Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Vol 16, 1991

METACOMPACT NEARNESS SPACES

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ABSTRACT. Metacompact nearness spaces and nearness maps are shown to form a bireflective subcategory of NEAR. The metacompact nearness structure generated by the collection of point finite open covers of a symmetric topological space is studied. Under suitable conditions, the completion of such a space is shown to be the smallest metacompact subspace of the Wallman compactification containing the original space.

INTRODUCTION

A nearness space is called a metacompact nearness space provided every uniform cover is refined by a point finite uniform cover. Metacompact nearness spaces were introduced by Bentley in [4].

It is shown that META, the subcategory of NEAR consisting of the metacompact nearness spaces and nearness maps, is bireflective in NEAR. From this it follows that the product of a family of metacompact nearness spaces is a metacompact nearness space and a subspace of a metacompact nearness space is a metacompact nearness space.

A particular metacompact nearness structure, denoted by μ_{PF}, is studied in detail. It is defined to be the nearness structure on a symmetric topological space X generated by the family of all point finite open covers on the space. It is shown that the full subcategory of NEAR consisting of objects of the form (X, μ_{PF}), where X is a symmetric topological space, is isomorphic to the category TOP, of symmetric topological spaces and continuous maps.

It is shown that if Y is a T_1 metacompact extension of X then $\mu_Y \subset \mu_{PF}$. An example is provided to show that (X, μ_{PF}) need not be concrete. However, under suitable conditions, $(X*, \mu_{PF}*)$, the completion of the space (X, μ_{PF}), is the smallest metacompact subspace of wX, the Wallman Compactification of X, containing X.

1. PRELIMINARIES.

We will assume that the reader is basically familiar with the concept of a nearness space as defined by Herlich in [7] and [8].

Definition 1.1. Let X be a set and μ a collection of covers of X, called uniform covers. Then (X, μ) is a nearness space provided:

N1. $\mathcal{H} \in \mu$ and $\mathcal{H}$ refines $\mathcal{L}$ implies $\mathcal{L} \in \mu$.

N2. $\{X\} \in \mu$ and $\emptyset \notin \mu$.

N3. If $\mathcal{H} \in \mu$ and $\mathcal{L} \in \mu$ then $\mathcal{H} \wedge \mathcal{L} = \{H \cap L : H \in \mathcal{H}$ and $L \in \mathcal{L}\} \in \mu$.

N4. $\mathcal{H} \in \mu$ implies $\{int(H) : H \in \mathcal{H}\} \in \mu$. $(int(H) = \{x : \{X - \{x\}, H\} \in \mu.)$

For a given nearness space (X, μ) the collection of sets that are "near" is given by $\xi = \{\mathcal{H} \subset \mathcal{P}(X) : \{X - H : H \in \mathcal{H}\} \notin \mu\}$. The closure operator generated by a nearness space is given by $cl_\xi(A) = \{x : \{\{x\}, A\} \in \xi\}$. If we are primarily using these "near" collections we will denote the nearness space by (X, ξ). The underlying topology of a nearness space is always symmetric: that is, $x \in \{\overline{y}\}$ implies $y \in \{\overline{x}\}$.

Definition 1.2. Let (X, ξ) be a nearness space. The nearness space is called:

(1) topological provided $\mathcal{H} \in \xi$ implies $\cap\overline{\mathcal{H}} \neq \emptyset$.

(2) complete provided each ξ-cluster is fixed; that is, $\cap\overline{\mathcal{H}} \neq \emptyset$ for each maximal element $\mathcal{H}$ in ξ.

(3) concrete provided each near collection is contained in some ξ-cluster.

(4) contigual provided $\mathcal{H} \notin \xi$ implies there exists a finite $\mathcal{L} \subset \mathcal{H}$ such that $\mathcal{L} \notin \xi$.
(5) totally bounded provided $\mathcal{H} \notin \xi$ implies there exist a finite $\mathcal{L} \subset \mathcal{H}$ such that $\cap\mathcal{L} = \emptyset$.

Let (X,t) be a symmetric topological space. Set:

$$\xi_t = \{\mathcal{H} \subset \mathcal{P}(X) : \cap\overline{\mathcal{H}} \neq \emptyset\}$$
$$\xi_P = \{\mathcal{H} \subset \mathcal{P}(X) : \overline{\mathcal{H}} \text{ has f. i. p .}\}$$
$$\xi_L = \{\mathcal{H} \subset \mathcal{P}(X) : \overline{\mathcal{H}} \text{ has c. i. p }\}$$

Each of these is a compatible nearness structure on X. They can be defined equivalently as follows:

$\mu_t = \{\mathcal{L} \subset \mathcal{P}(X) : \mathcal{L}$ is refined by an open cover of $X\}$
$\mu_P = \{\mathcal{L} \subset \mathcal{P}(X) : \mathcal{L}$ is refined by a finite open cover of $X\}$
$\mu_L = \{\mathcal{L} \subset \mathcal{P}(X) : \mathcal{L}$ is refined by a countable open cover of $X\}$.

ξ_P is called the Pervin nearness structure on X and ξ_L the Lindelöf nearness structure on X. They are discussed in [5] and [4], respectively.

Definition 1.3 Let $\mathcal{F}$ be a closed filter in a topological space (X,t). Set $\mathcal{A}(\mathcal{F}) = \{A : \overline{A} \in \mathcal{F}\}$. If $\mathcal{F}$ is a prime closed filter, set $\mathcal{O}(\mathcal{F}) = \{O \in t : X - O \notin \mathcal{F}\}$.

If $\mathcal{F}$ is a prime closed filter then $\mathcal{O}(\mathcal{F})$ is a prime open filter and if $\mathcal{F}$ is a closed ultrafilter then $\mathcal{O}(\mathcal{F})$ is a minimal prime open filter and in this case $\mathcal{O}(\mathcal{F}) = \{O \in t :$ there exists $F \in \mathcal{F}$ with $O \supset F\}$. That is; if $\mathcal{F}$ is a closed ultrafilter then the open envelope of $\mathcal{F}$ is a minimal prime open filter [6].

2. Metacompact nearness spaces

Definition 2.1 A nearness space (X,μ) is called a metacompact nearness space if for each $\mathcal{H} \in \mu$ there exists a $\mathcal{L} \in \mu$ such that $\mathcal{L}$ refines $\mathcal{H}$ and $\mathcal{L}$ is point finite. If (X,t) is a symmetric topological space, set $\mu_{PF} = \{\mathcal{L} \subset \mathcal{P}(X) :$ There exists a point finite open cover of X that refines $\mathcal{L}\}$.

Theorem 2.2. *Let (X,t) be a symmetric topological space. Then:*

(1) *μ_{PF} is a compatible metacompact nearness structure on X.*
(2) *$\xi_{PF} = \{\mathcal{H} \subset \mathcal{P}(X) : X\backslash\mathcal{H}$ is not refined by a point finite open cover of X $\}$.*
(3) *If μ is a compatible metacompact nearness structure on X then $\mu \subset \mu_{PF}$.*

Theorem 2.3. *Let (X,t) be a symmetric topological space. Then:*

(1) $\xi_t \subset \xi_{PF} \subset \xi_P$
(2) $\mu_P \subset \mu_{PF} \subset \mu_t$.

Theorem 2.4. (1) *If $\mathcal{H} \in \xi_{PF}$ then $\overline{\mathcal{H}}$ has the finite intersection property.*
(2) *If $\mathcal{H} \in \xi_{PF}$ then the closed filter $\mathcal{F}$ generated by $\overline{\mathcal{H}}$ belongs to ξ_{PF}.*
(3) *If $\mathcal{H}$ is a ξ_{PF}-cluster and $\mathcal{F}$ is the closed filter generated by $\mathcal{H}$ then $\mathcal{F}$ is a prime closed filter.*

Proof. (1) Since $\xi_{PF} \subset \xi_P$ it follows that if $\mathcal{H} \in \xi_{PF}$ then $\overline{\mathcal{H}}$ has the f. i. p. (2) Let $\mathcal{H} \in \xi_{PF}$. Then $\overline{\mathcal{H}}$ has the f. i. p. and let $\mathcal{F}$ denote the closed filter generated by $\overline{\mathcal{H}}$. If $\mathcal{F} \notin \xi_{PF}$ then $X\backslash\mathcal{F} \in \mu_{PF}$ and there exist a point finite open refinement $\mathcal{O}$ of $X\backslash\mathcal{F}$. Let $O \in \mathcal{O}$. Then there exists $A_1, \ldots, A_n \in \mathcal{H}$ such that

$O \subset X - (\overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_n) = (X - \overline{A}_1) \cup (X - \overline{A}_2) \cup \ldots \cup (X - \overline{A}_n)$.

Set $P_{O(k)} = O \cap (X - \overline{A}_k)$. Then the collection $\{P_{O(k)} : 1 \leq k \leq n(0); O \in \mathcal{O}\}$ is a point finite open refinement of $X\backslash\mathcal{H}$. But this is impossible since $\overline{\mathcal{H}} \in \xi_{PF}$. Hence $\mathcal{F} \in \xi_{PF}$.

(3) Let $\mathcal{H}$ be a ξ_{PF}-cluster and $\mathcal{F}$ the closed filter generated by $\overline{\mathcal{H}}$. To see that $\mathcal{F}$ is prime let F and G be closed sets such that $F \cup G \in \mathcal{F}$. If $F \notin \mathcal{F}$ and $G \notin \mathcal{F}$ is follows that $\{F\} \cup \overline{\mathcal{H}} \notin \xi_{PF}$ and $\{G\} \cup \overline{\mathcal{H}} \notin \xi_{PF}$ and thus

$$(\{F\} \cup \overline{\mathcal{H}}) \vee (\{G\} \cup \overline{\mathcal{H}}) \notin \xi_{PF}.$$

But $\mathcal{F} = (\{F\} \cup \overline{\mathcal{H}}) \vee (\{G\} \cup \overline{\mathcal{H}}) \in \xi_{PF}$ and we have a contradiction. Therefore $\mathcal{F}$ is a a prime closed filter.

Example 2.5 To see that ξ_{PF} need not be concrete consider the half-disc topology on $X = P \cup L$ where $P = \{(x,y) : x, y \in \mathbb{R}$ and $y > 0\}$ and L the real axis where P has the usual topology t and the topology on X is t together with all sets of the form $\{x\} \cup (P \cap U)$ where $x \in L$ and U is an Euclidean neighborhood of x in the plane. This is precisely example 78 in [9] where it is shown that X is countably metacompact but any covering of X by basis elements has no point finite refinement.

Now let $\mathcal{E}$ be any open cover of X by basic elements and let $\mathcal{L} = \mathcal{E} \cup \{P\}$. It is an easy observation that $\mathcal{L}$ also has no point finite open refinement. Let $\mathcal{F}$ be the closed filter generated by $\{X - O : O \in \mathcal{L}\}$. Since $\mathcal{L} \notin \mu_{PF}$ is follows that $\mathcal{F} \in \xi_{PF}$. Suppose ξ_{PF} is concrete. Then there exists a ξ_{PF}-cluster $\mathcal{A}$ containing $\mathcal{F}$. By Theorem 2.4, $\overline{\mathcal{A}} = \mathcal{H}$ is a prime closed filter. Since X is countably metacompact it follows that $\mathcal{H}$ has the countable intersection property.

Since $P \in \mathcal{L}$ it follows that $L \in \mathcal{F} \subset \mathcal{H}$. Since $\cap\mathcal{H} \subset \cap\mathcal{F} = \emptyset$ it follows that $\{x\} \notin \mathcal{H}$ for each $x \in L$.

Now each $(a,b) \subset L$ is closed in X. Either, for some integer $k, (k, k+1) \in \mathcal{H}$ or $(k, k+1) \notin \mathcal{H}$ for all integers k.

Suppose $(k, k+1) \in \mathcal{H}$. Then $(k, k+1) = (k, (2k+1)/2) \cup \{(2k+1)/2\} \cup ((2k+1)/2, k+1)$. Now $\{(2k+1)/2\} \notin \mathcal{H}$ and thus either $(k, (2k+1)/2) \in \mathcal{H}$ or $((2k+1)/2, k+1) \in \mathcal{H}$. Continuing in this manner, one constructs a sequence of nested intervals each of which belongs to $\mathcal{H}$ but whose intersection is empty. This is impossible since $\mathcal{H}$ has the c. i. p.

Hence the assumption that $(k, k+1) \in \mathcal{H}$ for some integer k leads to a contradiction and thus it follows that for all integers $k, (k, k+1) \notin \mathcal{H}$.

Now $L = (-\infty, 0) \cup \{0\} \cup (0, \infty) \in \mathcal{H}$. Since $\{0\} \notin \mathcal{H}$, either $(-\infty, 0) \in \mathcal{H}$ or $(0, \infty) \in \mathcal{H}$. Suppose $(0, \infty) \in \mathcal{H}$. Then $(0,1) \cup \{1\} \cup (1, \infty) \in \mathcal{H}$. Since $(0,1) \notin \mathcal{H}$ and $\{1\} \notin \mathcal{H}$ it follows that $(1, \infty) \in \mathcal{H}$. Repeating the argument, one has that $(n, \infty) \in \mathcal{H}$

for each positive integer n. Thus $\{(n,\infty) : n \in N\} \subset \mathcal{H}$ but $\mathcal{H}$ has the c. i. p. and therefore this is impossible. A similar contradiction follows from the assumption that $(-\infty, 0) \in \mathcal{H}$. Thus, $\mathcal{F}$ is not contained in a prime closed filter $\mathcal{H} \in \xi_{PF}$. Hence, ξ_{PF} is not concrete.

It is known [5], $\mu_P = \mu_L$ iff X is countably compact and $\mu_L = \mu_t$ iff X is Lindelöf.

Theorem 2.6. *Let X be a symmetric topological space. Then:*

(1) $\mu_t = \mu_{PF}$ *iff X is metacompact.*
(2) $\mu_L \subset \mu_{PF}$ *iff X is countably metacompact.*
(3) $\mu_{PF} \subset \mu_L$ *iff every point finite open cover of X has a countable subcover.*

Theorem 2.7. *let X be a symmetric topological space. The following statements are equivalent.*

(1) $\mu_P = \mu_{PF}$.
(2) *Every point finite open cover of X has a finite subcover.*
(3) μ_{PF} *is totally bounded.*
(4) μ_{PF} *is contigual.*

It is well-known that the category of symmetric topological spaces is isomorphic to the subcategory of Near consisting of all topological nearness spaces and nearness maps. Since they are isomorphic, we identify them and call it TOP. The isomorophism maps (X, t) to (X, ξ_t).

It was shown in [5] that the categories of Pervin Nearness Spaces and nearness maps and Lindelöf Nearness Spaces and nearness maps are each isomorphic to TOP.

Let META denote the full subcategory of NEAR consisting of all metacompact nearness spaces and nearness maps. Define

$$T : \text{NEAR} \to \text{META by } T(X,\mu) = (X, T(\mu)),$$

where $T(\mu) = \{\mathcal{H} \in \mu :$ there exists $\mathcal{L} \in \mu$ which refines $\mathcal{H}$ and $\mathcal{L}$ is point finite$\}$

Theorem 2.9. (1) *META is a bireflective full subcategory of NEAR.*

(2) *The restriction of T to TOP is an isomorphism.*

Proof. Let (X, μ) be a nearness space and $g : (X, \mu) \to (Y, v)$ be a near map where (Y, v) is a metacompact nearness space. Define $h : (X, T(\mu)) \to (Y, v)$ by $h(x) = g(x)$ for all $x \in X$. To see that h is a near map let $\mathcal{E} \in v$. Then there exists $\mathcal{P} \in v$ such that $\mathcal{P}$ is point finite and refines $\mathcal{E}$. Now $\text{int}(\mathcal{P})$ is in v and is also point finite. $g^{-1}(\text{int}(\mathcal{P}))$ is an open point finite cover of X. Hence $g^{-1}(\text{int}(\mathcal{P}))$ belongs to $T(\mu)$ and therefore $h^{-1}(\mathcal{E}) \in T(\mu)$.

To see (2), note that $T(X, \mu_t) = (X, \mu_{PF})$. The only part of the proof that is not immediately evident is if $f : (X, t) \to (Y, s)$ is a continuous map then $f : (X, \mu_{PF}) \to (Y, v_{PF})$ is a near map. Let $\mathcal{E} \in v_{PF}$. Then there exists a point finite open cover $\mathcal{P}$ of Y that refines $\mathcal{E}$. Since f is continuous, $f^{-1}(\mathcal{P})$ is a point finite open cover of X. Hence $f^{-1}(\mathcal{E}) \in \mu_{PF}$ and f is a near map.

Since NEAR is a properly fibred topological category, it is known, Herrlich [8], that an epireflective subcategory of N-EAR is closed under the formation of subobjects and products. Thus, we have the following result.

Corollary 2.10. *The product of a nonempty family of metacompact nearness space is a metacompact nearness space. The subspace of a metacompact nearness space is a metacompact nearness space.*

3. Extensions

All spaces in this section are assumed to be T_1. An extension Y of a space X is a space in which X is densely embedded. Unless otherwise noted, we will assume for notational convenience that $X \subset Y$. It is well known that for any extension Y of X there exists an equivalent extension Y' with $X \subset Y'$.

If Y is an extension of X then $\xi = \{\mathcal{A} \subset \mathcal{P}(X) : \cap\, cl_Y \mathcal{A} \neq \emptyset\}$ is called the nearness structure on X induced by Y. Equivalently, $\mu = \{\mathcal{U} \subset \mathcal{P}(X) : \cup\, op(\mathcal{U}) = Y\}$ where $op(\mathcal{U}) = \{op(U) : U \in \mathcal{U}\}$ and $op(U) = Y - cl_Y(X - U)$.

Let (Y,t) be a topological space and $\overline{X} = Y$. For each $y \in Y$, set $\mathcal{O}_y = \{O \cap X : y \in O \in t\}$. Then $\{\mathcal{O}_y : y \in Y\}$ is called the filter trace of Y on X.

The strict extension topology (See Banaschewski [1]) on Y is generated by the base $\{O^* : O \in t(X)\}$, where $O^* = \{y \in Y : O \in \mathcal{O}_y\}$. Let Y be a T_1 extension of X. Then Y is a strict extension of X if and only if $\{cl_Y A : A \subset X\}$ is a base for the closed sets in Y.

Herrlich and Bentley's completion of a nearness space [2] can be described as follows. Let (X,ξ) be a T_1 nearness space. Let X^* be the set of all ξ-clusters and for $A \subset X$ let $cl(A) = \{\mathcal{A} \in X^* : A \in \mathcal{A}\}$. A nearness structure ξ^* is defined on X^* as follows:

$\mathcal{B} \in \xi^*$ provided $\{A \subset X$: there exists $B \in \mathcal{B}$ with $B \subset cl(A)\} \in \xi$. (X^*,ξ^*) is a complete nearness space and $cl_\xi X = X^*$. Also, for $A \subset X, cl_{\xi^*}(A) = cl(A)$.

Herrlich and Bentley [3], also describe the completion in the following equivalent manner. The function $e : X \to X^*$ maps every $x \in X$ onto the cluster $\mathcal{A}_x$, consisting of those subsets of X which have x as an adherent point. For any subset B of X, the set B^* denotes the set of all $p \in X^*$ such that B meets every member of the cluster p. A cover $\mathcal{E}$ of X^* belongs to μ^* provided there exists $\mathcal{P} \in \mu$ such that $\{P^* : P \in \mathcal{P}\}$ refines $\mathcal{E}$.

The following two important theorems are due to Bentley and Herrlich [2].

Theorem A. *For any T_1 nearness space (X,ξ) the following conditions are equivalent.*

(1) *ξ is a nearness structure induced on X by a strict extension.*
(2) *The completion X^* of X is topological.*
(3) *ξ is concrete.*

Theorem B. *Strict extensions are equivalent if and only if they induce the same nearness structure.*

Theorem C. (Carlson [6]) *Let (X,t) be a T_1 topological space, Then:*

(1) *ξ_P is a compatible concrete contigual nearness structure on X.*
(2) *The ξ_P-clusters are of the form $\mathcal{A}(\mathcal{F}) = \{A \subset X : \overline{A} \in \mathcal{F}\}$, where $\mathcal{F}$ is a closed ultrafilter on X.*
(3) *$\xi_P = \{\mathcal{A} \subset \mathcal{P}(X) : \mathcal{A} \subset \mathcal{A}(\mathcal{F})$ for some closed ultrafilter $\mathcal{F}\}$.*
(4) *(X^*, ξ_{P^*}) is the Wallman compactification of (X,t).*
(5) *If (X,t) is normal then (X^*, ξ_{P^*}) is the Stone-Čech compactification of (X,t).*

Theorem D. (Carlson [6]) *Let (X,ξ) be a T_1 nearness space and (X^*,ξ^*) its completion. Then the trace filters on X are given by*

(1) *$\mathcal{O}_x = \{O \in t(\xi) : x \in O\}$ for $x \in X$, and*
(2) *$\mathcal{O}_\mathcal{A} = \{O \in t(\xi) : X - O \notin \mathcal{A}\}$ for $\mathcal{A} \in X^* - X$.*

Thus, if X is a T_1 topological space and ξ_P the Pervin nearness structure on X then the trace filters of (X^*, ξ_P^*) are of the form $\mathcal{O}_{\mathcal{A}(\mathcal{F})} = \mathcal{O}(\mathcal{F})$ where $\mathcal{F}$ is a closed ultrafilter on X.

Let (X,ξ) be a T_1 nearness space and (X^*,ξ^*) its completion. Recall that $\text{op}(A) = X^* - cl(X - A)$ for $A \subset X$. Moreover, the notation U^*, for $U \subset X$, has been used to indicate the collection of ξ-clusters $\mathcal{A}$ such that $U \cap A \neq \emptyset$ for each $A \in \mathcal{A}$ and also, if U is open in X, to denote the family of ξ-clusters $\mathcal{A}$ such that $\mathcal{O}_\mathcal{A}$, the trace filter of $\mathcal{A}$, contains U. The following theorem notes that this notation is consistent.

Theorem 3.1. *Let (X,ξ) be a T_1 nearness space and O an open set in X. Then:*

$$\begin{aligned} op(O) &= \{\mathcal{A} \in X^* : O \cap A \neq \emptyset \text{ for each } A \in \mathcal{A}\} \\ &= \{\mathcal{A} \in X^* : O \in \mathcal{O}_\mathcal{A}\} = O^*. \end{aligned}$$

Theorem 3.2. *Let (X,μ) be a T_1 nearness space. Then:*

(1) *$\mathcal{U} \in \mu$ iff $\mathcal{U}^* \in \mu^*$*

(2) *Let $\mathcal{O}$ be an open cover of X. Then $\mathcal{O} \in \mu$ iff $\mathcal{O}^* \in \mu^*$.*

It follows that if (X, μ_{PF}) is concrete then $\mathcal{E} \in \mu_{PF^*}$ if and only if there exists a point finite open cover $\mathcal{P}$ of X such that $\mathcal{P}^*$ refines $\mathcal{E}$.

It was shown in Example 2.5 that ξ_{PF} need not be concrete. The following theorem show that when ξ_{PF} is concrete and each ξ_{LF}-cluster is a ξ_P-cluster then its completion is a metacompact strict extension of X.

Theorem 3.3. *If ξ_{PF} is concrete and each ξ_{PF}-cluster is a ξ_P-cluster then (X^*, ξ^*_{PF}) is metacompact.*

Proof. Let $\mathcal{P}^* = \{P_{\beta^*} : \beta \in \Omega\}$ be a basic open cover of X^*. Then $\mathcal{P} = \{P_\beta : \beta \in \Omega\} \in \mu_{PF}$. Then there exists a point finite open cover $\mathcal{O} = \{O\alpha : \alpha \in \Lambda\}$ that refines $\mathcal{P}$.

Since $\mathcal{O} \in \mu_{PF}$ it follows that $\mathcal{O}^* = \{O^*_\alpha : \alpha \in \Lambda\}$ is an open cover of X^*. Easily, $\mathcal{O}^*$ refines $\mathcal{P}^*$.

Claim: $\mathcal{O}^*$ is point finite. Suppose not. Then there exists $\mathcal{A}(\mathcal{F}) \in X^*$ such that $\mathcal{A}(\mathcal{F})$ belongs to infinitely many members of $\mathcal{O}^*$. Thus, there exists a countably infinite collection $\{O^*_{\alpha(i)} : i \in N\} \subset \mathcal{O}^*$ such that $\mathcal{A}(\mathcal{F}) \in O^*_{\alpha(i)}$ for each $i \in N$. Now, by Definition 1.3 and theorem D, $\mathcal{A}(\mathcal{F}) \in O^*_{\alpha(i)}$ iff $O_{\alpha(i)} \in \mathcal{O}(\mathcal{F})$.

Case 1. $\cap\mathcal{F} \neq \emptyset$. Then $\mathcal{F} = \mathcal{F}_x$ for some $x \in X$ and hence $x \in \cap\{O_{\alpha(i)} : i \in N\}$ which contradicts the fact that $\mathcal{O}$ is point finite.

Case 2. $\cap\mathcal{F} = \emptyset$. Now for each $i \in N$ there exists $F_i \in \mathcal{F}$ such that $F_i \subset O_{\alpha(i)}$. For each $i \in N$, set $N_i = F_1 \cap F_2 \cap \ldots \cap F_i$ and $S_i = X - N_i$. Then:

(A) $N_1 \supset N_2 \supset \ldots \supset N_i \supset \ldots$

(B) $N_i \in \mathcal{F}$ and $O_{\alpha(i)} \supset N_i$, for each $i \in N$.

Now set $Q_1 = S_1$ and for $i \in N\backslash\{1\}$ set $Q_i = O_{\alpha(i-1)} \cap S_i$.

Claim: $\{Q_i : i \in N\}$ is a point finite open cover of X. For each $x \in X$, there exists $i(x) = \min\{k : x \notin N_k\}$. If $i(x) = 1$ then $x \in Q_1$. If $i(x) > 1$ then $x \in N_{i(x)-1} \subset O_{\alpha(i(x)-1)}$ and

$x \notin N_{i(x)}$ which implies $x \in X - N_{i(x)} = S_i(x)$. Thus, $x \in O_{\alpha(i(x)-1)} \cap S_{i(x)} = Q_{i(x)}$. Thus, $\{Q_i : i \in N\}$ is an open cover. It is point finite since $\{O_{\alpha(i)} : i \in N\}$ is point finite.

Now $\{Q_i : i \in N\} \subset X \backslash \mathcal{F}$ which implies that $X \backslash \mathcal{F} \in \mu_{PF}$ which is impossible since $\mathcal{F} \in \xi_{PF}$.

Hence, $\mathcal{A}(\mathcal{F})$ can not be contained in infinitely many members of $\mathcal{O}^*$ and then $\mathcal{O}^*$ is a point finite open refinement of $\mathcal{P}^*$. Therefore, X^* is metacompact.

Theorem 3.4. *Let Y be a T_1 extension of X. If Y is metacompact then $\mu_Y \subset \mu_{PF}$.*

Proof. Let $\mathcal{S} = \{S_\alpha : \alpha \in \Lambda\} \in \mu_Y$. Then $\mathcal{E} = \{op(S_\alpha) : \alpha \in \Lambda\}$ is an open cover of Y. Then there exists a point finite open refinement $\mathcal{P} = \{P_\beta : \beta \in \Omega\}$ of $\mathcal{E}$. Then $\{P_\beta \cap X : \beta \in \Omega\}$ is a point finite open refinement of $\mathcal{S}$ and therefore $\mathcal{S} \in \mu_{PF}$.

Corollary 3.5. *Let $X \subset Y \subset wX$. If Y is metacompact then $\mu_Y \subset \mu_{PF}$.*

Definition 3.6 Let X be a T_1 topological space. Let $\tilde{X}$ be the subspace of $(X^*, \xi_P^*) = wX$, the Wallman compactification of X, defined as follows: $\tilde{X} = \{\mathcal{A}(\mathcal{F}) : X \backslash \mathcal{F}$ contains no point finite open refinement$\}$.

Theorem 3.7. *(1) $\xi_t \subset \xi_{\tilde{X}} \subset \xi_{PF} \subset \xi_P$*
(2) $\mu_P \subset \mu_{PF} \subset \mu_{\tilde{X}} \subset \mu_t$

Theorem 3.8. *The following statements are equivalent.*

(1) *$\tilde{X}$ is metacompact.*
(2) *ξ_{PF} is concrete and each ξ_{PF}-cluster is a ξ_p-cluster.*
(3) *$\xi_{PF} = \xi_{\tilde{X}}$.*
(4) *$\mu_{PF} = \mu_{\tilde{X}}$.*

Proof. Statements (3) and (4) are easily equivalent.

(1) implies (2). Since $\tilde{X}$ is metacompact, it follows by theorems 3.4 and 3.7, that $u_{\tilde{X}} = \mu_{PF}$. Thus $\xi_{PF} = \xi_{\tilde{X}}$ and hence ξ_{PF} is concrete and each ξ_{PF}-cluster is a ξ_P-cluster.

(2) implies (3). By (2), each $\mathcal{H} \in \xi_{PF}$ is contained in a ξ_{PF}-cluster, say $\mathcal{E}$, and $\mathcal{E}$ is a ξ_p-cluster. Hence $\mathcal{E}$ is a $\xi_{\tilde{X}}$-cluster and thus $\mathcal{H} \in \xi_{\tilde{X}}$. Hence $\xi_{PF} = \xi_{\tilde{X}}$.

(3) implies (1). If $\xi_{PF} = \xi_{\tilde{X}}$ it follows that ξ_{PF} is concrete and hence $\tilde{X} = (X^*, \xi^*_{PF})$ is metacompact by theorem 3.1.

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