Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Vol 16, 1991

A FIXED-POINT THEOREM FOR TREE-LIKE CONTINUA

CHARLES L. HAGOPIAN

ABSTRACT. Let M be a tree-like continuum and let H be a map of $M \times [0,1]$ onto M such that $H(x,0) = x$ for each point x of M. Let f_t be the map of M into M defined by $f_t(x) = H(x,t)$. We prove that for some $t > 0$, the map f_t has a fixed point.

1. INTRODUCTION.

Recently, Minc [Mi] used an example of Bellamy [B] to construct a tree-like continuum that admits fixed-point-free self-maps arbitrarily close to the identity. We show that it is impossible to define a homotopy on a tree-like continuum that is made up of such maps. This is a partial solution to the problem of determining whether every deformation of a tree-like continuum has a fixed point [L, Problem 29]. Our proof is based on the dog-chases-rabbit principle [Bi2]. The author previously used this principle to prove that all deformations of uniquely arcwise connected continua [H2] and nonseparating plane continua [H3] have fixed points. For more information about fixed points of tree-like continua see [Bo], [C], [FM], [H4], [Ha], [M], [Ma], and [OR].

2. DEFINITIONS.

A map f of a space S is a *deformation* if there exists a map H of $S \times [0,1]$ onto S such that $H(x,0) = x$ and $H(x,1) = f(x)$ for each point x of S.

A *continuum* is a nondegenerate compact connected metric space.

A continuum is *uniquely arcwise connected* if it is arcwise connected and does not contain a simple closed curve.

A *nonseparating plane continuum* is a continuum in the plane whose complement relative to the plane is connected.

A *chain* is a finite collection $\{L_i : 1 \leq i \leq n\}$ of open sets such that $L_i \cap L_j \neq \emptyset$ if and only if $| i-j | \leq 1$. If $n > 2$ and the conditions of the preceding definition hold with the exception that $L_1 \cap L_n \neq \emptyset$, then the collection is called a *circular chain*. Each L_i is called a *link*.

A collection $\mathcal{R}$ of sets is *coherent* if for each nonempty proper subcollection $\mathcal{S}$ of $\mathcal{R}$, an element of $\mathcal{S}$ intersects an element of $\mathcal{R} \setminus \mathcal{S}$.

A finite coherent collection $\mathcal{T}$ of open sets is a *tree-chain* if no three elements of $\mathcal{T}$ have a point in common and no subcollection of $\mathcal{T}$ is a circular chain.

A continuum M is *tree-like* if for each positive number ϵ there is a tree chain covering M such that each element has diameter less than ϵ [Bil, p. 653].

A continuum is *hereditarily unicoherent* if each pair of its intersecting subcontinua has a connected intersection. Each tree-like continuum is hereditarily unicoherent.

3. The result.

Theorem *Suppose M is a tree-like continuum and H is a map of $M \times [0,1]$ onto M such that $H(x,0) = x$ for each point x of M. Let f_t be the map of M into M defined by $f_t(x) = H(x,t)$. Then for some $t > 0$, the map f_t has a fixed point.*

Proof. Assume each f_t with $t > 0$ is fixed-point free.

By the compactness of M, there is a positive number ϵ such that

(3.1) $\rho(x, f_1(x)) > 2\epsilon$ for each point x of M.

For each point x of M, let $\mathbb{Z}(x)$ denote the arc-component of M that contains x. Since M is hereditarily unicoherent, $\mathbb{Z}(x)$

does not contain a simple closed curve. Thus $\mathbb{Z}(x)$ is uniquely arcwise connected.

Notation Let y be a point of $\mathbb{Z}(x)\backslash(x)$. The unique arc and half-open arc in $\mathbb{Z}(x)$ with endpoints x and y are denoted by $[x, y]$ and $[x, y)$, respectively. We define $[x, x]$ to be $\{x\}$.

For each point x of M, Borsuk [Bo] showed there exists a unique sequence $a_1(x), a_2(x), \ldots$ of points of $\mathbb{Z}(x)$ such that $a_1(x) = x$ and for each positive integer n,

(3.2) $\rho(a_n(x), a_{n+1}(x)) = \epsilon$ [Bo,(4_n)],

(3.3) if $y \in [a_n(x), a_{n+1}(x))$, then $\rho(a_n(x), y) < \epsilon$ [Bo, (5_n)],

(3.4) $[x, a_n(x)] \cup [a_n(x), a_{n+1}(x)] = \{a_n(x)\}$ [Bo, (11)], and

(3.5) $[a_n(x), a_{n+1}(x)] \subset [x, f_1(a_n(x))]$ [Bo, (7_n), (13)].

For each positive integer n, let ψ_n be a homeomorphism of the half-open real line interval $[n-1, n)$ onto $[a_n(x), a_{n+1}(x))$. For each nonnegative real number r, let $\psi(r) = \psi_n(r)$ if $n-1 \leq r < n$.

Let $\mathbb{P}(x) = \cup\{[x, a_n(x)) : n = 2, 3, \ldots\}$. By (3.4), ψ is a one-to-one map of the nonnegative real line $[0, +\infty)$ onto $\mathbb{P}(x)$. The map ψ determines a linear ordering $<<$ of $\mathbb{P}(x)$ with x as the first point. The set $\mathbb{P}(x)$ is called a *Borsuk ray*.

By [H2, (3.6)].

(3.6) $\mathbb{P}(x)$ is $\{z \in \mathbb{Z}(x) : [x, y] \cap [z, f_1(z)] = \{z\}\}$.

For each point y of $\mathbb{P}(x)$,

(3.7) the Borsuk ray $\mathbb{P}(y)$ is $\{z \in \mathbb{P}(x) : y = z \text{ or } y << z\}$.

To see this let z be a point of $\mathbb{P}(x)$ such that $y = z$ or $y << z$. Then $[x, y] \cap [y, z] = \{y\}$. By (3.6), $[x, z] \cap [z, f_1(z)] = \{z\}$. Thus $[y, z] \cap [z, f_1(z)] = \{z\}$. It follows from (3.6) that $z \in \mathbb{P}(y)$. Hence $\mathbb{P}(y)$ contains $\{z \in \mathbb{P}(x) : y = z \text{ or } y << z\}$. Now (3.7) follows from (3.2) and the fact that $\mathbb{P}(y)$ is a one-to-one continuous image of $[0, +\infty)$.

By (3.2) and (3.5) with $n = 1$, for each point x of M,

(3.8) $\mathbb{P}(x) \cap [x, f_1(x)]$ is an arc of diameter at least ϵ.

Let τ be a positive number such that for each point x of M,

(3.9) the diameter of $H(x \times [0, \tau])$ is less than $\epsilon/10$.

By the compactness of M, there is a positive number δ less than $\epsilon/144$ such that
(3.10) $\rho(x, f_t(x)) > \delta$ for each point x of M and each number $t \geq \tau$.

Let $\mathcal{T}$ be a tree chain covering M such that each element of $\mathcal{T}$ has diameter less than δ.

Notation. For each point x of M, define $A(x) = \{y \in M : \epsilon/9 \leq \rho(x,y) \leq \epsilon/4\}$ and $C(x) = \{y \in M : \rho(x,y) = \epsilon/4\}$.

Let x_0 be a point of M. By (3.2), $\mathbb{P}(x_0) \cap C(x_0) \neq \emptyset$. Let x_1 be the first point of $\mathbb{P}(x_0)$ in $C(x_0)$. Let $\mathcal{C}_0$ be the chain in $\mathcal{T}$ that goes from x_0 to x_1. Since $\mathcal{T}$ is a tree chain, each link of $\mathcal{C}_0$ intersects $[x_0, x_1]$. By (3.2), $\mathbb{P}(x_1) \cap C(x_1) \neq \emptyset$. Let x_2 be the first point of $\mathbb{P}(x_1)$ in $C(x_1)$.

Note that
(3.11) $[x_1, x_2]$ misses $A(x_1) \cap \bigcup \mathcal{C}_0$.

To see this assume the contrary. Let y be a point of $[x_1, x_2] \cap A(x_1) \cap \bigcup \mathcal{C}_0$. Let L be a link of $\mathcal{C}_0$ that contains y. Let x be a point of $L \cap [x_0, x_1]$. By (3.7), $[x_1, x_2] \subset \mathbb{P}(x)$. Since the diameter of $[x, x_2]$ is less than ϵ, by (3.8), $[x, x_2] \subset [x, f_1(x)] \cap \mathbb{P}(x)$. It follows from (3.9) and the hereditary unicoherent of M that $[x_1, x_2] \cap H(x \times [0, \tau]) = \emptyset$. Thus $f_t(x) = y$ for some $t > \tau$, and since $\{x, y\} \subset L$, this contradicts (3.10). Hence (3.11) is true.

Let $\mathcal{C}_1$ be the chain in $\mathcal{T}$ that goes from x_1 to x_2.

Since $\mathcal{T}$ is a tree chain,
(3.12) each link of $\mathcal{C}_1$ intersects $[x_1, x_2]$.

Let $\mathcal{D}_1$ be the collection of links of $\mathcal{C}_1$ that lie in $A(x_1)$.

The sets $\bigcup \mathcal{D}_1$ and $\bigcup \mathcal{C}_0$ are disjoint; for otherwise, by (3.11) and (3.12), there exists a circular chain in $\mathcal{C}_0 \cup \mathcal{C}_1$, and this contradicts the fact that $\mathcal{T}$ is a tree chain. Hence $\mathcal{D}_1 \backslash \mathcal{C}_0 \neq \emptyset$.

We proceed inductively. Let n be a positive integer. Assume that for each positive integer $m \leq n$, there exist points x_m and x_{m+1} of M and subcollections $\mathcal{C}_m$ and $\mathcal{D}_m$ of $\mathcal{T}$ such that
(3.13m) x_{m+1} is the first point of $\mathbb{P}(x_m)$ in $C(x_m)$,
(3.14m) $\mathcal{C}_m$ is a chain from x_m to x_{m+1},

(3.15m) $\mathcal{D}_m$ is the collection of links of $\mathcal{C}_m$ that lie in $A(x_m)$,
(3.16m) $(\cup\mathcal{D}_m) \cap (\cup \cup \{\mathcal{C}_i : 0 \leq i < m\}) \neq \emptyset$, and
(3.17m) $\mathcal{D}_m \setminus \cup \{\mathcal{C}_i : 0 \leq i < m \neq \emptyset$.

Let x_{m+2} be the first point of $\mathbb{P}(x_{n+1})$ in $C(x_{n+1})$.

It follows from the argument for (3.11) that

(3.18) $[x_{n+1}, x_{n+2}]$ misses $A(x_{n+1}) \cap \cup\mathcal{C}_n$.

Let $\mathcal{C}_{n+1}$ be the chain in $\mathcal{T}$ that goes from x_{n+1} to x_{n+2}.

Since $\mathcal{T}$ is a tree chain,

(3.19) each link of $\mathcal{C}_{n+1}$ intersects $[x_{n+1}, x_{n+2}]$.

Let $\mathcal{D}_{n+1}$ be the collection of links of $\mathcal{C}_{n+1}$ that lie in $A(x_{n+1})$.

Since $\mathcal{T}$ is a tree chain, it follows from (3.18) and (3.19) that

(3.20) $(\cup\mathcal{D}_{n+1}) \cap (\cup\mathcal{C}_n) = \emptyset$.

Furthermore,

(3.21) $(\cup\mathcal{D}_{n+1}) \cap (\cup \cup \{\mathcal{C}_i : 0 \leq 1 < n\}) = \emptyset$.

To see this suppose an element L of $\mathcal{D}_{n+1}$ intersects $\cup \cup \{\mathcal{C}_i : 0 \leq i < n\}$. Let $\mathcal{E}$ be a chain in $\cup\{\mathcal{C}_i : 0 \leq i < n\}$ such that L intersects one end link of $\mathcal{E}$ and x_n belongs to the other end link of $\mathcal{E}$. Let $\mathcal{F}$ be the subchain of $\mathcal{C}_{n+1}$ with the property that x_{n+1} belongs to one end link of $\mathcal{F}$ and the other end link of $\mathcal{F}$ is L. It follows from (3.16n) and (3.20) that $\mathcal{C}_n \cup \mathcal{E} \cup \mathcal{F}$ contains a circular chain. This contradicts the fact that $\mathcal{T}$ is a tree chain. Hence (3.21) is true.

By (3.20) and (3.21), $\mathcal{D}_{n+1} \setminus \cup \{\mathcal{C}_i : 0 \leq i \leq n\} \neq \emptyset$. This completes the inductive step.

For each positive integer m, there exist points x_m and x_{m+1} of M and subcollections $\mathcal{C}_m$ and $\mathcal{D}_m$ of $\mathcal{T}$ that satisfy (3.13m)-(3.17m). Therefore $\mathcal{T}$ is infinite, and this contradicts the fact that $\mathcal{T}$ is a tree chain. Hence for some $t > 0$, the map f_t has a fixed point.

Comment. Parts of the preceding argument were derived from the author's proof that no homogeneous tree-like continuum contains an arc [H1].

References

[B] D. P. Bellamy, *A tree-like continuum without the fixed point property*, Houston J. Math. 6 (1979), 1-13.

[Bi1] R. H. Bing, *Snake-like continua*, Duke Math. J. **18** (1951), 653-663.

[Bi2] R. H. Bing, *The elusive fixed point property*, Amer. Math. Monthly **76** (1969), 119-132.

[Bo] K. Borsuk, *A theorem on fixed points*, Bull. Acad. Polon. Sci. **2** (1954), 17-20.

[C] H. Cook, *Tree-likeness of dendroids and λ-dendroids*, Fund. Math. **68** (1970), 19-22.

[FM] J. B. Fugate and L. Mohler, *A note on fixed points in tree-like continua*, Topology Proceedings, **2** (1977), 457-460.

[H1] C. L. Hagopian, *No homogeneous tree-like continuum contains an arc*, Proc. Amer. Math. Soc. **88** (1983), 560-564.

[H2] —, *The fixed-point property for deformations of uniquely arcwise connected continua*, Topology and its Appl. **24** (1986), 207-212.

[H3] —, *Fixed points of arc-component-preserving maps*, Trans. Amer. Math. Soc. **306** (1988), 411-420.

[H4] —, *Fixed points of tree-like continua*, Contemporary Math. **72** (1988), 131-137.

[Ha] O. H. Hamilton, *A fixed point theorem for pseudo-arcs and certain other metric continua*, Proc. Amer. Math. Soc. **2** (1951), 173-174.

[L] I. W. Lewis, *Continuum theory problems*, Topology Proc. **8** (1983), 361-394.

[M] R. Manka, *End continua and fixed points*, Bull. Acad. Polon. des Sci. ser. Math. Astronom. Phys. **7** (1975), 761-766.

[Ma] M. M. Marsh, *ϵ-mappings onto a tree and the fixed point property*, Topology Proceedings **14** (1989), 265-277.

[Mi] P. Minc, *A tree-like continuum admitting fixed point free maps with arbitrarily small trajectories*, to appear in Topology and its Appl.

[OR] L.G. Oversteegen and J. T. Rogers, Jr., *Fixed-point-free maps on tree-like continua*, Topology and its Appl. **13** (1982), 85-95.

California State University
Sacramento, California 95819