Topology Proceedings

CONCERNING IMAGES OF CONTINUA

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0. INTRODUCTION

In the American Journal of Mathematics in 1959, M. K. Fort
[2] wrote a beautiful paper in which he proved that the dyadic
solenoid is not a continuous image of any plane continuum. In
that paper he also showed that a certain spiral to two circles
is not a continuous image of any non-separating plane contin-
uum. These results relied on a theorem of Morton Curtis [1]
on locally trivial fibre spaces with totally disconnected fibres.
The author has always found these arguments somewhat less
than satisfying because, in particular, they cannot be used to
show that a spiral to a single circle is not a continuous image
of any non-separating plane continuum. In this paper, we iso-
late a theorem from which all these results can be obtained.
Specifically, this theorem avoids the use of the results on locally
trivial fibre spaces.

   Throughout this paper we use the term mapping to mean
continuous function and by a continuum we mean a compact,
connected metric space. Suppose Y is a continuum. A sub-
continuum K of Y is said to be a W-set provided if f is a
mapping of a continuum X onto Y, there is a subcontinuum
H of X such that $f[H] = K$. For information on mappings
to the circle, see Whyburn [7, pp. 219-229]. We use exp to
denote the mapping of the reals, R, onto the circle, S^1, defined

by $exp(t) = e^{it}$. We use the notation, $f \colon X \twoheadrightarrow Y$ to denote that f is a surjective mapping.

1. MAIN THEOREM.

In this section we state and prove the main theorem of this paper. In the next section, we give a couple of applications of this theorem.

Theorem. *Suppose K is a continuum, $p \colon K \twoheadrightarrow S^1$ is a mapping and $K_1, K_2, K_3, \cdots$ is a sequence of W-sets in K such that, for each positive integer j, there is a mapping $\phi_j \colon K_j \to R$ such that $p = exp \circ \phi_j$ and $diam(\phi_j[K_j]) \to \infty$ as $j \to \infty$. Then, if f is a mapping of a continuum M onto K, $p \circ f$ is essential.*

Proof. Suppose $p \circ f$ is homotopic to a constant and ϕ is a mapping of M into R such that $p \circ f = exp \circ \phi$. Since, for each positive integer j, the continuum K_j is a W-set in K there is a subcontinuum M_j of M such that $f[M_j] = K_j$. Since ϕ and $\phi_j \circ f$ are mappings of M_j into R so that $exp \circ \phi = exp \circ \phi_j \circ f$, there is an integer k_j such that $\phi(x) = \phi_j(f(x)) + 2k_j\pi$ for each x in M_j. Thus, $diam(\phi[M_j]) = diam(\phi_j[K_j])$. Hence, $diam(\phi[M_j]) \to \infty$ as $j \to \infty$. This contradicts the fact that, since M is compact, $diam(\phi[M]) < \infty$.

Remark. A mapping f of a continuum M onto a continuum Y is said to be *weakly confluent with respect to* the subcontinuum K of Y provided there is a subcontinuum H of M such that $f[H] = K$. Using this language, there is a slightly more general statement of this theorem. Instead of requiring that the continua K_i be W-sets in K, we only need that the mapping f be weakly confluent with respect to each term of a sequence of continua $K_1, K_2, K_3, \cdots$ such that, for each positive integer j, there is a mapping $\phi_j \colon K_j \to R$ such that $p = exp \circ \phi_j$ and $diam(\phi_j[K_j]) \to \infty$ as $j \to \infty$.

2. APPLICATIONS

We first show that a single spiral to a circle is not a continuous image of any continuum with property (b). (A continuum is said to have *property (b)* provided every mapping of it to S^1 is inessential. Continua with property (b) are also called *a-cyclic.*) We give an inverse limit description of a single spiral to a circle, identify a sequence of arcs in the continuum which we show are W-sets in the continuum and show that they satisfy the hypothesis of our theorem.
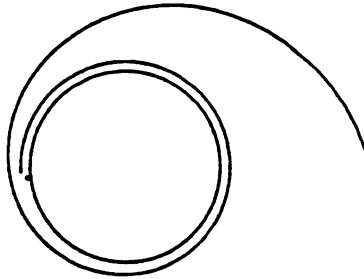

Figure 1

Let S^1 be the unit circle in the plane and let X denote $S^1 \cup [1,2]$. Define a mapping f from X onto itself by $f(x) = x$ for each x in S^1, $f(x) = exp(4\pi x)$ for $1 \leq x \leq \frac{3}{2}$ and $f(x) = 2(x-1)$ for $\frac{3}{2} \leq x \leq 2$. For each positive integer i, let $X_i = X$ and $f_i = f$. Denote by K the inverse limit of the inverse sequence $\{X_i, f_i\}$. Geometrically, K is a spiral to a circle, homeomorphic to the continuum indicated in Figure 1. Let p be $r \circ \pi_1$ where π_1 is the projection of K onto X_1 and r is the retraction of X onto S^1 which collapses $[1,2]$ to its left end point. Denote by K_i the arc in K defined by $\pi_i[K_i] = [1,2]$. Note that $f \mid [\frac{3}{2}, 2]$ is a homeomorphism, so specifying the ith projection of K_i to be $[1,2]$ completely determines K_i. Suppose f is a mapping of a continuum M onto K. Choose a positive integer i and let j be an integer greater than i. There is a subcontinuum M_j of M such that $\pi_j \circ f[M_j]$ is $\pi_j[K_i]$. The sequence $M_{i+1}, M_{i+2}, M_{j+3}, \cdots$ has a subsequence which converges to a subcontinuum H_i of M and $f[H_i] = K_i$ (see,

[3]). Consequently, each arc K_i is a W-set in K. Further, if $\phi_j \colon K_j \to R$ and $p(x) = exp(\phi_j(x))$ for each x in K_j, then $diam(\phi_j[K_j]) = 2(j-1)\pi$ and the hypotheses of the Theorem are met. Thus $p \circ f$ is essential, so M cannot have property (b). Consequently, K is not a continuous image of any continuum with property (b).

Among continua with property (b) are the non-separating plane continua, the tree-like continua, the Case-Chamberlin continuum and the continua in Pam Roberson's uncountable collection of Case-Chamberlin type continua [4]. As a result, K is not a continuous image of any continuum from any of these classes.

Remark. It is easy to see that the spiral to two circles can be mapped onto K, so this argument generalizes Fort's result. Alternatively, Fort's result could be obtained directly from our Theorem by identifying a similar sequence $K_1, K_2, K_3, \cdots$ of arcs in it.

Remark. Let X be defined to be $S^1 \cup [1,2]$ as above, f be defined as above and g be defined from X onto itself by $g(x) = x$ for each x in $S^1, g(x) = exp(-4\pi x)$ for $1 \leq x \leq \frac{3}{2}$ and $g(x) = 2(x-1)$ for $\frac{3}{2} \leq x \leq 2$. One obtains the collection of *Waraszkiewicz spirals* [6] by taking all inverse limits of inverse limit sequences $\{X_i, h_i\}$ where, for each i, $X_i = X$ and h_i is in $\{f, g\}$. In any of these spirals where the sequence $h_1, h_2, h_3, \cdots$ contains arbitrarily long subsequences of $f's$ (resp., $g's$), one can identify a similar sequence $K_1, K_2, K_3, \cdots$ of arcs in it such that, for each positive integer j, there is a mapping $\phi_j \colon K_j \to R$ such that $r \circ \pi_1 = exp \circ \phi_j$ and $diam(\phi_j[K_j]) \to \infty$ as $j \to \infty$. Thus, none of these Waraszkiewicz spirals is a continuous image of any continuum with property (b). Of course, we do not obtain the interesting property of these spirals that there is no model for the collection [5,6].

We now turn our attention to *non-planar solenoids*. To each sequence $P_1, P_2, P_3, \cdots$ of positive integers, each greater than

1, there corresponds a non-planar solenoid obtained as an inverse limit of an inverse limit sequence, $\{X_i, f_i\}$, where, for each i, $X_i = S^1$ and f_i is the mapping of S^1 onto itself defined by $f_i(z) = z^{P_i}$. By examining the arguments of Fort for Results 1 through 9, one can see that they can be carried directly over with only obvious, minor changes to show that any mapping of a plane continuum onto a non-planar solenoid followed by the projection, π_1, to the circle is inessential. (Perhaps, the most obscure change is relative to his property D. However, replacing 2^n by $P_1 \cdot P_2 \cdot P_3 \cdots P_n$ is sufficient.) Identifying a sequence of arcs $K_1, K_2, K_3, \cdots$ in a non-planar solenoid determined by the sequence $P_1, P_2, P_3, \cdots$ so that the sequence of arcs satisfies the hypothesis of our Theorem can be achieved by letting K_i denote an arc in the solenoid determined by $\pi_i[K_i] = \{z \mid 0 \leq \arg z \leq \pi\}$.

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