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SEMI-BOUNDARIES IN HYPERSPACES

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ABSTRACT.
Let $C(X)$ be the hyperspace of all subcontinua of a continuum X. In this paper we introduce the concept of semi-boundary. Given $A \in C(X)-\{X\}$, a subcontinuum B of A is in the semi-boundary of $C(A)$ if there exists a map $\alpha : [0,1] \to C(X)$ such that $\alpha(0) = B$ and $\alpha(t)$ in not contained in A for every $t > 0$. Using semi-boundaries we obtain characterizations of the interval, simple closed curves, local connectedness, acyclic finite graphs, hereditarily indecomposable continua, atriodic continua and continua containing n-ods.

INTRODUCTION.

The X will denote a *continuum* (i. e. a compact, connected space with metric d). The *hyperspace* $C(X)$ consists of all subcontinua of X with the Hausdorff metric H. Continuous functions are called *maps*. The unit closed interval in the real line is denoted by I. Given $A \in C(X)-\{X\}$, the *semi-boundary* of $C(A)$ is defined by $SB(A) = \{B \in C(A)$: there exists a map $\alpha : I \to C(X)$ such that $\alpha(0) = B$ and $\alpha(t)$ is not contained in A for all $t > 0\}$. Notice that $SB(A)$ depends on the containing space X. For simplicity, this dependence is suppressed in the notation.

Let us consider some examples: (a) Taking parametrized order arcs (see Def.1.1), it follows that $SB(\{x\}) = \{x\}$ for every $x \in X$ and $A \in SB(A)$ for all $A \in C(X) - \{X\}$. (b) If $A \in C(I) - \{I\}$ and A is not a one-point set, then $SB(A)$

is an arc and A is an end point of this arc if and only if $0 \in A$ or $1 \in A$. (c) If X is a single closed curve and $A \in C(X) - \{X\}$, then $SB(A)$ is an arc and A is not an end point of this arc. (d) Let X be the subspace of the Euclidean plane defined by $X = (\cup\{\overline{(0,1/n)(1,1/n)} : n \geq 1\}) \cup \overline{(0,0)(1,0)} \cup \overline{(0,0)(0,1)}$, where $\overline{pq}$ denotes the arc joining p and q. Define $A = (\cup\{\overline{(0,0)(1-1/n,1/n)} : n \geq 1\}) \cup \overline{(0,0)(1,0)} \cup \overline{(0,0)(0,1)}$. Then $\{(1-1/n,1/n)\} \in SB(A)$ for each n and $\{(1,0)\}$ is not in $SB(A)$. This show that $SB(A)$ is not necessarily closed.

In [12], S.B. Nadler, Jr., introduced and developed the concepts of arcwise, segmentwise and continuumwise accessiblility. All of them are inserted in the following context: Given $B \in \mathcal{A} \subset \mathcal{B} \subset 2^X = \{A \subset X : A$ is a nonempty closed subset of $X\}$, under which conditions is B arcwise (resp. segmentwise, continuumwise) accessible from $\mathcal{B} - \mathcal{A}$? That is, when does there exist an arc α in $\mathcal{B}$ (resp. a segment α in $\mathcal{B}$, a continuum α in $\mathcal{B}$) such that $\alpha \cap \mathcal{A} = \{B\}$? These concepts have been useful for the study of the structure of hyperspaces (see [12] and [4]). Semi-boundaries are also inserted in this context. Using Nadler's terminology, we could say that $C \in SB(A)$ if C is arcwise accessible from $C(X) - C(A)$.

Restricted semi-boundaries were used in [5]. Although semi-boundaries can be defined in every topological space, in this paper, we show that their use in hyperspaces has a special utility. The structure and characteristics of the semi-boundaries in $C(X)$ reflect many properties of the space X, using them we obtain characterizations of the interval, simple closed curves, local connectedness, acyclic finite graphs, hereditarily indecomposable continua, atriodic continua and continua containing n-ods.

The notions not defined here will be taken as in the book of S. B. Nadler, JR. [13]. I acknowledge many fruitful discussions with César Jiménez Espinosa on the topic of this paper. I wish to thank the referee for his useful comments and for suggesting to me Theorem 4.3 and its corollary.

1 BASIC FACTS.

Definition. Given $A, B \in C(X)$ such that $A \subset B \neq A$, a parametrized order arc from A to B is a map $\alpha : I \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$ and if $s < t$, then $\alpha(s) \subset \alpha(t) \neq \alpha(s)$. The existence of parametrized order arcs follows from [13, Thm. 1.8.]. Let $C^\wedge(X) = C(X) - \{X\}$ and $F_1(X) = \{ \{x\} \in C(X) : x \in X\}$. We write $X \simeq Y$ to denote that X is homeomorphic to Y.

1.2 Theorem. *Let $A \in C^\wedge(X)$.*
(a) $B \in SB(A)$ if and only if $B \in C(A)$ and there exists a map $\alpha : I \to C(X)$ such that $\alpha(0) = B, \alpha(t)$ is not contained in A for each $t > 0$ and if $s < t$, then $\alpha(s) \subset \alpha(t)$. Such a map will be named a removing map for B
(b) If $B \in SB(A)$ and $B \subset D \subset A$, then $D \in SB(A)$.
(c) $A \in SB(A)$.
(d) $SB(A)$ is pathwise connected.
(e) $SB(A) \subset Fr(C(A))$ (boundary of $C(A)$ in $C(X)$).
(f) If $B, D \in C(X)$, $B \cap D \neq \emptyset$, $B - D \neq \emptyset$, $D - B \neq \emptyset$ and E is a component of $B \cap D$, then $E \in SB(B) \cap SB(D)$.

Proof.
(a) ($\Rightarrow$) Let $\beta : I \to C(X)$ be a map such that $\beta(0) = B$ and $\beta(t)$ is not contained in A for every $t > 0$. Define $\alpha : I \to C(X)$ by $\alpha(t) = \cup\{\beta(s) \in C(X) : s \in [0, t]\}$.
(b) Let α be a removing map for B, define $\beta : I \to C(X)$ by $\beta(t) = D \cup \alpha(t)$. Clearly, β is a removing map for D.
(c) Every parametrized order arc from A to X is a removing arc for A.
(d) Let $B \in SB(A) - \{A\}$, let α be a parametrized order arc from B to A. By (b), Im $\alpha \subset SB(A)$.
(f) Take a parametrized order arc α from E to B. Given $t > 0$, $E \subset \alpha(t) \neq E$ and $\alpha(t) \subset B$, then $\alpha(t)$ is not contained in D. Thus $E \in SB(D)$. Similarly, $E \in SB(B)$.

1.3 Theorem. *Let $A \in C^\wedge(X)$ and $B \in C(A)$. Let $(B_n)n \subset C(X)$ be a sequence such that $B_n \to B$. Then each one of the*

following conditions implies that $B \in SB(A)$:
(a) If B_n is not contained in A and $B_{n+1} \subset B_n$ for each n;
(b) If B_n is not contained in A and $B_n \cap B \neq \emptyset$ for each n;
(c) If $B_n \in SB(A)$ and $B_n \cap B \neq \emptyset$ for each n.

Proof. (a) For each n, choose a map $\alpha_n : [1/(n+1), 1/n] \to C(X)$ such that $\alpha_n(1/(n+1)) = B_{n+1}, \alpha_n(1/n) = B_n$ and if $s < t$, then $\alpha_n(s) \subset \alpha_n(t)$. Define $\alpha : I \to C(X)$ by $\alpha(t) = \alpha_n(t)$ if $t \in [1/(n+1), 1/n]$ and $\alpha(0) = B$. Then α is a removing map for B.
(b) For each n, define $C_n = B \cup B_n \cup B_{n+1} \cup \ldots$. Then $C_n \in C(X)$, $C_n \to B$ and C_n is not contained in A and $C_{n+1} \subset C_n$ for every n. Thus (a) implies that $B \in SB(A)$.
(c) For each n, let α_n be a removing map for B_n. Choose $t_n > 0$ such that $H(B_n, \alpha_n(t_n)) < 1/n$. Then $\alpha_n(t_n) \to B, \alpha_n(t_n)$ is not contained in A and $B \cap \alpha_n(t_n) \neq \emptyset$ for each n. By (b), $B \in SB(A)$.

1.4 Theorem. *If $A \in C^\wedge(X)$ and $B \in SB(A)$, then there exists a minimal element (with respect to the inclusion) $C \in SB(A)$ such that $C \subset B$.*

Proof. By Theorem 1.3 (c), the intersection of a countable nest of elements in $SB(A)$ is in A. Then the proof follows from the Brower Reduction Theorem (see [7, p. 61])

2. Hereditarily Indecomposable Continua.

2.1 Definition. A continuum is *indecomposable* provided that X is not the union of two proper subcontinua. It is *hereditarily indecomposable* provided that each of its subcontinua is indecomposable.

It is easy to see that a continuum X is hereditarily indecomposable if and only if whenever A and B are subcontinua of X such that $A \cap B \neq \emptyset$, then $A \subset B$ or $B \subset A$.

The following theorem is related to Kelley's Theorem 8.2 in [6] and it is easy to prove using Theorem 1.2 (f).

Theorem. *X is hereditarily indecomposable if and only if $SB(A) = \{A\}$ for every $A \in C^\wedge(X)$.*

ARCS.

3.1 Theorem. *If X is a nondegenerate continuum then $X \simeq I$ if and only if $X \simeq SB(A)$ for every $A \in C^\wedge(X) - F_1(X)$.*

Proof. ($\Leftarrow$) Let $A \in C^\wedge(X) - F_1(X)$ and let $B \in SB(A) - \{A\}$. Since $SB(A)$ is arcwise connected, there exists an arc α in X. Since $SB(\alpha) \subset C(\alpha) \simeq$ *Disc* X, $SB(\alpha)$ is a plane continuum. Identify α with the interval $[0,1]$. If there exists $E \in SB(\alpha)$ such that $0, 1 \notin E$, then E is of the form $E = [a, b]$ with $0 < a \leq b < 1$. Define $\sigma : [0, a] \times [b, 1] \to C(X)$ by $\sigma(s,t) = [s,t]$. Then σ is an injective map and Theorem 1.2 (b) implies that $\text{Im}(\sigma) \subset SB(\alpha)$. Thus $X \simeq SB(\alpha)$ contains a subspace C which is homeomorphic to the square $I \times I$. Consider a simple triod Z (a space of the form of the letter Y) contained in C. Then every subcontinuum of Z is in $SB(Z)$. Then $C(Z) \subset SB(Z)$. But $C(Z)$ contains a cube $I \times I \times I$. It follows that $X \simeq SB(Z)$ contains a cube. This contradiction proves that if $[a, b] \in SB(\alpha)$, then $a = 0$ or $b = 1$. Then $SB(\alpha)$ is a subcontinuum of the set $\{[0, b] : b \in I\} \cup \{[a, 1] : a \in I\}$ which is homeomorphic to I. Therefore $X \simeq SB(\alpha) \simeq I$.

4. N-ODS.

Definition. An $n-od$ (resp. $\infty - od$) in X is an element $B \in C(X)$ for which there exists an $A \in C(X)$ such that $B - A$ contains at least n components (resp. infinitely many components). X is said to be *atriodic* if it does not contain 3-*ods*. Given $A \in C^\wedge(X)$, we will denote by $m(A)$ the set of minimal elements in $SB(A)$.

In [5] it was proved that X contains $n-ods$ (resp. $\infty - ods$) if and only if $C(X)$ contains $n-$cells (resp. Hilbert cubes).

4.2 Theorem. *Let $n \geq 1$. Then X contains $n-ods$ if and only if there exists $E \in C^\wedge(X)$ such that $m(E)$ has at least n elements.*

Proof. ($\Rightarrow$) Suppose that $n \geq 2$. Let $A, B \in C(X)$ be such that $A \subset B$ and $B - A$ has at least n components. Let $D_1, \ldots, D_n$ be components of $B - A$. Then $Cl(D_i) \cap A \neq \emptyset$ and $A \cup D_i \in C(X)$ for each i. Fix an open subset U of X such that $A \subset U$ and $D_i - Cl(U) \neq \emptyset$ for each i. Let E be the component of $Cl(U)$ such that $A \subset E$. Given i and taking a parametrized order arc from A to $A \cup D_i$, we can find a continuum which properly contains A and which is properly contained in $(A \cup D_i) \cap U$. Hence, $E \cap D_i \neq \emptyset$. By Thm 1.2 (f), there exists an element $F_i \in SB(E)$ contained in $Cl(D_i)$. Let E_i be a minimal element in $SB(E)$ such that $E_i \subset F_i$. Since U is open, it is not possible that $E_i \subset A$, so $E_i \cap D_i \neq \emptyset$. Hence $E_1, \ldots, E_n$ are pairwise different.

($\Leftarrow$) Let $E \in C^\wedge(X)$ be such that $m(E)$ has at least n elements. Let $E_1, \ldots, E_n \in m(E)$. If there exist A and $B \in C(X)$ such that $A \cap B$ has infinitely many components, then X contains $\infty-ods$ ([11, Thm, 14]). Suppose then that $A \cap B$ has finitely many components for every $A, B \in C(X)$. For each i, choose a removing map α_i for E_i. Given i, E_j is not contained in E_i for every $j \neq i$, so a number $t_i > 0$ can be choosen in such a way that E_j is not contained in $\alpha_i(t_i)$ for every $j \neq i$. Given $i \neq j$, we will show that there exists $s > 0$ such that $\alpha_i(s) \cap \alpha_j(t_i) \subset E$. If $E_i \cap \alpha_j(t_j) = \emptyset$, then it is easy to find s. Suppose then that $C_1, \ldots, C_r$ are the components of $E_i \cap \alpha_j(t_j)$. Suppose also that there is not such an s. Given $k \geq 1$, $\alpha_i(1/k) \cap \alpha_j(t_j)$ is not contained in E. Let C be the union of the components of $\alpha_i(1/k) \cap \alpha_j(t_j)$ which do not intersect E_i. Then C is a compact set disjoint from E_j, and there exists a number $z_k \in (0, 1/k)$ such that $\alpha_i(z_k) \cap C = \emptyset$. Choose a point x in $\alpha_i(z_k) \cap \alpha_j(t_j) - E$. Let D_k be the component of $\alpha_i(1/k) \cap \alpha_j(t_j)$ such that $x \in D_k$. Then $D_k \cap E_i \neq \emptyset$, so there exists $1 \leq 1_k \leq r$ such that $C_{1_k} \subset D_k$.

Let $1_0 \in \{1, \ldots, r\}$ be such that $1_0 = 1_k$ for infinitely many k. Suppose that $k_1 < k_2 < \ldots$ are such that $1_0 = 1_{k_m}$ for all m. Given $m \geq 2$, $D_{k_m} \in C(X), D_{k_m}$ is not contained in E, $C_{1_0} \subset D_{k_m}$ and D_{k_m} is a component of $\alpha_i(1/k_m) \cap \alpha_j(t_j) \subset \alpha_i(1/k_{m-1}) \cap \alpha_j(t_j)$, so D_{k_m} is contained in $D_{k_{m-1}}$. Since $D_{k_m} \subset \alpha_i(1/k_m) \to E_i \subset E$, we have that (Thm. 1.3 (a)) $D = \cap\{D_{k_m} : m \geq 1\} \in SB(E)$ and $D \subset E_i \cap \alpha_j(t_j)$. Thus $E_i = D \subset \alpha_j(t_j)$. This contradicts the choice of t_j and proves the existence of s.

Then, given i, we may choose $s_i \in (0, t_i)$ such that $\alpha_i(s_i) \cap \alpha_j(t_j) \subset E$ for every $j \neq i$. Define $B = E \cup \alpha_1(s_1) \cup \ldots \cup \alpha_n(s_n) \in C(X)$. Then B is an $n-od$..

4.3 Theorem. *X is an atriodic continuum if and only if $SB(A)$ is either a point or an arc for every $A \in C^{\wedge}(X)$.*

Proof. We will use the following consequence of Thm. 1.8 in [1]: X is an atriodic continuum if and only if there is not three subcontinua of X with nonempty intersection and such that no one of them is contained in the union of the other two.

($\Rightarrow$) By theorems 1.2 (c), 1.4 and 4.2, $m(A)$ has only one or two elements. We only analyze the case $m(A) = \{B_1, B_2\}$ where $B_1 \neq B_2$, the proof of the other one is analogous. Choose two parametrized order arcs β_1 and β_2 from B_1 to A and B_2 to A, respectively. And fix removing maps α_1 and α_2 for B_1 and B_2, respectively.

We assert that if $B_1 \subset B \subset A$ and $B \in C(X)$, then $B \in Im\ \beta_1$. Suppose, on the contrary, that $B \notin Im\ \beta_1$. Let $t_0 = \max\ \{t \in I : \beta_1(t) \subset B\}$. Then $\beta_1(t_0) \subset B \neq \beta_1(t_0)$. Fix a point $p \in B - \beta_1(t_0)$ and let $t_1 > t_0$ be such that $p \notin \beta_1(t_0)$. Choose a point q in $\beta_1(t_1) - B$. Then there exists $t_2 > 0$ such that $p, q \notin \alpha_1(t_2)$. Then $\alpha_1(t_2)$ is not contained in A. Thus $B, \beta_1(t_1)$ and $\alpha_1(t_2)$ are three subcontinua of X with nonempty intersection and no one of them is contained in the union of the other two. This is a contradiction which proves the assertion.

A similar assertion holds for B_2. Then, $SB(A) = Im\ \beta_1 \cup Im\ \beta_2$ (see theorems 1.2 (b)and 1.4).

If there exists and element $E \in Im\beta_1 \cap Im\beta_2$ such that $E \neq A$, choose a point $x_0 \in A - E$. Notice that $B_1, B_2 \in SB(E)$ and

α_1 and α_2 are removing maps for B_1 and B_2, respectively (with respect to E). As in the proof of the sufficiency of Theorem 4.2, it is possible to find $s_1, s_2 > 0$ such that $\alpha_1(s_1) \cap \alpha_2(s_2) \subset E$. We may suppose that $x_0 \notin \alpha_1(s_1) \cup \alpha_2(s_2)$. Thus $A, E \cup \alpha_1(s_1)$ and $E \cup \alpha_2(s_2)$ are three subcontinua of X with nonempty intersection and no one of them is contained in the union of the other two. This contradiction proves that $Im\ \beta_1 \cap Im\ \beta_2 = \{A\}$.

Hence $SB(A) = Im\ \beta_1 \cup Im\ \beta_2$ is an arc.

($\Leftarrow$) Suppose that X is not atriodic. Theorem 4.2 implies that there exists $E \in C^\wedge(X)$ such that $m(E)$ has at least three different elements B_1, B_2, and B_3. Let $\beta_1 : I \to C(X)$ (resp. β_2 and β_3) be a parametrized order arc from B_1 (resp. B_2 and B_3) to E. Then for each $i = 1, 2, 3$, $Im\ \beta_i$ is a subarc of the arc $SB(E)$ joining B_i to E. This implies that $B_i \in Im\ B_j$ for some $i \neq j$. Thus $B_j \subset B_i$ which is a contradiction. Therefore X is atriodic.

4.4 Corollary. *If X is a Peano continuum, then X is either an arc or a simple closed curve if and only if $SB(A)$ is an arc for all proper nondegenerate subcontinua A of X.*

Proof. ($\Rightarrow$) Is immediate, ($\Leftarrow$) By Theorem 4.3, X is an atriodic Peano continuum. Then X is either an arc or a simple closed curve.

4.5 Theorem. *If there exists $E \in C^\wedge(X)$ such that $m(E)$ is infinite, then X contains $\infty-$ods.*

Proof. Let $E \in C^\wedge(X)$ be such that $m(E)$ is infinite. Choose a sequence $E_1, E_2, \ldots$ of pairwise different elements in $m(E)$. We may suppose that $E_n \to E_0$ for some $E_0 \in C(X)$. Suppose that $A \cap B$ has finitely many components for every A and B in $C(X)$.

If $E_0 \subset E_n$ for infinitely many n, by Thm. 1.3 (c), $E_0 \in SB(E)$. But each E_n is minimal, so infinitely many of them are equal to E_0. This contradiction proves that this case is not possible. Then we may suppose that E_0 is not contained in E_n for every n.

For each n, let α_n be a removing map for E_n. Let $t > 0$ be such that $\alpha_n(t)$ does not contain E_0, then there exists $M > 0$ such that E_m is not contained in $\alpha_n(t)$ for every $m \geq M$. Thus there exists $t_n > 0$ such that E_m is not contained in $\alpha_n(t_n)$ for every $m \neq n$. Proceeding as in the proof of the necessity in Thm. 4.2, a number $s_n \in (0, t_n)$ can be found in such a way that $\alpha_n(s_n) \cap (\alpha_1(t_1) \cup \ldots \cup \alpha_{n-1}(t_{n-1})) \subset E$ and $H(E_n, \alpha_n(s_n)) < 1/n$.

Define $B = E \cup \alpha_1(s_1) \cup \alpha_2(s_2) \cup \ldots$. Then $\alpha_n(s_n) - E$ is open and closed in $B - E$ for each n. Therefore, B is an ∞*−od*.

The converse of Theorem 4.5 is not true as it is shown in the following example.

4.6 Example. Choose an hereditarily indecomposable continuum Z contained in the Euclidean plane $\mathbb{R}^2$ such that $(0,0) \in Z$. For each n, let $Z_n = \{(1/n)(x, y, (1/n) \parallel (x,y) \parallel) \in \mathbb{R}^3 : (x,y) \in Z\}$. Then each Z_n is a subcontinuum in $\mathbb{R}^3$ such that $Z_n \simeq Z$. If $n \neq m$, then $Z_n \cap Z_m = \{0\}$ and $Z_n \to \{0\}$.

Define $X = Z_1 \cup Z_2 \cup \ldots$. Then X is a continuum and $X - \{0\}$ has infinitely many components. Thus X is an $\infty-$*od*.

Let $E \in C^\wedge(X)$, we will show that $m(E)$ is finite. We analyze three cases:

(a) $0 \notin E$. Then there exists n such that $E \subset Z_n - \{0\}$. Since Z_n is hereditarily indecomposable, then $SB(E) = \{E\}$ and $m(E) = \{E\}$.

(b) $0 \in E$. Then $E \cap Z_n \in C(X)$ for each n. Let $A \in m(E)$, if $0 \notin A$, then $A \subset Z_n \cap E - \{0\}$ for some n. Then if $B \in C(X)$ is such that $A \subset B$ and $0 \notin B$, then $B \subset Z_n \cap E$. This implies that $A \notin SB(E)$. This contradiction proves that $0 \in A$.

(b.1) Z_n is not contained in E for infinitely many n. Take $A \in m(E)$. Then $0 \in A$. Since $Z_n \to \{0\}$, by Thm. 1.3 (b) we have $\{0\} \in SB(E)$. Thus $\{0\} = A$. Hence $m(E) = \{\{0\}\}$.

(b.2) The set of n such that Z_n is not contained in E is a finite set $\{n_1, \ldots, n_r\}$. Notice that $E \cap Z_{n_i} \in SB(E)$ for every i. Given $A \in m(E)$, we assert that $E \cap Z_{n_i} \subset A$ for some i. Suppose that this is not true. Let α be a removing map for A. So there exists $t > 0$ such that $\alpha(t)$ does not contain any $E \cap Z_{n_i}$. Since $E \cap Z_{n_i}$ and $\alpha(t) \cap Z_{n_i}$ are intersecting subcontinua of Z_{n_i}, we have that $\alpha(t) \cap Z_{n_i} \subset E \cap Z_{n_i}$ for each i. Then $\alpha(t) \subset E$. This contradiction proves that $E \cap Z_{n_i} \subset A$ for some i. Then A belongs to the set $\{E \cap Z_{n_i}, \ldots, E \cap Z_{n_r}\}$. Therefore $m(E)$ is finite.

5. Locally Connected Continua.

5.1 Theorem. *The following assertions are equivalent.*

(a) X is locally connected,

(b) If $A \in C^\wedge(X)$ and $p \in Fr(A)$, then $\{p\} \in SB(A)$ and,

(c) $SB(A) = Fr(C(A))$ for every $A \in C^\wedge(X)$.

Proof. (a) $\Rightarrow$ (b) follows from Thm. 1.3 (a).

(b) $\Rightarrow$ (a). Suppose that X is not locally connected. Then there exists an open subset U of X and there exists a component D of U such that D is not open. Let $p \in D - Int(D)$. Then $p \in Fr(Cl(D))$ and $\{p\} \notin SB(Cl(D))$.

(a) $\Rightarrow$ (c). Let $B \in Fr(C(A))$, then there exists $b \in Fr(A)$ such that $b \in B$. By (b), $\{b\} \in SB(A)$. Thm 1.2 (b) implies that $B \in SB(A)$.

(c) $\Rightarrow$ (b). Let $A \in C^\wedge(X)$ and $p \in Fr(A)$, then $\{p\} \in Fr(C(A)) = SB(A)$.

Using the Baire Category Theorem, the following lemma is easy to prove.

5.2 Lemma. *Let $\{A_n : n \geq 1\}$ be a countable family of pairwise disjoint closed subsets of X and let U be an open subset of X such that $A_n \cap U \neq \emptyset$ for every n and $U \subset \cup\{A_n : n \geq 1\}$. Then $(Int(A_n)) \cap U \neq \emptyset$ for infinitely many n.*

5.3 Theorem. *Suppose that X is a pathwise connected continuum. Then X is locally connected if and only if $m(A) \subset F_1(A)$ for every $A \in C^\wedge(X)$.*

Proof. ($\Rightarrow$) Let $A \in C^\wedge(X)$ and let $B \in m(A)$. Then $B \in Fr(C(A))$, so there exists $b \in B$ such that $b \in Fr(A)$,. By Thm. 5.1, $\{b\} \in SB(A)$, so $B = \{b\} \in F_1(A)$.

($\Leftarrow$) Suppose that X is not locally connected. Then there exists an open subset U of X and there exists a component D^* of U such that D^* is not open. Choose $p_0 \in D^* - Int(D^*)$. Let W, V be open subsets of X such that $p_0 \in W \subset Cl(W) \subset V \subset Cl(U) \subset U$. Notice that infinitely many components of $Cl(U)$ intersect W.

Let $D = \{D : D$ is a component of $Cl(U)$ and $D \cap Cl(V) \neq \emptyset\}$. Let L be an arc in X. Given $x \in L \cap Cl(V)$, there exists a subarc L_x of L such that $s \in L_x$, x is not an end point of L_x and $L_x \subset U$. Since $L \cap Cl(V)$ is a compact subset of L, $L \cap Cl(V)$ can be covered by finitely many sets of the form L_x. Thus $L \cap Cl(V)$ can be covered by finitely many elements of D. Therefore L intersects at most finitely many sets of the form $D \cap Cl(V)$ with $D \in D$.

Choose a point $x_0 \in X - Cl(U)$. Given $D \in D$, we assert that there exists finitely many elements $D_1, \ldots, D_n$ in D and there exists arcs $L_1, \ldots, L_n$ in $X - V$ such that $D_n = D$, $x_0 \in L_1, L_1 \cap D_1 \cap Cl(V) \neq \emptyset$ and $i \in \{2, \ldots, n\}, L_i \cap D_{i-1} \cap Cl(V) \neq \emptyset$ and $L_i \cap D_i \cap Cl(V) \neq \emptyset$.

To prove this, let $x \in D \cap Cl(V)$ and let $\gamma : I \to X$ be an injective map such that $\gamma(0) = x_0$ and $\gamma(1) = x$. Let $\{C_1, \ldots, C_m\} = \{E \in D : Im\, \gamma \cap Cl(V) \cap E \neq \emptyset\}$ with $C_1 = D$. Let $t_1 = \min \gamma^{-1}(D \cap Cl(V))$. If $\gamma([0, t_1))$ does not intersect $Cl(V)$, put $n = 1, D_1 = D$ and $L_1 = \gamma([0, t_1])$. If $\gamma([0, t_1))$ intersects $Cl(V)$, let $s_1 = \max \gamma^{-1}(Cl(V)) \cap [0, t_1)$. Then we may suppose that $\gamma(s_1) \in C_2$. Let $t_2 = \min \gamma^{-1}(C_2 \cap Cl(V))$. If $\gamma([0, t_2))$ does not intersect $Cl(V)$, put $n = 2$, $D_2 = D$, $L_2 = \gamma([s_1, t_1])$, $D_1 = C_2$ and $L_1 = \gamma([0, t_2])$. If $\gamma([0, t_2)) \cap Cl(V) \neq$

$\emptyset$, this procedure can be continued to obtain $n, D_1, \dots, D_n$ and $L_1, \dots, L_n$.

Given $D \in D$, a sequence $C = (n, L_1, \dots, L_n, D_1, \dots, D_n)$ with the properties mentioned above will be called a *chain ending in D*. The size of C is n. Define $n(D) = \min \{n :$ there exists a chain C ending in D such that n is the size of $C\}$. Choose a chain $C(D) = (n(D), L_1(D), \dots, L_{n(D)}(D), E_1(D), \dots, E_{n(D)}(D))$ ending in D.

Let $D_1 = \{D : D$ is a component of $Cl(U)$ and $D \cap Cl(W) \neq \emptyset\}$. Then $D_1 \subset D$. For each m, let $E_m = \{D \in D_1 : n(D) = m\}$ and $F_m = \{D \in D : n(D) = m\}$. Then $E_m \subset F_m$.

We will construct an $A \in C^\wedge(X)$ such that $m(A)$ is not contained in $F_1(A)$. We consider three cases:

(a) There exists m such that E_m is infinite.

Since every element of E_m intersects $Cl(W)$, we have that there exists a point $y_0 \in Cl(W)$ such that every neighborhood of y_0 intersects infinitely many elements of E_m. Consider $m_0 = \min \{n :$ every neighborhood of y_0 intersects infinitely many elements of $F_n\}$. Let D_0 be the component of $Cl(U)$ such that $y_0 \in D_0$. Then there exist an open subset W_0 of X such that $y_0 \in W_0 \subset V$. Since there exists a neighborhood of y_0 which intersects only finitely many elements of $\{D \in D : n(D) < m_0\}$, then we may suppose that either $W_0 \cap (\{D \in D : n(D) < m_0\}) = \emptyset$ or $W_0 \cap (\{D \in D : n(D) < m_0\} = W_0 \cap D_0$. In both cases $W_0 \cap (\cup\{D \in D : n(D) < m_0\}) \subset D_0$.

Choose a sequence $(D_r)r$ of pairwise different elements of F_{m_0} such that $D_r \cap N_{1/r}(y_0) \neq \emptyset$ (if $\varepsilon > 0, N\varepsilon(y_0)$ denotes the open ball of radius ε about y_0) and $D_r \neq D_0$ for each r and $D_r \to D'$ for some $D' \in C(X)$. Since $y_0 \in D' \subset Cl(U)$, then $D' \subset D_0$.

Given r, let $F_r = L_1(D_r) \cup \dots \cup L_{m_0}(D_r) \cup E_1(D_r) \cup \dots \cup E_{m_0}(D_r)$ and let $B_r = Cl(F_r \cup F_{r+1} \cup \dots)$. Then $F_r, B_r \in C(X), D_r \subset F_r \subset B_r$ and $B_{r+1} \subset B_r$ for each r. Thus $B_r \to B_0 = \cap\{B_r : r \geq 1\}$. For each r, notice that $n(E_i(D_r)) = i$ for all $i = 1, \dots, m_0$, thus $F_r \cap W_0 \subset D_0 \cup D_r$. Then $W_0 \cap B_r = W_0 \cap Cl(F_r \cup F_{r+1} \cup \dots) \subset Cl(W_0 \cap (F_r \cup F_{r+1} \cup \dots)) \subset$

$Cl(D_0 \cup D_r \cup D_{r+1} \cup \dots) = D_0 \cup D_r \cup D_{r+1} \cup \dots$. Thus $W_0 \cap B_r \subset D_0 \cup D_r \cup D_{r+1} \cup \dots$. This implies that $W_0 \cap B_0 \subset D_0$.

Let W_1 be an open subset of U such that $y_0 \in W_1 \subset Cl(W_1) \subset W_0$. Let A be the component of $D_0 \cup (X - W_1)$ which contains B_0. Then $A \in C(X)$ and $B_0 \in C(A)$. To see that $B_0 \in SB(A)$, take $r \geq 1$. Let $s > r$ be such that $N_{1/s}(y_0) \subset W_1$. Let $x \in N_{1/s}(y_0) \cap D_s$, then $x \notin D_0 \cup (X - W_1)$. Thus $\emptyset \neq B_s \cap (X - A) \subset B_r \cap (X - A)$. By Thm. 1.3 (a), we have that $B_0 \in SB(A)$.

Let $B* \in m(A)$ be such that $B* \subset B_0$ (Thm. 1.4). If $B* \in F_1(A)$, then there exists $a \in B_0$, such that $\{a\} \in SB(A)$. Let α be a removing map from $\{a\}$. If $a \in W_0$, then $a \in D_0$ and there exists $t > 0$ such that $\alpha(t) \subset Cl(U)$. Thus $\alpha(t) \subset D_0$. So $\alpha(t) \subset A$ which is absurd. If $a \notin W_0$, then there exists $t > 0$ such that $\alpha(t) \cap Cl(W_1) = \emptyset$. This implies that $\alpha(t) \subset A$. This contradiction proves that $B*$ is an element of $m(A) - F_1(A)$ and completes this case.

(b)E_m is finite for each m.

Then D_1 is countable. Let $D_2 = \{D \in D_1 : D \cap W \neq \emptyset\}$, then D_2 is countable. Since $W \subset \cup\{D : D \in D_2\}$, Lemma 5.2 implies that $\text{Int}(D) \cap W \neq \emptyset$ for infinitely many $D \in D_2$. Choose a sequence $D_1, D_2, \dots$ of pairwise different elements of D_2 such that $\text{Int}(D_r) \cap W \neq \emptyset$ for each r. Choose points $x_r \in \text{Int}(D_r) \cap W$. We may suppose that $D_r \to D'$ for some $D' \in C(X)$ and $x_r \to y_0$ for some $y_0 \in D'$. Given $r > s$, r is called a *son of s* if, considering the chosen chain $C(D_r) = (n(D_r),\ L_1(D_r), \dots, L_{n(D_r)}(D_r),\ E_1(D_r), \dots E_{n(D_r)}(D_r))$ for D_r, we have that $D_s = E_i(D_r)$ for some $i \in \{1, \dots, n(D_r)\}$. Now we consider two cases.

(b.1) $\{s : s$ has finitely many sons$\}$ is finite.

Then there exists a sequence (s_r) such that $s_1 < s_2 < \dots$ and s_{r+1} is son of s_r for every r.

Given r, let $F_r = D_{s_r}$ and $z_r = x_{s_r}$. Since s_{r+1} is son of s_r, then $F_r = E_i(F_{r+1})$ for some $i \in \{1, \dots, n(F_{r+1}) - 1\}$. Since $n(E_i(F_{r+1})) = i$, then $n(F_r) < n(F_{r+1})$ and $F_r = E_{n(F_r)}(F_{r+1})$.

Define $G_r = E_{n(F_r)}(F_{r+1}) \cup L_{n(F_r)+1}(F_{r+1}) \cup E_{n(F_r)+1}(F_{r+1}) \cup \ldots \cup L_{n(F_{r+1})}(F_{r+1}) \cup E_{n(F_{r+1})}(F_{r+1})$. Then $G_r \in C(X)$ and $F_r, F_{r+1} \subset B_r$. Define $B_r = Cl(G_r \cup G_{r+1} \cup \ldots)$. Then $B_r \in C(X)$ and $B_{r+1} \subset B_r$. Thus $B_r \to B_0 = \cap\{B_r : r \geq 1\}$.

If $r < s$ and $i \in \{n(F_s), \ldots, n(F_{s+1})\}$, $n(E_i(F_{s+1})) = i \geq n(F_s) > n(F_r)$. Then $F_r \cap W \cap G_s = \emptyset$. Hence $\text{Int}(F_r) \cap W \cap B_s = \emptyset$. Therefore $\text{Int}(F_r) \cap W \cap B_0 = \emptyset$ for every r. For each r, fix $\varepsilon_r > 0$ such that $\varepsilon_r < 1/r$ and $Cl(N\varepsilon_r(z_r)) \subset \text{Int}(F_r) \cap W$. Then $Cl(N\varepsilon_r(z_r)) \cap B_0 = \emptyset$. Let $R = \{y_0\} \cup (\cup\{Cl(N\varepsilon_r(z_r)) : r \geq 1\})$. Then R is compact and $R \cap B_0 \subset \{y_0\}$.

For each $b \in B_0 - \{y_0\}$, let $\delta_b > 0$ be such that $Cl(N_{\delta_b}(b)) \cap R = \emptyset$. If $b = y_0 \in B_0$, then choose $\delta_b > 0$ such that $Cl(N_{\delta_b}(b)) \subset V$. For each $b \in B_0$, let Q_b be the component of $Cl(N_{\delta_b}(b))$ such that $b \in Q_b$.

Define $A = Cl(\cup\{Q_b : b \in B_0\})$. Then $B_0 \subset A, A \in C(X)$ and if $b \in B_0$, then $\{b\} \notin SB(A)$.

Given r, we will show that B_r is not contained in A. Let D_0 be the component of $Cl(U)$ such that $y_0 \in D_0$. Then $D' \subset D_0$. Suppose that $F_k \neq D_0$ with $k = r$ or $r + 1$. Then $F_k \subset B_r$. Given $b \in B_0 - \{y_0\}$, $N\varepsilon_k(z_k) \cap Q_b = \emptyset$ and if $b = Y_0 \in B_0$, then $Q_b \subset D_0$. Therefore $Q_b \cap N\varepsilon_k(z_k) \subset D_0 \cap F_k = \emptyset$. This implies that $N\varepsilon_k(z_k) \cap A = \emptyset$. Thus $z_k \in B_r - A$.

By Thm. 1.3 (a), $B_0 \in SB(A)$. Let $B* \in m(A)$ be such that $B* \subset B_0$. Then $B* \notin F_1(X)$ and $B* \in m(A) - F_1(A)$.

(b.2) $\{s : s$ has finitely many sons $\}$ is infinite.

Let $(s_r)r$ be a sequence such that $s_1 < s_2 < \ldots, s_r$ has finitely many sons and s_{r+1} is not a son of $s_1, \ldots, s_r$ for every r.

For each r, let $F_r = D_{s_r}$, $z_r = x_{s_r}$,$G_r = L_1(F_r) \cup \ldots \cup L_{n(F_r)}(F_r) \cup E_1(F_r) \cup \ldots \cup E_{n(F_r)}(F_r)$ and $B_r = Cl(G_r \cup G_{r+1} \cup \ldots)$. Thus $G_r, B_r \in C(X)$ and $B_{r+1} \subset B_r$. Then $B_r \to B_0 = \cap\{B_r : r \geq 1\}$. If $r < k$, s_k is not a son of s_r, then F_r is different from each one of the sets $E_{1(F_k)}, \ldots, E_{n(F_k)}(F_k)$. Thus $F_r \cap W \cap G_k = \emptyset$. Hence $\text{Int}(F_r) \cap W \cap B_0 = \emptyset$.

For each r, let $\varepsilon_r > 0$ be such that $\varepsilon_r < 1/r$ and $Cl(N\varepsilon_r(z_r)) \subset \text{Int}(F_r) \cap W$. Define $R = \{y_0\} \cup (\cup\{Cl(N\varepsilon_r(z_r)) : r \geq 1\})$. Then

R is compact and $R \cap B_0 \subset \{y_0\}$.

For each $b \in B_0 - \{y_0\}$, let $\delta_b > 0$ be such that $Cl(N_{\delta_b}(b)) \cap R = \emptyset$. If $b = y_0 \in N_0$, then choose $\delta_b > 0$ such that $Cl(N_{\delta_b}(b)) \subset V$. For each $b \in B_0$, let Q_b be the component of $Cl(N_{\delta_b}(b))$ such that $b \in Q_b$.

Define $A = Cl(\cup\{Q_b : b \in B_0\})$. Then $B_0 \subset A$, $A \in C(X)$ and if $b \in B_0$, then $\{b\} \notin SB(A)$.

Proceeding as in (b.1), $B_r - A \neq \emptyset$ for every r Then $B_0 \in SB(A)$ and if $B* \in m(A)$ is such that $B* \subset B_0$, then $B* \in m(A) - F_1(A)$.

This completes the proof of the theorem.

6. Acyclic Finite Graphs.

6.1 Theorem. *Suppose that X is pathwise connected. If lim sup $m(A_n)$ is at most countable for every sequence $(A_n)n$ in $C^\wedge(X)$ which converges in $C(X)$, then X is a dendrite.*

Proof. First, we will prove that if $A \in C(X)$, then each arc α with end points in A is contained in A. Suppose, on the contrary, that there exists an arc α with end points a and b, such that $a, b \in A$ and α is not contained in A. We may suppose that $(\alpha - \{a, b\}) \cap A = \emptyset$. Let $\{x_n : n \geq 1\}$ be a countable dense subset of $\alpha - \{a, b\}$. For each n, choose a subarc α_n of $\alpha - \{a, b\}$ such that if p_n and q_n are the end points of α_n, then $x_n \in \alpha_n - \{p_n, q_n\}$ and diameter $(\alpha_n) < 1/n$. Define $A_n = A \cup (\alpha - (\alpha_n - \{p_n, q_n\}))$. Then $A_n \to A \cup \alpha$, $\{p_n\}$, $\{q_n\} \in SB(A_n)$ and $\alpha \subset lim\ sup\{p_n\} \subset lim\ sup\ m(A_n)$. Then $lim\ sup\ m(A_n)$ is uncountable. This contradiction shows that $\alpha \subset A$.

Then we have the following consequences:

(a) A is pathwise connected for every $A \in C(X)$,

(b) X does not contain simple closed curves,

(c) If a, $b \in X$ and $a \neq b$, there exists a unique arc in X joining them. This arc will be denoted by $\overline{ab}$ and $\overline{aa}$ will denote the set {a},

(d) X is hereditarily unicoherent and,

(e) X is a dendroid.

Now we will prove that X is locally connected. Suppose that this is not true. Then there exists an open subset U of X and there exists a component D of U such that D is not open. Choose a point $p \in D - Int(D)$. Let V be an open subset of X such that $p \in V \subset Cl(V) \subset U$. Let $(D_n)n$ be a sequence of pairwise different components of $Cl(V)$ such that $D_n \cap D = \emptyset$ and $N1/n(p) \cap D_n \neq \emptyset$ for each n. We may suppose that $D_n \to D_0$ for some $D_0 \in C(X)$. Then $p \in D_0 \subset D$ and since $D_n \cap Fr(V) \neq \emptyset$, then $D_0 \cap Fr(V) \neq \emptyset$. Hence D_0 has uncountably many points.

Choose a countable dense subset $\{a_n : n \geq 1\}$ of D_0. Choose a point $x_1 \in U - D$ such that $d(a_1, x_1) < 1$. Let $q_1 \in Cl(D)$ be such that $\overline{x_1 q_1} \cap Cl(D) = \{q_1\}$ and let $z_1 \in \overline{x_1 q_1} - \{x_1, q_1\}$ be such that diameter $(\overline{x_1 z_1}) < 1$. Notice that $a_2 \notin \overline{x_1 q_1}$, then there exists a point $x_2 \in U - (D \cup \overline{x_1 q_1})$ such that $d(a_2,\ x_2) < 1/2$. Let $q_2 \in Cl(D)$ be such that $\overline{x_2 q_2} \cap Cl(D) = \{q_2\}$ and let $z_2 \in \overline{x_2 q_2} - \{x_2 q_2\}$ be such that diameter $(\overline{x_2 z_2}) < 1/2$ and $\overline{x_2 z_2} \cap \overline{x_1 q_1} = \emptyset$. Proceeding in this way it is possible to construct sequences of points $(x_n)n, (q_n)n$ and $(z_n)n$ of x such that, for each n, diameter $(\overline{x_n z_n}) < 1/n$, $z_n \in \overline{x_n q_n} - \{x_n, q_n\}$, $x_n \in U - (D \cup \overline{x_1 q_1} \cup n \ldots \cup \overline{x_{n-1} q_{n-1}}), d(a_n, x_n) < 1/n, \overline{x_n q_n} \cap Cl(D) = \{q_n\}$ and $\overline{x_n z_n} \cap (\overline{x_1 q_1} \cup \ldots \cup \overline{x_{n-1} q_{n-1}}) = \emptyset$

For each n, define $A_n = Cl(D) \cup \overline{z_1 q_1} \cup \ldots \cup \overline{z_n q_n}$. Then $A_n \in C(X)$ and $A_n \subset A_{n+1}$. Thus $(A_n)n$ is a sequence in $C^\wedge(X)$ which converges in $C(X)$. Since $\overline{x_n z_n} \cap A_n = \{z_n\}$, we have that $\{z_n\} \in m(A_n)$.

Each point in D_0 is an accumulation point of the set $\{a_n : n \geq 1\}$. Then every point in D_0 is an accumulation point of the set $\{z_n : n \geq 1\}$. This implies that $D_0 \subset \mathit{lim\ sup}\ m(A_n)$. Hence $\mathit{lim\ sup}\ m(A_n)$ is an uncountable set. This contradiction proves that X is locally connected. Therefore X is a dendrite.

The converse of Theorem 6.1 is not true as it is shown in the following example.

6.2 Example. For each rational number $z = r/s$ in the interval $(0, 1)$ with r and s relatively prime positive integers, let L_z be the segment $L_z = \{(z,y) \in \mathbb{R}^2 : 0 \leq y \leq 1/s\}$. Let $X = (I \times \{0\}) \cup (\cup\{L_z : z \text{ is a rational number in } (0,1)\})$. Then X is a dendrite. For each n, let $A_n = I \times \{0\}$. Then $F_1(A_n) \subset SB(A_n)$, so $F_1(A_n) = m(A_n)$. Thus *lim sup* $m(A_n) =$ *lim sup* $F_1(A_n) = I \times \{0\}$. Therefore lim sup $m(A_n)$ has uncountably many points.

The following theorem is related to [6, Lemma 5.2].

6.3 Theorem. *Let x be a pathwise connected continuum. Then X is an acyclic finite graph if and only if lim sup $m(A_n)$ is finite for every sequence $(A_n)n$ in $C^\wedge(X)$ which converges in $C(X)$.*

Proof. ($\Leftarrow$) By Thms. 6.1, 5.1 and 5.3, X is a dendrite. Taking constant sequences, we have that $m(A)$ is finite for every $A \in C^\wedge(X)$ and $m(A) = \{\{x\} : x \in Fr(A)\}$. We will prove then that if X is a dendrite where every $A \in C(X)$ has a finite boundary, then X is an acyclic finite graph.

Choose a convex metric d for X (see [1] and [8]), then $D\varepsilon(x) \in C(X)$ for every $\varepsilon > 0$ and $x \in X$, where $D\varepsilon(x) = \{y \in X : d(x,y) \leq \varepsilon\}$ If $x \neq y$, $\overline{xy}$ will denote the unique arc in X joining x and y, and $\overline{xx}$ will denote the set $\{x\}$. If $\varepsilon > 0$ and $x \in X$, let $L\varepsilon(x) = \cup\{\overline{xy} : y \in Fr(D\varepsilon(x)\}$. Since $Fr(D\varepsilon(x))$ is finite, $L\varepsilon(x)$ is a finite union of arcs and $L\varepsilon(x) \subset D\varepsilon(x)$.

Given $x \in X$, we will show that there exists $\varepsilon_x > 0$ such that $D\varepsilon_x(x) = L\varepsilon_x(x)$. Suppose, on the contrary, that $D\varepsilon(x)$ is not contained in $L\varepsilon(x)$ for every $\varepsilon > 0$. Then it is possible to construct sequences $(\varepsilon_n)n \subset (0,\infty)$ and $(x_n)n$, $(z_n)n \subset X$ such that $x_n \in D\varepsilon_n(x) - L\varepsilon_n(x)$, $\varepsilon_{n+1} < \min\{d(x,x_n),\ 1/n\}$, $z_n \in \overline{xx_n} - \{x, x_n\}$ and $\overline{x_n z_n} \cap (D\varepsilon_{n+1}(x) \cup L\varepsilon_n(x)) = \emptyset$ for every n.

Define $A = \cup\{\overline{xz_n} : n \geq 1\}$. Then $A \in C(X)$. Notice that $D\varepsilon_n(x) \cap (\overline{xx_1} \cup \ldots \cup \overline{xx_{n+1}}) \subset L\varepsilon_n(x)$. Then $\overline{x_n z_n} \cap (\overline{xx_1} \cup \ldots \cup \overline{xx_{n-1}}) = \emptyset$. This implies that $\overline{x_n z_n} \cap A = \{x_n\}$ and $z_n \notin \{z_1, \ldots, z_{n-1}\}$. Thus $\{z_1,\ z_2,\ \ldots\}$ is an infinite subset of $Fr(A)$. This is a contradiction because $Fr(A)$ is finite. Then we have shown the existence of ε_x.

Taking finitely many sets of the form $L\varepsilon_x(x)$ covering X, we have that X is a dendrite which is a finite union of arcs. It is easy to prove that X is an acyclic finite graph.

($\Rightarrow$) We are supposing that X does not contain simple closed curves and X is of the form: $X = L_1 \cup \ldots \cup L_m$ where each L_i is an arc and $L_i \cap L_j = \emptyset$ or $L_i \cap L_j$ is a point which is an end point of both L_i and L_j. For each i, let a_i and b_i be the end points of L_i and let $J_i = L_i - \{a_i, b_i\}$.

Let $(A_n)n$ be a sequence in $C^\wedge(X)$ which converges in $C(X)$. Let $A = \lim A_n$. We will show that (*lim sup* $m(A_n)) \cap \{ \{w\} : w \in J_i\}$ has at most two points for each i. Suppose, on the contrary, that there exist three different one-point sets $\{x\}$, $\{y\}$ and $\{z\}$ in the set lim sup $m(A_n) \cap \{ \{w\} : w \in J_i\}$ for some i. We will identify J_i with $I = [0,1]$. We may suppose that $x < y < z$. Then the intervals $(0, y)$ and $(y, 1)$ are open subsets of X.

By Theorems 5.1 and 5.3, $m(B) = \{ \{w\} \in F_1(X) : w \in Fr(B)\}$ for all $B \in C^\wedge(X)$. Since $\{x\}, \{z\} \in$ lim sup $m(A_n) =$ lim sup $\{ \{w\} : w \in Fr(A)\} \subset$ lim sup $F_1(A_n)$, then x, $z \in$ lim sup $A_n = A =$ lim A_n. Thus sequences $(x_n)n$ and $(z_n)n$ can be chosen such that $x_n \to x$, $z_n \to z$ and x_n, $z_n \in A_n$ for every n. Fix two points $y_1 \in (x, y)$ and $y_2 \in (y, z)$. Then there exists N such that $x_n \in (0, y_1)$ and $z_n \in (y_2, 1)$ for every $n \geq N$. Since X contain no simple closed curves, $(x_n, z_n) \subset A_n$. Then (y_1, y_2) is an open subset of X contained in A_n. Hence $y \in (y_1, y_2)$ and $(y_1, y_2) \cap Fr(A_n) = \emptyset$ for each $n \geq N$. Thus $\{y\} \notin$ lim sup $\{ \{w\} \in F_1(X) : w \in Fr(A_n)\}) =$ lim sup $m(A_n)$. This contradiction proves that (lim sup $m(A_n)) \cap \{ \{w\} : w \in J_i\}$ has at most two points for each i.

Since lim sup $m(A_n) \subset F_1(X)$ and $F_1(X) = \{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_m\} \cup \{ \{w\} : w \in J_1 \cup \ldots \cup J_m\}$, we conclude that *lim sup* $m(A_n)$ is finite.

7. Simple Closed Curves.

7.2 Definition. Define $\mathcal{S} : C^\wedge(X) \to C(C(X))$ by $\mathcal{S}(A) = Cl(SB(A))$.

7.2 Theorem. *Let $A \in C^{\wedge}(X)$. If $\mathcal{S}|C(A)$ is continuous, then A is hereditarily unicoherent.*

Proof. Suppose that there exists H and $K \in C(X)$ such that $H \cap K$ is disconnected and $H \cup K \neq X$. Let P and Q be two nonempty closed subset of X such that $P \cup Q = H \cap K$. Fix a point $p \in P$ and let $\alpha : I \to C(X)$ be a parametrized order arc joining $\{p\}$ and H. Let $t_0 = \min \{t : \alpha(t) \cap Q \neq \emptyset\}$. Then $\alpha(t_0) \cap Q \neq \emptyset$ and $t_0 > 0$. Choose an increasing sequence $(t_n)n$ in $[0, t_0)$ such that $t_n \to t_0$.

Let $B_0 = \alpha(t_0) \cup K$ and let $B_n = \alpha(t_n) \cup K \in C(X)$, then $B_n \to B_0$. For all n, let C_n be the component of $B_n \cap \alpha(t_0)$ which contain $\alpha(t_n)$. Since $B_n \cap \alpha(t_0) = \alpha(t_n) \cup (\alpha(t_0) \cap K) = [\alpha(t_n) \cup (\alpha(t_0) \cap P)] \cup (\alpha(t_0) \cap Q)$ and $\alpha(t_n) \cup (\alpha(t_0) \cap P)$ and $\alpha(t_0) \cap Q$ are two nonempty disjoint closed subsets of X, we have that B_n is not contained in $\alpha(t_0)$ and $\alpha(t_0)$ is not contained in B_n. Thus, by Thm. 1.2 (f), $C_n \in SB(B_n) \subset \mathcal{S}(B_n)$. Since $\alpha(t_n) \subset C_n \subset \alpha(t_0), C_n \to \alpha(t_0)$. Therefore, by hypothesis, $\alpha(t_0) \in \mathcal{S}(B_0)$.

Let U and V be disjoint open subsets of X such that $P \subset U$, $Q \subset V$ and $Cl(U) \cap Cl(V) = \emptyset$. Then $\alpha(t_0) - (U \cup V)$ and $K - (U \cup V)$ are disjoint nonempty closed subsets of X. Let W and Z be open subsets of X such that $\alpha(t_0) - (U \cup V) \subset W$ and $K - (U \cup V) \subset Z$.

Since $\alpha(t_0) \in \mathcal{S}(B_0)$, there exists a sequence $(C_n)n$ in $SB(B_0)$ such that $C_n \to \alpha(t_0)$. Since $\alpha(t_0)$ is not contained in K, there exists R such that C_R is not contained in $K, C_R \subset U \cup V \cup W, C_R \cap U \neq \emptyset$ and $C_R \cap V \neq \emptyset$. Then $C_R \cap \alpha(t_0) \neq \emptyset$.

Choose a point $q \in \alpha(t_0) \cap Q$. Let L_1 be the component of $\alpha(t_0) \cap Q$ such that $q \in L_1$. Then $L_1 \subset V$ and by taking a parametrized order arc from L_1 to K it is possible to find $L \in C(X)$ such that $L_1 \subset L \subset K, L_1 \neq L$ and $L \subset V$. Then L is not contained in $\alpha(t_0)$. So $L \cap (V - \alpha(t_0)) \neq \emptyset$.

Define $M = C_R \cup \alpha(t_0) \cup L \in C(B_0)$. Since $C_R \in SB(B_0)$, by Thm. 1.2 (b), $M \in SB(B_0) \subset \mathcal{S}(B_0) = \lim \mathcal{S}(B_n)$ and M is not contained in $\alpha(t_0)$. Thus there exists a sequence $(M_n)n \subset$

$C(X)$ such that $M_n \in SB(B_n)$ for each n and $M_n \to M$. Then there exists N such that $M_n \subset U \cup V \cup W$, $M_n \cap U \neq \emptyset, M_n \cap V \neq \emptyset$ and $M_n \cap (V - \alpha(t_0)) \neq \emptyset$.

Notice that $M_N \subset (K \cap Cl(V)) \cup (\alpha(t_N) \cup Cl(U)), \emptyset \neq M_N \cap (V - \alpha(t_0)) \subset K \cap Cl(V) \cap M_N$, $\emptyset \neq M_N \cap U \subset M_N \cap (\alpha(t_N) \cup Cl(U))$, $(K \cap Cl(V)) \cap (\alpha(t_N) \cup Cl(U)) \subset K \cap Cl(V) \cap \alpha(t_N) \subset (K \cap H \cap Cl(V)) \cap \alpha(t_N) = Q \cap \alpha(t_N) = \emptyset$. This contradicts the connectivity of M_N and completes the proof of the theorem.

7.3 Corollary. *If S is continuous, then every proper subcontinuum of X is unicoherent.*

7.4 Definition. A *generalized Warzaw circle* is an arcwise connected circle like continuum which is not a simple closed curve. By Theorem 6 in [9], X is a generalized Warzaw circle if and only if there exists a bijective map $f : [0, \infty) \to X$ such that $f[0,1] = Cl(f[t,\infty)) - f[t,\infty)$ for every $t > 1$. Such an f is said to be a *rolling map* for X.

7.5 Lemma. *Let X be generalized Warzaw circle with a rolling map f. Then $C(X) = \{f[a,b] : 0 \leq a \leq b\} \cup \{f[0,b] \cup f[a,\infty) : b \geq 1\}$.*

Next we will restate Theorem 2 in [10] of Nadler and Quinn:

7.6 Theorem. *X is an atriodic pathwise connected space if and only if X is a simple closed curve, an arc or a generalized Warzaw circle.*

7.7 Lemma. *Let X be a generalized Warzaw circle with a rolling map f. Let $A = f[0,1]$. Then $SB(A) = \{f[a,1] : a \in I\}$.*

7.8 Theorem. *Let X be a pathwise connected space, then the following assertions are equivalent:*
(a) X is a simple closed curve,
(b) S is continuous,
(c) $SB(A) - \{A\}$ is disconnected for every $A \in C^\wedge(X) - F_1(X)$ and,
(d) $SB(A) \cap F_1(X)$ has exactly two elements for every $A \in C^\wedge(X) - F_1(X)$.

Proof. Clearly (a) $\Rightarrow$ (b), (c) and (d).

(b) $\Rightarrow$ (a). By Corollary 7.3, every proper subcontinuum of X is unicoherent. If X contains a simple closed curve S, then X is equal to S. Suppose then that X does not contain simple closed curves. Then every pair of different points x and y in X can be joined by a unique arc which will be denoted by $\overline{xy}, \overline{xx}$ will denote the set $\{x\}$.

First, we will prove that X is hereditarily pathwise connected. Suppose, on the contrary, that there exists $A \in C(X)$ and there exist two different point x_0 and y_0 in A such that $\overline{x_0y_0} \cap A = \{x_0, y_0\}$. Then $A - \{x_0, y_0\}$ and $\overline{x_0y_0} - \{x_0, y_0\}$ are separated sets and $A \cup \overline{x_0y_0}$ is not unicoherent, so $A \cup \overline{x_0y_0} = X$, Cor. 7.3.

Choose a point $p_0 \in \{\overline{x_0y_0}\} - x_0y_0$. Given $a \in A, x_0$ or y_0 is in $\overline{ap_0}$. Define $H = \{a \in A : x_0 \in \overline{ap_0}\}$ and $K = \{a \in A : y_0 \in \overline{ap_0}\}$. Then $A = H \cup K$ and $H \cap K = \emptyset$. Since A is connected, we may suppose that $Cl(H) \cap K \neq \emptyset$. Given $a \in H, \overline{ax_0} - \{x_0\}$ is a connected subset of $(A - \{x_0, y_0\}) \cup (\overline{x_0y_0} - \{x_0, y_0\})$. Then $\overline{ax_0} \subset A$. Thus $\overline{ax_0} \subset H$. Hence H is pathwise connected. Similarly, K is pathwise connected.

Choose a point $x_1 \in Cl(H) \cap K$ and we may suppose that $\overline{x_1y_0} \cap Cl(H) = \{x_1\}$. Then $x_1 \neq x_0$. Notice that $p_0 \in \overline{x_1x_0} - \{x_1, x_0\}$ and $Cl(H) \cap \overline{x_1x_0} = \{x_1, x_0\}$. Then $Cl(H) \cup \overline{x_1x_0}$ is not unicoherent. Therefore $X = Cl(H) \cup \overline{x_1x_0}$.

We assert that $Cl(H) \cap K = \{x_1\}$. To see this, suppose that there exists a point $x_2 \in Cl(H) \cap K - \{x_1\}$. Then $x_2 \notin \overline{x_1x_0}$. Hence $x_2 \in \mathrm{Int}(Cl(H))$. This implies that $\{x_2\} \notin \mathcal{S}(Cl(H))$. Choose a dense subset $\{a_1, a_2, \dots\}$ of H. Given n, let $B_n = \overline{x_0a_1} \cup \dots \cup \overline{x_0a_n} \subset H$. Then $x_2 \notin B_n$. Let m_n be such that $d(x_2, a_{m_n}) < 1/n$ and $a_{m_n} \notin B_n$, then $m_n > n$. Choose a point $c_n \in \overline{x_0a_{m_n}} - \{a_{m_n}\}$ such that $\overline{c_na_{m_n}} \cap B_n = \emptyset$ and $d(a_{m_n}, c_n) < 1/n$. Define $C_n = B_n \cup \overline{x_0c_n} \in C(Cl(H))$, then $\overline{c_na_{m_n}} \cap c_n = \{c_n\}$. Thus $\{c_n\} \in SB(C_n) \subset \mathcal{S}(C_n)$. Notice that $C_n \to Cl(H)$ and $\{c_n\} \to \{x_2\}$. By hypothesis, $\mathcal{S}(C_n) \to \mathcal{S}(Cl(H))$, so $\{x_2\} \in \mathcal{S}(Cl(H))$. This contradiction proves that $Cl(H) \cap K = \{x_1\}$.

Choose a parametrized order arc α from $\{x_1\}$ to $Cl(H)$. For each n, let $t_n = 1/n$. For each n choose a point $p_n \in \alpha(t_n) - (\alpha(t_{n+1}) \cup \overline{x_0x_1})$. Then $p_n \in Cl(H) \subset A$ and $p_n \neq x_1$. Thus $p_n \in H$, $x_0 \in \overline{p_np_0}$ and $p_0 \in \overline{p_nx_1}$. Since $\alpha(t_n) \cap \overline{p_nx_1} \subset (\overline{p_np_0} - \{p_0\}) \cup (\overline{p_0x_1}) - \{p_0\})$ and intersects both sets, we have that $\alpha(t_n) \cup \overline{p_nx_1}$ is a not unicoherent. Then $X = \alpha(t_n) \cup \overline{p_nx_1}$. Therefore, for each n, $p_n \in \overline{p_{n+1}x_1} - \{p_{n+1}\}$ and this implies that $(\overline{p_{n+1}p_n} - \{p_n\}) \cap \overline{p_nx_1} = \emptyset$. Then $\overline{p_{n+1}p_n} \subset \alpha(t_n)$. Hence $\overline{p_np_{n+1}} \to \{x_1\}$.

Consider the set $S = \overline{x_1p_1} \cup \overline{p_1p_2} \cup \overline{p_2p_3} \cup \dots$. Since $p_n \in \overline{p_{n+1}x_1}$ for each n and $\overline{p_np_{n+1}} \to \{x_1\}$, we have that S is a simple closed curve. This contradicts our supposition and proves that X is hereditarily pathwise connected.

Then X is an hereditarily pathwise connected continuum which does not contain simple closed curves. This implies that X is hereditarily unicoherent. Therefore X is a dendroid. From lemma 3 in [2], it follows that there exist two points w_0, z_0 in X such that $\overline{w_0z_0}$ is a maximal arc in X.

Since $\mathcal{S}$ is not continuous for $X \simeq$ Interval, $X \neq \overline{w_0z_0}$. Let U be an open subset of X such that $\overline{w_0z_0} \subset Cl(U) \neq X$. Let A be the component of $Cl(U)$ such that $\overline{w_0z_0} \subset A$. Choose $\varepsilon > 0$ such that $z_0 \notin N\varepsilon(w_0)$. Given n, let A_n be the component of $A - N\varepsilon/n(w_0)$ which contains z_0. Then $A_n \subset A_{n+1}$ for each n and $A_n \to A_0 = Cl(\cup\{A_n : n \geq 1\})$

Given n, choose a point $w_n \subset \overline{w_0z_0}$ such that $\overline{w_0w_n} \cap A_n = \emptyset$ and $d(w_0, w_n) < 1/n$. Define $B_n = A_n \cup \overline{w_nz_0}$. $B_n \in C(X)$ and $\{w_n\} \in SB(B_n) \subset \mathcal{S}(B_n)$. Moreover $w_n \to w_0$.

Given $x \in A - \{w_0, z_0\}$, there exists $y \in \overline{w_0z_0}$ such that $\overline{xy} \cap \overline{w_0z_0} = \{y\}$. The maximality of $\overline{w_0z_0}$ implies that $y \neq w_0, z_0$ so there exists n such that $N\varepsilon/n(w_0) \cap \overline{xy} = \emptyset$. Then $x \in A_n$. This proves that $A - \{w_0, z_0\} \subset \cup\{A_n : n \geq 1\}$ and $A = A_0$. By the continuity of $\mathcal{S}$, we have that $\{w_0\} \in \mathcal{S}(A)$. Then there exists $B \in SB(A)$ such that $B \subset U$. Let β be a removing map for B. Then there exists $t > 0$ such that $\beta(t) \subset U$ and this implies that $\beta(t) \subset A$ which is absurd.

This contradiction proves that X must contain a simple closed curve and so X is a simple closed curve.

(c) or (d) $\Rightarrow$ (a). Suppose (c) or (d). First, we will prove that X is atriodic. Suppose, on the contrary, that X contains a triod. Since X is arcwise connected, it is easy to prove that there exists $C \in C(X)$ and there exist arcs γ_1, γ_2, and γ_3 in X such that $\gamma_1 - C$, $\gamma_2 - C$ and $\gamma_3 - C$ are disjoint subsets of X and $\gamma_i \cap C$ is an end point a_i of γ_i for $i = 1, 2, 3$.

For $i = 1, 2, 3$, let b_i be the end point of γ_i such that $b_i \neq a_i$. Choose a point $c_i \in \gamma_i - \{a_i, b_i\}$. Let B_i be the subarc of γ_i joining a_i and c_i. Define $A = C \cup \beta_1 \cup \beta_2 \cup \beta_3$. Then $A \in C^\wedge(X)$ and $\{c_1\}, \{c_2\}, \{c_3\} \in SB(A)$.

We will show that $SB(A) - \{A\}$ is pathwise connected. Let $\mathcal{C}$ be the path component of $\{c_1\}$ in the space $SB(A) - \{A\}$. Taking a parametrized order arc from $\{c_1\}$ to $\beta_1 \cup \beta_2 \cup C$. we have that $\beta_1 \cup \beta_2 \cup C \in \mathcal{C}$.

With a parametrized order arc from $\{c_2\}$ to $\beta_1 \cup \beta_2 \cup C$, we obtain that $\{c_2\} \in \mathcal{C}$. Similarly, $\{c_3\} \in \mathcal{C}$.

Let $D \in SB(A) - \{A\}$. If $c_1 \in D$, taking a parametrized order arc from $\{c_1\}$ to D, we have that $D \in \mathcal{C}$. If $c_1 \notin D$, there exists $d_1 \in \beta_1 - \{c_1, a_1\}$ such that the subarc α of β_1 joining c_1 and d_1 is such that $\alpha \cap D = \emptyset$. Then $D_1 = (A - \alpha) \cup \{d_1\}$ is a proper subcontinuum of A such that $c_2, c_3 \in D_1$, $D \subset D_1$. By Thm. 1.2 (b), $D_1 \in SB(A)$. Taking parametrized order arcs from $\{c_2\}$ to D_1 and from D to D_1, we obtain that $D_1 \in \mathcal{C}$ and $D \in \mathcal{C}$.

Therefore $SB(A) - \{A\}$ is pathwise connected and $SB(A)$ contains three one-point sets. These conclusions are contrary to (c) and (d) respectively. Hence X must be atriodic.

Clearly an interval does not satisfy (c) nor (d). If X is a generalized Warzaw circle with a rolling map f, let $A = f[0, 1]$. By lemma 7.7, $SB(A) = \{f[a, 1] : a \in I\}$. Then the unique one-point set in $SB(A)$ is $f(1)$ and $SB(A) - \{A\}$ is a semi-open interval. Thus X does not satisfy (c) nor (d).

Then Thm. 7.6 implies that X is a simple closed curve.

8. Two Examples

8.1 Example. Let X be an hereditarily indecomposable continuum. Then, by Thm. 2.1, $SB(A) = \{A\}$ for each $A \in C^\wedge(X)$. Thus $\mathcal{S}(A) = \{A\}$ and $m(A) = \{A\}$. Therefore $\mathcal{S}$ is continuous and lim sup $m(A_n)$ is finite for every sequence $(A_n)n$ in $C^\wedge(X)$ which converges in $C(X)$. Thus pathwise connectedness is a necessary condition in Theorems 6.1 and 6.3 and the equivalence between (a) and (b) in Theorem 7.8.

8.2 Example. Let X be a solenoid. Then every element in $C(A)$ is an arc which can be enlarged through both end points. Thus X satisfies (c) and (d) in Theorem 7.8 and $m(A)$ consists of two one-point sets for every $A \in C^\wedge(X) - F_1(X)$. Then pathwise connectedness is a necessary condition in Theorem 5.3 and in the equivalences (a) $\Leftrightarrow$ (c) and (a) $\Leftrightarrow$ (d) in Theorem 7.8.

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