finite products of Sorgenfrey lines are D-spaces. As stated in [9], E. van Douwen proved that GO-spaces are paracompact if and only if they are D-spaces; the proof of this result is known to a privileged few, but it has not yet been published. Our results, which follow, establish that paracompact p-spaces and semistratifiable spaces are D-spaces. Curiously, van Douwen's result does not imply ours (it does not even imply that metrizable spaces are D-spaces) and ours do not appear to imply his. Therefore, the search for a "common denominator" is an open problem.

Several interesting problems, relating to D-spaces, remain open: Are subparacompact spaces D-spaces? Is the countable cartesian product of Sorgenfrey lines a D-space? Is the finite product of irrational Sorgenfrey lines (i.e. the subspace of irrational numbers of the Sorgenfrey line) a D-space?

Theorem 1. *Semistratifiable spaces are D-spaces.*

Proof. Let (X,τ) be a semistratifiable space. To each $U \in \tau$ assign a sequence $\{F(U,n)\}_{n<\omega}$ of closed subsets of X such that $U = U_{n<\omega}F(U,n)$ and $F(U,n) \subset F(V,n)$ whenever $U \subset V$.

Let $\mathcal{U} = \{U_X \mid x \in X\}$ be a neighborhood assignment in X. For each $n \in \omega$, let $\mathcal{U}_n = \{U_x \mid x \in F(U_x, n)\}$ and $X_n = \{x \in X \mid U_x \in \mathcal{U}_n\}$. Clearly, each $X_j \subset X_{j+1}$ and $X = \cup_{j<\omega}X_j$.

Let j_0 be the smallest element of ω such that $\mathcal{U}_{j_0}$ (and X_{j_0}) is nonempty. By transfinite induction, pick $U_{x_\alpha} \in \mathcal{U}_{j_0}, \alpha < \gamma_0$, such that

(i) $\alpha < \beta < \gamma_0$ implies $x_\beta \notin U_{x_\alpha}$,

(ii) $X_{j_0} \subset \cup_{\alpha<\gamma_0} U_{x_\alpha}$.

It follows that $D_0 = \{x_\alpha \mid \alpha < \gamma_0\}$ is a closed discrete subset of X: Pick $z \in D_0^-$ and note that $z \in \cup_{\alpha<\gamma_0}U_{x_\alpha}$, because $D_0 \subset F(\cup_{\alpha<\gamma_0}U_{x_\alpha}, j_0) \subset \cup_{\alpha<\gamma_0}U_{x_\alpha}$.

Let α_0 be the smallest element of γ_0 such that $z \in U_{x_{\alpha_0}}$. Then $V = U_{x_{\alpha_0}} - F(U_{\alpha<\alpha_0}U_{x_\alpha}, j_0)$ is a neighborhood of z such that $V \cap D_0 = \{x_{\alpha_0}\}$, and this also shows that D_0 is discrete.

Letting $X_0' = \cup_{\alpha<\gamma_0}U_{x_\alpha}$, let j_i be the smallest element of ω such that $X_{j_1} - X_0' \neq \emptyset$. Then, let $\mathcal{U}_{j_1} = \{U_x \in \mathcal{U}_{j_1} \mid x \in X_{j_1} - X_0'\}$. Again, by transfinite induction, pick $U_{x_\alpha} \in \mathcal{U}_{j_1}, \alpha < \gamma_1$, such that

(i) $\alpha < \beta < \gamma_1$ implies $x_\beta \notin U_{x_\alpha}$,

(ii)$X_{j_1} - X_{j_0}' \subset \cup_{\alpha<\gamma_1}U_{x_\alpha}$.

Again, we get that $D_1 = \{x_\alpha \mid \alpha < \gamma_1\}$ is a closed discrete subset of X and, letting $X_1' = \cup_{\alpha<\gamma_1}U_{x_\alpha}$, we get that $X_{j_1} \subset X_0' \cup X_1'$.

It is now clear that, by (ordinary) induction, we can find integers $j < 0, j < 1, j < 2, \ldots$, and closed discrete subsets $D_i \subset X_{j_i}, i \in \omega$, such that

(1) each $X_{j_i} \subset \cup\{U_x \mid x \in \cup_{k=0}^{i} D_k\}$,

(2) each $D_{i+1} \cap (\cup\{U_x \mid x \in \cup_{k=0}^{i} D_k\}) = \emptyset$.

Finally, let $D = \cup_{i\in\omega}D_i$ and note that $\{U_x \mid x \in D\}$ covers X, because of (1).

Next, let us prove that D is a closed discrete subset of X: Pick $z \in X$, and let n be the smallest integer such that $z \in$ some U_{x_α} with $x_\alpha \in D_n$. Then $V = (U_{x_\alpha} - \cup_{k=0}^{n} D_k) \cup \{x_\alpha\}$ is a neighborhood of z such that $V \cap D = \{x_\alpha\}$, which proves that D is closed and discrete.

Consequently, since D is closed and discrete and $\{U_x \mid x \in D\}$ covers X we get that X is a D-space, which completes the proof.

Theorem 2. *The perfect inverse image of a D-space is a D-space.*

Proof. Let $f : X \to\!\!\to Y$ be a perfect map from X onto a D-space Y. let $\{U_x \mid x \in X\}$ be a neighborhood assignment in X. For each $x \in X$, pick finitely many $x_1, \ldots, x_{i(x)} \in f^{-1}f(x)$ such that $f^{-1}f(x) \subset U_{x_1} \cup \ldots \cup U_{i(x)}$, and then pick open inverse set U_x^* such that $f^{-1}f(x) \subset U_x^* \subset U_{x_1} \cup \ldots \cup U_{i(x)}$ (recall that f is closed). Let $\varphi : Y \to X$ be a function such that $\varphi(y) \in f^{-1}(y)$ (no continuity required). Let $\{V_y \mid y \in Y\}$ be the neighborhood assignment in Y defined by $V_y = f(U^*_{\varphi(y)})$. Since Y is a D-space, let D be a closed discrete subset of Y such that $\{V_y \mid y \in D\}$ covers Y. Let $D^* = \cup\{\{x_1, \ldots, x_{i(x)}\} | y \in D$ and $x = \varphi(y)\}$.

D^* is closed and discrete: Note that $F^{-1}(D)$ is closed subset of X which is the union of the discrete collection $\{f^{-1}(y) | y \in D\}$ of compact subsets of X. Since $D^* \cap f^{-1}(y)$ is a finite set, for each $y \in D$, it follows immediately that D^* is closed and discrete.

Since $\{U_x \mid x \in D^*\}$ clearly covers X, the proof is complete.

Remark. Note that Theorem 2 remains valid if the fibers of f are D-spaces, a condition which is much weaker than compactness.

Corollary 3. *If X is a D-space and Y is a compact space, then $X \times Y$ is a D-space.*

Proof. The projection map $X \times Y \to X$ is a perfect map. Theorem 2 completes the proof.

Corollary 4. *Paracompact p-spaces (or M-spaces) are D-spaces.*

Proof. It is well-known that a space X is a paracompact p-space if and only if X is a paracompact M-space if and only if X is the perfect inverse image of a metrizable space. Theorem 2 completes the proof.

Corollary 5. *For an M-space X, the following are equivalent.*

(a) X is paracompact,

(b) X is a D-space,

(c) X is subparacompact,

(d) X is metric-refinable (i.e. for each open cover $\mathcal{U}$ of X there exists a metric space M and a continuous function $f : X \twoheadrightarrow M$ such that $\{f^{-1}(y) \mid y \in M\}$ refines $\mathcal{U}$).

Proof. From [10] we get that X is an M-space if and only if there exists a quasi-perfect map $f : X \twoheadrightarrow M$ onto a metrizable space M (i.e. f is closed, continuous and the fibers of f are countably compact). Since countably compact D-space (or subparacompact spaces) are easily seen to be compact, Theorem 2 completes the proof that (a), (b) and (c) are equivalent. Theorem 2.2 of [12] says that (d) implies (a) and Theorem 1 of [11] proves that (a) implies (d). This completes the proof.

Theorem 6. *The closed continuous image of a D-space is a D-space.*

Proof. Let $f : X \twoheadrightarrow Y$ be a closed continuous map from a D-space X onto a space Y. Let $\{U_y \mid y \in Y\}$ be a neighborhood assignment in Y. For each $y \in Y$ and $x \in f^{-1}(y)$, let $V_x = f^{-1}(U_y)$. Then $\{V_x \mid x \in X\}$ is a neighborhood assignment in X. Let D be a closed discrete subset of X such that $\{V_x \mid x \in D\}$ covers X. Pick $D' \subset D$ such that $D' \cap f^{-1}(y)$ is a singleton, for each $y \in Y$. Thus $\{V_x \mid x \in D'\}$ also covers X (note that,. if $x_1, x_2 \in D \cap f^{-1}(y)$ then $V_{x_1} = V_{x_2}$).

Since D' is still closed and discrete, we easily get that $f(D')$ is also closed and discrete. Since $f|D'$ is one-to-one, let $D'' = \{y(x) \in Y \mid x \in D' \text{ and } f(x) = y(x)\}$ Since $\{U_{y(x)} \mid y(x) \in D''\}$ covers Y and D'' is closed and discrete subset of Y, we are done.

It is obvious that closed subspaces of D-spaces are D-spaces. A non-trivial fact is that F_σ-subspaces of D-spaces are also D-spaces. Open subspaces of D-spaces may fail to be D-spaces. (The space ω_1 of countable ordinals, with the order topology, is countably compact but not compact; hence, ω_1 is not a D-space, but $\omega_1 + 1$ is a D-space.)

Proposition 7. *If X is the countable union of closed D-subspaces then X is a D-space.*

Proof. Let $X = U_{n<\omega} F_n$, where each F_n is closed D-subspace of X, and let $\{U(x) \mid x \in X\}$ be a neighborhood assignment.

Pick a closed discrete subset D_1 of F_1 such that $\mathcal{D}_1 = \{U(x) \mid x \in D_1\}$ covers F_1. Since $\{U(x) \mid x \in F_2 - \cup\mathcal{D}_1\}$ covers $F_2 - \cup\mathcal{D}_1$ (a closed subset of F_2), pick a closed discrete subset D_2 of $F_2 - \cup\mathcal{D}_1$ such that $\mathcal{D}_2 = \{U(x) \mid x \in D_2\}$ covers $F_2 - \cup\mathcal{D}_1$. Note that $D_2^\# = D_1 \cup D_2$ is a closed discrete subset of $F_1 \cup F_2$ such that $D_2^\# = \mathcal{D}_1 \cup \mathcal{D}_2$ covers $F_1 \cup F_2$. Inductively, since $\{U(x) \mid x \in F_n - \cup D_{n-1}^\#\}$ covers $F_n - \cup D_{n-1}^\#$ (a closed subset of F_n), pick a closed discrete subset D_n of $F_N - \cup\mathcal{D}_{n-1}^\#$ such that $\mathcal{D}_N^\# = \{U(x) \mid x \in D_n\}$ covers $F_n - \cup\mathcal{D}_{n-1}^\#$. Note that $\mathcal{D}_n^\# = \mathcal{D}_{n-1}^\# \cup D_n$ is a closed idscrete subset of $F_1 \cup \ldots \cup F_n$ such that $\mathcal{D}_n^\# = \mathcal{D}_{n-1}^\# \cup \mathcal{D}_n$ covers $F_1 \cup \ldots F_n$.

Letting $D = \cup_{n<\omega} D_n$ and $\mathcal{D} = \cup_{n<\omega} \mathcal{D}_n$, it is easily seen that D is a closed discrete subset of X such that $\{U(x) \mid x \in D\}$ covers X. This completes the proof.

Corollary 8. *F_σ-subsets of D-spaces are D-spaces.*

Proof. The proof of Theorem 7 is readily adapted to proving that if a space X is dominated by a family $\{F_\alpha \mid \alpha \in \Lambda\}$ of closed D-subspaces, where Λ is an ordinal number, then X is a D-space. (One can use transfinite induction, because $\cup_{\alpha<\beta} F_\alpha$

is a closed subspace of X, for each $\beta < \Lambda$.) Therefore, we get the much stronger result that follows.

Theorem 9. *If a space X is dominated by closed D-subspaces then X is a D-space.*

Again, the space ω_1 of countable ordinals shows that Theorem 9 does not generalize to weak topologies.

Corollary 10. *If a space X is the union of a hereditarily closure-preserving (in particular, locally finite) collection of closed D-subspaces then X is a D-space.*

Proof. This follows immediately from Theorem 9. It also follows immediately from Theorem 6 of [2] and Theorem 6.

Theorems 1 and 6 suggest that one look for a characterization of perfect inverse images of semistratifiable spaces. Unfortunately, there are subparacompact spaces which are not perfect inverse images of semi-stratifiable spaces. For example, the Sorgenfrey line S is actually paracompact and submetrizable; if S were the perfect inverse image of a semistratifiable space then, using the argument in the proof of Theorem 8.1 of [3], one would get that S is semistratifiable, a contradiction. These remarks, combined with Theorem 3.2 of [5], prove the following result.

Proposition 11. *For regular spaces, the class of perfect inverse images of semistratifiable spaces is properly contained in the class of subparacompact spaces.*

The difficulty with proving or disproving that subparacompact spaces are D-spaces stems from the fact that they "look like they are", as the following result shows.

Theorem 12. *If (X,τ) is a subparacompact space then, for each open cover $\mathcal{U}$ of X, there exists a closed discrete subspace D of X and a function $\gamma : D \to \mathcal{U}$ such that $x \in \gamma(x)$, for each $x \in D$, and $\{\gamma(x) \mid x \in D\}$ is a subcover of $\mathcal{U}$.*

Proof. (This result can be proved by using Theorem 1.2(c) of [5] and a straightforward modification of the proof of Theorem 1; however, after we did so, we realized that this is a well-known result in disguise.) Say $\mathcal{U} = \{U_\alpha \mid \alpha \in \Lambda\}$. Since subparacompact spaces are irreducible (i.e. open covers have minimal open refinements; see [6]), by Theorem 2.3 of [1], there exists a discrete collection of nonempty closed sets $\{T_\beta \mid \beta \in \Lambda'\}$, with $\Lambda' \subset \Lambda$, such that each $T_\beta \subset U_\beta$ and $\{U_\beta \mid \beta \in \Lambda'\}$ covers X. To complete the proof, pick a point x_β in each T_β, let $D = \{x_\beta \mid \beta \in \Lambda'\}$ and $\gamma : D \to \mathcal{U}$ be defined by $\gamma(x_\beta) = U_\beta$.

Remark. Clearly, thanks to Theorem 2.3 of [1] , the necessary condition of Theorem 12 is equivalent to irreducibility. Henceforth, we will think of irreducibility accordingly.

We conclude out study of D-spaces with a few applications which show their usefulness.

Proposition 13. *A space is $\aleph_1$−compact and irreducible if and only if it is Lindelöf.*

Proof. Since the "if" part is trivial, let us prove the "only if" part. Let $\mathcal{U}$ be an open cover of an irreducible and $\aleph_1$-compact space X. Pick a closed discrete subspace D of X and a subcover $\{U_x \mid x \in D\}$ of $\mathcal{U}$. Since X is $\aleph_1$-compact, D is countable: therefore $\{U_x \mid x \in D\}$ is a countable subcover of $\mathcal{U}$, which completes the proof.

The next result generalizes Theorem 3.5 of [5].

Proposition 14. *A space is countably compact and irreducible if and only it if is compact.*

Proof. Analogous to the proof of Proposition 13; note that, in this case, D must be finite.

Similarly, one can prove the following two results.

Proposition 15. *Collectionwise Hausdorff, irreducible spaces with CCC are Lindelöf.*

Proposition 16. *Collectionwise Hausdorff Moore spaces with CCC are metrizable.*

Finally, we prove the following result, which is noteworthy because it is false for irreducible spaces; it also raises an interesting question.

Theorem 17. *A monotonically normal D-space X is paracompact.*

Proof. let $\mathcal{U}$ be an open cover of (X,τ). For each $x \in X$, let $U(x)$ be an element of $\mathcal{U}$ such that $x \in U(x)$ (i.e. $\varphi(x) = U(x)$, where $\varphi : X \to \tau$ is a neighborhood assignment.) Then, $x \to U(x)_x$ is also a neighborhood assignment. Therefore, since X is a D-space, let D be a closed discrete subspace of X such that $\{U(x)_x \mid x \in D\}$ covers X. It follows that $\{U(x)_x \mid x \in D\}$ is cushioned in $\{U(x) \mid x \in D\}$ by the assignment $U(x)_x \to U(x)$: Let $D' \subset D$ and pick $z \notin \cup_{x \in D'} U(x)$. Pick a neighborhood V of z such that $V \cap D' = \emptyset$ (this can be done even if $z \in D - D'$, because D is discrete). Note that if $V_z \cap U(w)_w \neq \emptyset$, for some $w \in D'$, then $w \in V$ or $z \in U(w)$, and either is impossible. This proves that $\overline{\cup_{x \in D'} U(x)_x} \subset \cup_{x \in D'} U(x)$. Hence, $\{U(x)_x \mid x \in D\}$ is a cushioned open refinement of $\mathcal{U}$, and this proves that (X,τ) is paracompact.

Remark. There are monotonically normal irreducible spaces which are not paracompact: Let $X = \omega_1 \times (\omega + 1)$ with the points of $\omega_1 \times \omega$ discretified (see Example 3.9 of [7], where it is proved that this space is irreducible). To show that X is monotonically normal, let us use Theorem 1.2(e) of [4], as follows: For each $(x,n) \in \omega_1 \times \omega$, let $\{(x,n)\}_{(x,n)} = \{(x,n)\}$: for each $]y,x] \subset \omega_1$ and $n \in \omega$, let $(]y,n] \times]n,\omega])_{(x,\omega)} =]y,x] \times]n,\omega]$.

Question. Are monotonically normal paracompact spaces D-spaces?

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