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$C_p(X)$-REPRESENTATION OF CERTAIN BOREL ABSORBERS*

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ABSTRACT. For each even ordinal α, we construct a countable completely regular space X_α such that the function space $C_p(X_\alpha)$ is universal for the collection $\mathcal{M}_\alpha$ of all absolute Borel sets of multiplicative class α and, moreover, $C_p(X_\alpha)$ is homeomorphic to the $\mathcal{M}_\alpha$-absorber Ω_α.

1. INTRODUCTION.

If α is a countable ordinal greater than 1, then $\mathcal{M}_\alpha$ and $\mathcal{A}_\alpha$ denote the classes of all absolute Borel sets of the multiplicative and additive class α, respectively. In Bestvina and Mogilski [2] it was shown that in each of the classes $\mathcal{M}_\alpha$ and $\mathcal{A}_\alpha$ there exist unique maximal objects Ω_α and Λ_α which are $\mathcal{M}_\alpha$-absorbers and $\mathcal{A}_\alpha$-absorbers, respectively. Moreover, Ω_α and Λ_α can be represented as linear subspaces of the topological Hilbert space $\mathbb{R}^\mathbb{N}$.

There is another way of obtaining linear subspaces of $\mathbb{R}^\mathbb{N}$ of arbitrarily high Borel complexity. If X is a space then $C_p(X)$ denotes the space of continuous, real-valued functions on X endowed with the topology of pointwise convergence. If X is countable then $C_p(X)$ is obviously linearly homeomorphic to a linear subspace of $\mathbb{R}^\mathbb{N}$. According to Lutzer, van Mill, and Pol [16] the space $C_p(X)$ may be of arbitrarily high Borel complexity – it may even be a non-analytic set. The following theorem

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was proved by Dobrowolski, Marciszewski, and Mogilski [11] (see also [1,5,7,8,9,10]).

Theorem 1.1. *Let X be a countable, nondiscrete, completely regular space. If $C_p(X) \in \mathcal{M}_2$ then it is homeomorphic to Ω_2.*

According to Dijkstra et al. [6] $C_p(X)$ cannot be an element of $\mathcal{A}_2$ and hence this theorem gives a complete classification of the spaces $C_p(X)$ of Borel class not higher than 2. Theorem 1.1 suggests the following:

Conjecture 1.2. *If $C_p(X) \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$ then it is homeomorphic to Ω_α.*

As we mentioned above all multiplicative classes $\mathcal{M}_\alpha$, where $\alpha > 1$, are represented among spaces $C_p(X)$. Moreover, Marciszewski recently showed that if $C_p(X)$ is Borel then $C_p(X) \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$ for some α, improving a result of Calbrix [3,4]. The method of absorbers was employed in the proof of Theorem 1.1 and the crucial step was to show that a space $C_p(X)$, which is in $\mathcal{M}_2 \setminus \mathcal{A}_2$, is an $\mathcal{M}_2$-absorber (and hence homeomorphic Ω_2). For higher Borel classes $\mathcal{M}_\alpha$ it is not clear how to exibit at least one space X such that $C_p(X)$ is an $\mathcal{M}_\alpha$-absorber. In this paper we prove that if α is an even ordinal then there exists a countable completely regular space X_α such that $C_p(X_\alpha)$ is homeomorphic to Ω_α.

2. Borel absorbers.

Since we are dealing with function spaces it is convenient to represent the Hilbert cube Q by $[-\infty, \infty]^{\mathbf{N}}$ and its pseudointerior s by $\mathbb{R}^{\mathbf{N}}$. Let Y stand for either s or Q. Let us recall that a closed subset A of Y is a Z-set if given an open cover $\mathcal{U}$ of Y there exists a $\mathcal{U}$-close to the identity map $f : Y \to Y$ such that $f(Y)$ misses A. A countable union of Z-sets is called a σZ-set. Let $Y = Q$ or s and let $\mathcal{C}$ be a collection of subsets of Y that is topological (i.e., invariant under homeomorphisms of Y) and closed hereditary. We say that a subset X of Y is a $\mathcal{C}$-absorber if

(1) $X \in \mathcal{C}$,
(2) X is contained in a σZ-set of Y,
(3) X is strongly $\mathcal{C}$-universal, i.e., for every A in Y such that $A \in \mathcal{C}$ and for every map $f : Y \to Y$ that restricts to a Z-embedding (an embedding onto a Z-set) on a closed set K, there exists a Z-embedding $g : Y \to Y$ that can be chosen arbitrarily close to f with the properties: $g|K = f|K$ and $g^{-1}(X) \setminus K = A \setminus K$.

The notion of $\mathcal{C}$-absorber generalises concepts of [18] and [19]. Their most important property is uniqueness: if X and X' are $\mathcal{C}$-absorbers in Y then the pairs (Y, X) and (Y, X') are homeomorphic. Bestvina and Mogilski in [2] constructed two transfinite sequences $(\Omega_\alpha)_{1<\alpha<\omega_1}$ and $(\Lambda_\alpha)_{1<\alpha<\omega_1}$ of subsets in the Hilbert cube Q such that Ω_α is an $\mathcal{M}_\alpha$-absorber and Λ_α is an $\mathcal{A}_\alpha$-absorber for $1 < \alpha < \omega_1$. In this section we will present a slightly modified construction of $\mathcal{M}_\alpha$-absorbers for even α and $\mathcal{A}_\alpha$-absorbers for odd α based on the use of a special product of spaces described in [14] (see also [15] and [4]).

If X_n is a subset of Y_n for $n = 1, 2, \ldots$, then the set

$$\mathop{\mathbf{F}}_{n=1}^{\infty} X_n = \{(x_n) \in \prod_{n=1}^{\infty} Y_n : \exists\, k\ \forall n > k\ x_n \in X_n\}$$

is called the Fréchet product (the product with respect to the Fréchet filter) of the sequence $(X_n)_{n=1}^{\infty}$. If $X_n = X$ for all n, then we write $\mathbf{F}(X) = \mathbf{F}_{n=1}^{\infty} X_n$ and $X^{\mathbf{N}} = \prod_{n=1}^{\infty} X_n$.

Lemma 2.1. *Let $Y = Q$ or s and let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$, where, for $n = 1, 2, \ldots$, $\mathcal{C}_n$ is a collection of subsets of Y which is topological, closed hereditary, and closed under finite intersections and unions, and $\mathcal{C}_n \subset \mathcal{C}_{n+1}$. Then:*

(1) *If X_n is strongly $\mathcal{C}_n$-universal for $n = 1, 2, \ldots$, then $\prod_{n=1}^{\infty} X_n$ is strongly $\mathcal{C}_\delta$-universal, where $\mathcal{C}_\delta$ stands for the collection of countable intersections of elements of $\mathcal{C}$;*
(2) *If X_n is strongly $\mathcal{C}_n$-universal then $\mathbf{F}_{n=1}^{\infty} X_n$ is strongly $\mathcal{C}_\sigma$-universal, where $\mathcal{C}_\sigma$ stands for the collection of*

countable unions of elements of $\mathcal{C}$.

Part (1) follows from [12, Lemma 2.3] or [8, Theorem 3.1]. Part (2) is in essence a complementary formulation of the $S'_\infty(X)$ part of [8, Theorem 3.1] (where complementary means that all sets are replaced by their complements).

Let

$$\sigma = \{(x_n) \in \mathbb{R}^{\mathbf{N}} : x_n = 0 \text{ for all but finitely many } n\}.$$

Identifying countable products of Q with Q we construct Borel absorbers in Q as follows:

$$\Omega_2 = \sigma^{\mathbf{N}},$$

$$\Lambda_\alpha = \mathbf{F}(\Omega_{\alpha-1}),$$

if α is an odd ordinal, and

$$\Omega_\alpha = \begin{cases} (\Lambda_{\alpha-1})^{\mathbf{N}}, & \text{if } \alpha \text{ is an even nonlimit ordinal} \\ \prod_{n=1}^\infty \Lambda_{\alpha_n}, & \text{if } \alpha = \lim \alpha_n \end{cases}$$

Let us point out that both Ω_α and Λ_α are subsets of s. The next result follows from Lemma 6.3 of [2] (which is based on an idea of Sikorski [17,13]) and Lemma 2.1.

Proposition 2.2. *For each even ordinal $\alpha < \omega_1$, the spaces Ω_α and $\Lambda_{\alpha+1}$ are $\mathcal{M}_\alpha$-absorbers and $\mathcal{A}_{\alpha+1}$-absorbers, respectively, in both Q and s.*

We have the following easy observation for even ordinals.

Remark 2.3. The transfinite sequence (Ω_α), where α is an even ordinal and $1 < \alpha < \omega_1$ is defined by the choice of Ω_2 and the following recurrence relation:

$$\Omega_\alpha = \begin{cases} (\mathbf{F}(\Omega_{\alpha-2}))^{\mathbf{N}}, & \text{if } \alpha - 1 \text{ is not a limit ordinal} \\ \prod_{n=1}^\infty \mathbf{F}(\Omega_{\alpha_n}), & \text{if } \alpha = \lim \alpha_n. \end{cases}$$

3. Borel filters and corresponding sequence spaces

We recall that a family $\mathfrak{F}$ of subsets of $\mathbb{N}$ is a filter on $\mathbb{N}$ if $\emptyset \notin \mathfrak{F}$, $A \cap B \in \mathfrak{F}$ provided $A, B \in \mathfrak{F}$ and $A \subseteq C \subseteq \mathbb{N}$, $A \in \mathfrak{F}$ implies $C \in \mathfrak{F}$. All filters $\mathfrak{F}$ can be identified with subsets of the Cantor set $2^{\mathbb{N}}$. Hence the cartesian product and the Fréchet product of filters are filters on $\mathbb{N} \times \mathbb{N}$ and hence subsets of $2^{\mathbb{N}\times\mathbb{N}}$. We shall reduce these product filters to filters on $\mathbb{N}$ (and subsets of $2^{\mathbb{N}}$) via some fixed bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

The filter $\mathfrak{F}_1$ consisting of all cofinite subsets of $\mathbb{N}$ is called the Fréchet filter and we have $\mathfrak{F}_1 \in \mathcal{A}_1 \setminus \mathcal{M}_1$. Lutzer, van Mill, Pol [16] and Calbrix [3] showed that there are filters $\mathfrak{F}_\alpha \in \mathcal{A}_\alpha \setminus \mathcal{M}_\alpha$ for every α. Using the filter $\mathfrak{F}_1$ Calbrix [4] constructed filters $\mathfrak{F}_\alpha \in \mathcal{A}_\alpha \setminus \mathcal{M}_\alpha$ for odd ordinals α as follows

$$\mathfrak{F}_\alpha = \begin{cases} \mathbf{F}((\mathfrak{F}_{\alpha-2})^{\mathbb{N}}), & \text{if } \alpha \text{ and } \alpha - 1 \text{ are not limit ordinals} \\ \mathbf{F}(\prod_{n=1}^{\infty} \mathfrak{F}_{\alpha_n}), & \text{if } \alpha - 1 = \lim \alpha_n. \end{cases}$$

As was pointed out in [4], $(\mathfrak{F}_\alpha)^{\mathbb{N}} \in \mathcal{M}_{\alpha+1} \setminus \mathcal{A}_{\alpha+1}$.

Each filter $\mathfrak{F}$ generates the following sequence spaces in $s \subset Q$:

$$\sigma_{\mathfrak{F}} = \{(x_n) \in s : \exists A \in \mathfrak{F} \; x_n = 0 \text{ for } n \in A\},$$

$$c_{\mathfrak{F}} = \{(x_n) \in s : \forall \varepsilon > 0 \; \exists A \in \mathfrak{F} \; |x_n| < \varepsilon \text{ for } n \in A\},$$

and

$$\sigma_{\mathfrak{F}}(\delta) = \{(x_n) \in s : \exists \; A \in \mathfrak{F} \; \forall \; n \in A \; |x_n| \le \delta\},$$

where $\delta > 0$. Note that $c_{\mathfrak{F}}$ consists of all sequences that converge to zero with respect to the filter $\mathfrak{F}$. The next two lemmas are immediate consequences of our definitions.

Lemma 3.1. *Let $(\mathfrak{G}_n)_{n=1}^{\infty}$ be a sequence of filters and let $\mathfrak{G} = \prod_{n=1}^{\infty} \mathfrak{G}_n$, $\mathfrak{F} = \mathbf{F}_{n=1}^{\infty} \mathfrak{G}_n$. Then:*

(1) $\sigma_{\mathfrak{G}} = \prod_{n=1}^{\infty} \sigma_{\mathfrak{G}_n}$;

(2) $\sigma_{\mathfrak{G}}(\delta) = \prod_{n=1}^{\infty} \sigma_{\mathfrak{G}_n}(\delta)$;

(3) $\sigma_{\mathfrak{F}} = \mathop{\mathbf{F}}\limits_{n=1}^{\infty} \sigma_{\mathfrak{G}_n}$;

(4) $\sigma_{\mathfrak{F}}(\delta) = \mathop{\mathbf{F}}\limits_{n=1}^{\infty} \sigma_{\mathfrak{G}_n}(\delta)$;

(5) $c_{\mathfrak{G}} = \prod_{n=1}^{\infty} c_{\mathfrak{G}_n}$.

Lemma 3.2. *If $\mathfrak{F}$ is a filter then*

$$c_{\mathfrak{F}} = \bigcap_{k=1}^{\infty} \sigma_{\mathfrak{F}}(2^{-k}).$$

As a consequence of the fact that the constructions of transfinite sequences of Borel absorbers and Borel filters described above are almost identical we obtain the following lemma.

Lemma 3.3. *We have:*

(1) $\sigma = \sigma_{\mathfrak{F}_1}$ *and* $\Omega_2 = (\sigma_{\mathfrak{F}_1})^{\mathbf{N}}$;

(2) *If α is an odd ordinal then $\sigma_{\mathfrak{F}_\alpha} = \Lambda_\alpha$ (in particular $\sigma_{\mathfrak{F}_\alpha} \in \mathcal{A}_\alpha \setminus \mathcal{M}_\alpha$);*

(3) *If α is an even ordinal then*

$$\Omega_\alpha = \begin{cases} (\sigma_{\mathfrak{F}_{\alpha-1}})^{\mathbf{N}}, & \text{if } \alpha \text{ is not a limit ordinal} \\ \prod_{n=1}^{\infty} \sigma_{\mathfrak{F}_{\alpha_n}}, & \text{if } \alpha = \lim \alpha_n. \end{cases}$$

Proposition 3.4. *If α is an odd ordinal then there exists a homeomorphism $h_\alpha : Q \to Q^{\mathbf{N}}$ such that*

$$h_\alpha(\sigma_{\mathfrak{F}_\alpha}(2^{-k})) = \underbrace{\sigma_{\mathfrak{F}_\alpha} \times \sigma_{\mathfrak{F}_\alpha} \times \cdots \times \sigma_{\mathfrak{F}_\alpha}}_{k \text{ times}} \times Q \times Q \times \cdots$$

for $k = 1, 2, \ldots$.

Proof. The existence of the homeomorphism $h_1 : Q \to Q^{\mathbf{N}}$ sending $\sigma_{\mathfrak{F}_1}(2^{-k})$ onto

$$\underbrace{\sigma_{\mathfrak{F}_1} \times \sigma_{\mathfrak{F}_1} \times \cdots \times \sigma_{\mathfrak{F}_1}}_{k \text{ times}} \times Q \times Q \times \cdots$$

for $k = 1, 2, \ldots$ was proved in section 6 of [8].

The homeomorphisms h_α for $\alpha > 1$ are constructed inductively using the following elementary observation about products of homeomorphisms. Let k be a natural number. Assume that $(H_i)_i$ is a sequence of homeomorphisms from Y onto $Y^{\mathbf{N}}$ and let Z_i and X_i be subsets of Y such that

$$H_i(Z_i) = \underbrace{X_i \times X_i \times \cdots \times X_i}_{k \text{ times}} \times Y \times Y \times \cdots$$

for $i = 1, 2, \ldots$. If we define the homeomorphism $H = \mathbf{P}_{n=1}^{\infty} H_n : Y^{\mathbf{N}} \to (Y^{\mathbf{N}})^{\mathbf{N}}$ by

$$\left(\left(\mathop{\mathbf{P}}_{n=1}^{\infty} H_n\right)(y)\right)_{ij} = (H_i(y_i))_j, \qquad \text{for } y \in Y^{\mathbf{N}},$$

then

$$H\left(\prod_{i=1}^{\infty} Z_i\right) = \underbrace{\prod_{i=1}^{\infty} X_i \times \prod_{i=1}^{\infty} X_i \times \cdots \times \prod_{i=1}^{\infty} X_i}_{k \text{ times}} \times Y^{\mathbf{N}} \times Y^{\mathbf{N}} \times \cdots$$

and

$$H\left(\mathop{\mathbf{F}}_{i=1}^{\infty} Z_i\right) = \underbrace{\mathop{\mathbf{F}}_{i=1}^{\infty} X_i \times \mathop{\mathbf{F}}_{i=1}^{\infty} X_i \times \cdots \times \mathop{\mathbf{F}}_{i=1}^{\infty} X_i}_{k \text{ times}} \times Y^{\mathbf{N}} \times Y^{\mathbf{N}} \times \cdots$$

Let α be an odd ordinal for which h_α has been constructed. If we define $h_{\alpha+1} = \mathbf{P}_{n=1}^{\infty} h_\alpha$ then

$$h_{\alpha+1}(\sigma_{\mathfrak{Z}_\alpha}(2^{-k})^{\mathbf{N}}) = \underbrace{\sigma_{\mathfrak{Z}_\alpha}^{\mathbf{N}} \times \sigma_{\mathfrak{Z}_\alpha}^{\mathbf{N}} \times \cdots \times \sigma_{\mathfrak{Z}_\alpha}^{\mathbf{N}}}_{k \text{ times}} \times Q^{\mathbf{N}} \times Q^{\mathbf{N}} \times \cdots$$

for $k = 1, 2, \ldots$. Next we put $h_{\alpha+2} = \mathbf{P}_{n=1}^{\infty} h_{\alpha+1}$ (the role of Y is now played by $Q^{\mathbf{N}}$). We obtain:

$$h_{\alpha+2}(\mathbf{F}(\sigma_{\mathfrak{Z}_\alpha}(2^{-k})^{\mathbf{N}})) = \underbrace{\mathbf{F}(\sigma_{\mathfrak{Z}_\alpha}^{\mathbf{N}}) \times \mathbf{F}(\sigma_{\mathfrak{Z}_\alpha}^{\mathbf{N}}) \times \cdots \times \mathbf{F}(\sigma_{\mathfrak{Z}_\alpha}^{\mathbf{N}})}_{k \text{ times}} \times (Q^{\mathbf{N}})^{\mathbf{N}} \times (Q^{\mathbf{N}})^{\mathbf{N}} \times \cdots$$

for $k = 1, 2, \ldots$.

If we use Lemma 3.1 and we identify $(Q^{\mathbf{N}})^{\mathbf{N}}$ with Q as usual then we find

$$h_{\alpha+2}(\sigma_{\mathfrak{Z}_{\alpha+2}}(2^{-k})) = \underbrace{\sigma_{\mathfrak{Z}_{\alpha+2}} \times \sigma_{\mathfrak{Z}_{\alpha+2}} \times \cdots \times \sigma_{\mathfrak{Z}_{\alpha+2}}}_{k \text{ times}} \times Q \times Q \times \cdots$$

for $k = 1, 2, \ldots$, which finishes this step of the induction.

Now assume that $\alpha = \lim \alpha_n$ is a limit ordinal and that the h_{α_n}'s have been defined. If we put

$$h_{\alpha+1} = \mathop{\mathbf{P}}_{m=1}^{\infty} \left(\mathop{\mathbf{P}}_{n=1}^{\infty} h_{\alpha_n} \right)$$

then we have as above that

$$h_{\alpha+1}(\sigma_{\mathfrak{Z}_{\alpha+1}}(2^{-k})) = \underbrace{\sigma_{\mathfrak{Z}_{\alpha+1}} \times \sigma_{\mathfrak{Z}_{\alpha+1}} \times \cdots \times \sigma_{\mathfrak{Z}_{\alpha+1}}}_{k \text{ times}} \times Q \times Q \times \cdots$$

for $k = 1, 2, \ldots$. This finishes the induction.

From Proposition 3.4 we obtain

Corollary 3.5. *If α is odd then the pair $(Q, c_{\mathfrak{Z}_\alpha})$ is homeomorphic to $(Q, \Omega_{\alpha+1})$.*

Proof. The homeomorphisms h_α from Proposition 3.4 satisfy

$$h_\alpha(c_{\mathfrak{Z}_\alpha}) = \bigcap_{k=1}^{\infty} h_\alpha(\sigma_{\mathfrak{Z}_\alpha}(2^{-k})) = (\sigma_{\mathfrak{Z}_\alpha})^{\mathbb{N}} = \Omega_{\alpha+1}.$$

4. Function spaces $C_p(X)$ which are Borel absorbers.

First we describe the spaces X_α. If $\mathfrak{F}$ is a filter on $\mathbb{N}$, then $X_{\mathfrak{F}}$ denotes the space $\mathbb{N} \cup \{\infty\}$ topologized by isolating the points of $\mathbb{N}$ and using the family $\{A \cup \{\infty\}\}_{A \in \mathfrak{F}}$ as a neighbourhood base at ∞. If α is an even ordinal we let

$$X_\alpha = \begin{cases} X_{\mathfrak{Z}_{\alpha-1}}, & \text{if } \alpha \text{ is not a limit ordinal} \\ X_{\prod_{n=1}^{\infty} \mathfrak{Z}_{\alpha_n}}, & \text{if } \alpha = \lim \alpha_n. \end{cases}$$

Calbrix [3] observed that $C_p(X_\alpha) \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$. $C_p(X_\alpha)$ is a subset of the Hilbert cube $\tilde{Q} = Q \times [-\infty, \infty]$. We prove[1]

Theorem 4.1. *If α is an even ordinal then the pair $(\tilde{Q}, C_p(X_\alpha))$ is homeomorphic to (Q, Ω_α) and hence it is an $\mathcal{M}_\alpha$-absorber.*

[1] A similar result was proved independently by Cauty, Dobrowolski and Marciszewski.

Proof. Let α be an even countable ordinal. If α is not a limit then $(\tilde{Q}, C_p(X_\alpha))$ is homeomorphic to $(\tilde{Q}, c_{\mathfrak{F}_{\alpha-1}} \times \mathbb{R})$, cf. the proof of [8, Theorem 6.5]. By Corollary 3.5 the pair $(\tilde{Q}, c_{\mathfrak{F}_{\alpha-1}} \times \mathbb{R})$ is homeomorphic to $(\tilde{Q}, \Omega_\alpha \times \mathbb{R})$ and by [8, Lemma 6.4] $(\tilde{Q}, \Omega_\alpha \times \mathbb{R})$ is homeomorphic to (Q, Ω_α). If $\alpha = \lim \alpha_n$, then $(\tilde{Q}, C_p(X_\alpha))$ is homeomorphic to $(\tilde{Q}, c_{\mathfrak{F}} \times \mathbb{R})$, where $\mathfrak{F} = \prod_{n=1}^\infty \mathfrak{F}_{\alpha_n}$. By Lemma 3.1 $c_{\mathfrak{F}}$ is equal to $\prod_{n=1}^\infty c_{\mathfrak{F}_{\alpha_n}}$. Since by Corollary 3.5 each $(Q, c_{\mathfrak{F}_\alpha})$ is homeomorphic to (Q, Ω_{α_n+1}) the product $(Q^{\mathbb{N}}, \prod_{n=1}^\infty c_{\mathfrak{F}_{\alpha_n}})$ is homeomorphic to $(Q^{\mathbb{N}}, \prod_{n=1}^\infty \Omega_{\alpha_n+1})$. Since both $\prod_{n=1}^\infty \Omega_{\alpha_n+1}$ and $\Omega_\alpha = \prod_{n=1}^\infty \Lambda_{\alpha_n}$ are $\mathcal{M}_\alpha$-absorbers we have

$$
\begin{aligned}
(\tilde{Q}, C_p(X_\alpha)) &\cong (\tilde{Q}, c_{\mathfrak{F}} \times \mathbb{R}) \\
&\cong \left(Q^{\mathbb{N}} \times [-\infty, \infty], \left(\prod_{n=1}^{\infty} \Omega_{\alpha_n+1}\right) \times \mathbb{R}\right) \\
&\cong (\tilde{Q}, \Omega_\alpha \times \mathbb{R}) \cong (Q, \Omega_\alpha).
\end{aligned}
$$

It is possible to start the construction of the $\mathfrak{F}_\alpha$'s with an arbitrary sequence $(\mathfrak{G}_i)_i$ of $F_{\sigma\delta}$-filters instead of just the Fréchet filter. Then the basis step in the proof of Proposition 3.4 becomes

$$
h_2\left(\prod_{i=1}^{\infty} \sigma_{\mathfrak{G}_i}(2^{-k})\right) = \underbrace{\prod_{i=1}^{\infty} \sigma_{\mathfrak{G}_i} \times \prod_{i=1}^{\infty} \sigma_{\mathfrak{G}_i} \times \cdots \times \prod_{i=1}^{\infty} \sigma_{\mathfrak{G}_i}}_{k \text{ times}} \times Q^{\mathbb{N}} \times Q^{\mathbb{N}} \times \cdots ,
$$

which follows from [9, Proposition 3.1]. The result is a rich collection of spaces $C_p(X)$ that are topologically characterised by their Borel complexity, i.e. the α such that $C_p(X) \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$.

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