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RECENT RESULTS ON HOMOGENEOUS CURVES AND ANR'S

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ABSTRACT. In the first part of the paper we discuss some new results concerning homogeneous curves. The second part deals with higher dimensional locally connected, locally compact homogeneous spaces. They are shown to be local Cantor manifolds which allows us to derive some observations about homogeneous *ANR*'s. This is related to the conjecture that homogeneous *ANR*'s are generalized manifolds.

Throughout the paper all spaces are assumed to be metric separable and all mappings are continuous. A space X is homogeneous if for every two points $x, y \in X$ there is a homeomorphism $h : X \to X$ such that $h(x) = y$. The set of all self-homeomorphisms of X is denoted by $H(X)$.

1. New characterizations of homogeneous curves.

The reader is referred to an excellent presentation of the characterization problem and its history by Rogers [14] (see also [9]). We begin with three theorems obtained after [14]. The following definition will be convenient. Small subcontinua of the continuum X are said to have a property $\mathcal{P}$ if there exists a number $\varepsilon > 0$ such that any subcontinuum of X with diameter less than ε has property $\mathcal{P}$.

Theorem 1.1. *If X is a nondegenerate homogeneous continuum such that small subcontinua of X are unicoherent, then*

X contains either an arc or arbitrarily small nondegenerate indecomposable subcontinua.

Theorem 1.2 [13]. *A homogeneous continuum X is a solenoid if and only if small subcontinua of X are arcs.*

A subcontinuum K of X is called terminal if for any other subcontinuum L of X that meets K, either $K \subset L$ or $L \subset K$.

The following theorem obtained in [7] and [15] is known as the Terminal Decomposition Theorem.

Theorem 1.3. *Each homogeneous curve admits a unique continuous decomposition into terminal, hereditarily indecomposable, tree-like, homogeneous and mutually homeomorphic subcontinua such that the decomposition space is a homogeneous curve containing no nondegenerate proper terminal subcontinua.*

Theorem 1.1. was formulated in [7] for hereditarily unicoherent X; however the proof gives the conclusion if the hypothesis is satisfied only by small subcontinua. In the same spirit, Theorem 1.2 improves a characterization of solenoids by Hagopian [5] who assumed that all proper subcontinua were arcs.

In view of Theorem 1.3, it is important to know how to detect terminal subcontinua. A method is developed in [7]. The idea is to use Theorem 1.1 and imposing some additional conditions on X, produce terminal subcontinua from already existing (small) indecomposable ones.

We can get intersecting characterizations of some homogeneous curves by investigating the structure of small subcontinua and, then , by applying Theorems 1.2 and 1.3.

The following technical definition will be used. Let $\mathcal{P}$ be a class of polyhedra. A continuum X is said to be partially $\mathcal{P}$-like if there exist a number $\varepsilon > 0$ and a sequence of finite open covers $\mathcal{C}_n$ of X with mesh $\mathcal{C}_n \to 0$, each of them containing a subfamily $\mathcal{D}_n$ such that the nerve of $\mathcal{D}_n$ belongs to $\mathcal{P}$ and there is a point $d_n \in \bigcup\mathcal{D}_n$ satisfying $\rho(d_n, cl(\bigcup(\mathcal{C}_n \setminus \mathcal{D}_n))) > \varepsilon$, where ρ is a metric in X (if $\mathcal{C}_n = \mathcal{D}_n$ for each n, X is partially $\mathcal{P}$-like by definition). Obviously, $\mathcal{P}$-like continua are partially $\mathcal{P}$-like. If we are concerned with one-dimensional continua, then class $\mathcal{P}$ consists of certain graphs. In what follows, all graphs are assumed to be connected and nondegenerate.

A class $\mathcal{P}$ of graphs is called hereditary if each subgraph $\mathcal{P}'$ of a graph $P \in \mathcal{P}$ also belongs to $\mathcal{P}$. The classes of arcs, trees, graphs of genus $\leq k$ and k-junctioned graphs (i.e. with at most k ramification points) are examples of hereditary classes of graphs.

A variation of partially tree-like continua is considered in [8] under the name of somewhat tree-like continua. The definition differs in that we assume the $\mathcal{D}_n$'s are tree-chains, $diam \bigcup \mathcal{D}_n > \varepsilon$ and $\bigcup(\mathcal{C}_n \backslash \mathcal{D}_n \cap \bigcup \mathcal{D}_n$ is contained in only one end-link of $\mathcal{D}_n$.

Theorem 1.4. *If $\mathcal{P}$ is a hereditary class of graphs and X is a partially $\mathcal{P}$-like homogeneous continuum, then X is a curve whose small subcontinua are $\mathcal{P}$-like.*

Proof. Let $\varepsilon > 0$ and open covers $\mathcal{C}_n$ of X be such that mesh $\mathcal{C}_n \to 0$, $\mathcal{C}_n$ contains a subfamily $\mathcal{D}_n$ whose nerve belongs to $\mathcal{P}$ and there is a point $d_n \in \bigcup \mathcal{D}_n$ such that $\rho(d_n, cl(\bigcup(\mathcal{C}_n \backslash \mathcal{D}_n))) > \varepsilon$. Assume that $lim\ d_n = d$. Let Y be the component of d in the closure of the open $\varepsilon/6$-ball around d. Take n so large that there exists an $\varepsilon/3$-homeomorphism $h \in H(X)$ moving d to d_n (here we use the well-known Effros Theorem as stated, e.g. in [7, p.125]). Then $h(Y) \subset \bigcup \mathcal{D}_n$, whence Y is $\mathcal{P}$-like. Now, consider Effros' δ for $\varepsilon/18$ and assume, additionally, that $\delta < \varepsilon/36$. If Z denotes an arbitrary component of the closure of the open δ-ball around d, then there is an $\varepsilon/18$- homeomorphism $g \in H(X)$ such that $d \in g(Z)$. The continuum $g(Z)$ is contained in Y and thus, it is $\mathcal{P}$-like. Since each component of a closed neighborhood of d is $\mathcal{P}$-like, the homogeneity of X yields small subcontinua of X are $\mathcal{P}$-like

The next theorem was obtained in [8] by methods described earlier.

Theorem 1.5. *Any somewhat tree-like homogeneous curve contains arbitrarily small nondegenerate terminal subcontinua.*

One can notice that if a curve is not somewhat tree-like, then it can be represented as an inverse limit of graphs without end-points (i.e. graphs whose every vertex is of order > 1).

Theorem 1.6. *If a homogeneous curve contains no nondegenerate proper terminal subcontinua, then it is an inverse limit of graphs without end-points.*

Two classes of finitely cyclic and of finitely junctioned curves are considered in [8] and [4], respectively. A curve X is finitely cyclic if graphs in an inverse limit expansion of X have genus $\leq k$ for some k; X is finitely junctioned if the graphs have at most k ramification points for some k. It can be shown that if the graphs have no end-points then "finitely cyclic" implies finitely junctioned". Finitely junctioned homogeneous curves have small subcontinua being arc-like. If such a curve contains no terminal proper nondegenerate subcontinua, then one can show more (using Theorem 1.1): small subcontinua are arcs [4]. Thus the curve is a solenoid, by Theorem 1.2. Applying the Terminal Decomposition Theorem, we get two characterizations.

Theorem 1.7 [4]. *A homogeneous finitely junctioned curve must be a pseudo-arc, a solenoid of pseudo-arcs or a solenoid.*

Theorem 1.8 [8]. *A finitely cyclic homogeneous curve must be a tree-like continuum, a solenoid or it admits a decomposition as in Theorem 1.3 with quotient space a solenoid.*

One wished elements of the terminal decomposition were always pseudo-arcs, by this is just one of the hardest open problems in the theory of homogeneous curves. They are pseudo-arcs for finitely junctioned curves, because small subcontinua and, consequently, elements of the decomposition are arc-like. Now, besides that problem, it remains to study curves which are both infinitely junctioned and infinitely cyclic. It would also be natural to assume, in view of the Terminal Decomposition Theorem, that they contain no nontrivial terminal subcontinua. Examples are the Menger universal curve as well as other homogeneous curves involving the Menger curve in a construction [14]. Presumably, the Menger curve must be contained in such curves. So far however, it is not known if they contain arcs.

We end this chapter by deriving some observations on the curves.

In what follows X is a homogeneous curve that contains no nondegenerate proper terminal subcontinua.

Proposition 1.9. *The following statements are equivalent.*

(a) X is infinitely junctioned.

(b) X is infinitely cyclic.

(c) For every closed countable subset P of $X, X\setminus P$ contains a non-arc-like continuum.

Proposition 1.10. *X is not partially arc-like or X ia a solenoid.*

Proposition 1.11. *For any inverse limit expansion $X = \varprojlim(G_n, f_n)$ into graphs G_n, if X if infinitely junctioned, then $\bigcup_n p_n^{-1}(V(G_n))$ is dense in X where p_n is a projection of X onto G_n and $V(G_n)$ is the set of all vertices of G_n. Graphs G_n can be chosen to have no end-points.*

Proposition 1.9 follows from [4], Proposition 1.10 is a consequence of Theorem 1.4 and [4], while Proposition 1.11 follows from Proposition 1.10 and Theorem 1.6.

2. Local Cantor manifolds and homogeneous *ANR*'s.

A locally compact n-dimensional space is called a Cantor manifold if no subset of dimension $\leq n-2$ separates it. If the space is infinite-dimensional (strongly infinite-dimensional), then it is a Cantor manifold if it contains no finite-dimensional (weakly infinite-dimensional, resp.) separator.

It is proved in [6] that locally compact connected homogeneous spaces are Cantor manifolds. One can apply the idea from [6] to get a local version of the theorem.

A locally compact, locally connected space is said to be a local Cantor manifold if each connected open subset is a Cantor manifold of the same dimension (finite or infinite).

Theorem 2.1. *Any locally compact, locally connected homogeneous space is a local Cantor manifold.*

An essential role in the proof is played by a version of Effros' Theorem stated in [2, p.584] for compacta, which is however valid for locally compact spaces X with the same proof, because the group of homeomorphisms $H(X)$ is a Borel subset of the mapping space X^X with the compact-open topology [6].

Proposition 2.2. *If (X,ρ) is a locally compact homogeneous space, $a \in X$ and $\varepsilon > 0$, then there exists a $\delta > 0$ such that if $\rho(x,a) < \delta$, then there is $h \in H(X)$ such that $h(a) = x$ and $\hat{\rho}(h, id) < \varepsilon$, where $\hat{\rho}$ is a metric in X^X compatible with the compact-open topology.*

We can now sketch a proof of Theorem 2.1. Suppose X is a locally compact, locally connected homogeneous space which is not a local Cantor manifold. For simplicity consider the case $dim\ X = n$ (other are analogous). Let U be a connected open subset separated by a subset A of dimension $\leq n-2$ into two open subsets U_1 and U_2. We can assume that A is closed and nowhere dense in U. Pick $a \in A \cap \overline{U}_1 \cap \overline{U}_2$ and sufficiently small compact neighborhood V of a in U. By the Hurewicz Theorem V contains an n-dimensional Cantor manifold C. By Proposition 2.2 we can assure that $a \in C \subset U$. The set C intersects one of the sets U_1 or U_2, say U_1. Then, applying Proposition 2.2 once again, C can be slightly pushed toward U_2 so that it intersects both U_1 and U_2. This means that C is separated by A, a contradiction.

Theorem 2.1 includes a positive answer to the problem by Lysko[10] of whether finite-dimensional homogeneous compact *ANR*'s are local Cantor manifolds. We are going to show some other applications to homogeneous *ANR*'s. A famous problem posed by Bing and Borsuk [1] is whether an n-dimensional homogeneous compact *ANR* X must be an n-manifold for $n > 2$. A weaker version of the problem is whether X is a homology manifold, which means that $H_k(X, X\backslash\{x\}) = 0$ for $k < n$ and $H_n(X, X\backslash\{x\}) = Z$. Following Mitchell [12], we denote by $H(n)$ the property that $H_k(X, X\backslash\{x\}) = 0$ for all $x \in X$ and $k \leq n$, and by $D(n,m)$ the disjoint (n,m)-cells property which means that any two mappings f, g of the standard cells B^n and B^m, respectively, into X can be approximated by mappings $f' : B^n \to X$ and $g' : B^m \to X$ with the disjoint images. The reader is referred to [12] for various results concerning relationships between $D(n,m)$, $H(n)$ and other properties.

Theorem 2.3. *Any locally compact, locally connected homogeneous space X of dimension greater than 2 has the $D(1,1)$. If the space X is a compact* ANR, *then it also satisfies $H(2)$.*

To prove Theorem 2.3 it suffices to observe that no arc separates a connected open subset of X, by Theorem 2.1. Then one can repeat the proof of [3, Proposition 2.2] to get the $D(1,1)$. The property $H(2)$ follows from $D(1,1)$ by [12].

Recall that a subset A of X is called LCC^k (locally coconnected in dimensions $\leq k$) if for each $a \in A$ and each neighborhood $U \subset X$ of a there is a neighborhood $V \subset X$ of a such that for $i \leq k$ any map $f : S^i \to V - A$ has an extension $\overline{f} : B^{i+1} \to U - A$ (S^i denotes the standard i-sphere).

In the sequel we shall consider the following properties of a space X.

(A_n) $X \in D(0,n)$;

(B_n) for every $x \in X$ any mapping $f : B^n \to X$ of the n-cell is approximated by mappings with images in $X\backslash\{x\}$;

(C_n) for every $x \in X\{x\}$ is LCC^{n-1};

(D) for every $x \in X\{x\}$ is a Z-set in X;

(E) for every $\varepsilon > 0$ there exists $f : X \to X$ such that f is non-surjective and f is ε-close to the identity.

Theorem 2.4. *If X is a locally compact, homogeneous space, then*

(1) $(A_n) \Leftrightarrow (B_n)$
and
(2) $\forall n(A_n) \Leftrightarrow (D)$.
If X is a homogeneous compact ANR, *then*
(3) $(D) \Leftrightarrow (E)$
and if dim $X > 2$, then
(4) $(A_n) \Leftrightarrow (C_n)$.

Proof. A proof of $(A_n) \implies (B_n)$ is a matter of a simple application of Proposition 2.2. The inverse implication is clear. Statement (2) follows from (1) and form the definition of a Z-set (see [11]). If X is an *ANR*, then $\{x\}$ is a Z-set if and only if for every $\varepsilon > 0$ there exists a mapping $f : X \to X\backslash\{x\}$ which is ε-close to the identity [11]. To show (3) is now a routine application of Effros' Theorem.

(A_n) implies (C_n) because (B_n) does. This is true for any *ANR* (not necessarily homogeneous) and follows from the homotopy extension property. For a proof of $(C_n) \implies (A_n)$ we need the following lemma.

Lemma. *If each mapping $f : B^k \to X$ is approximated by a mapping $f' : B^k \to X\backslash\{x\}$ then each mapping $g : S^k \to X$ is approximated by a mapping $g' : S^k \to X\backslash\{x\}$.*

Indeed, suppose $x \in g(S^k)$. If $g(S^k) = \{x\}$, then define $g'(S^k) - \{x'\}$, where a point x' is close to x. If $g^{-1}(x) \neq S^k$, then take a k-dimensional disk $D^k \subset S^k$ which contains $g^{-1}(x)$ in its interior. Put $f_1 = f|D_k$ and $f_2 = g|cl(S^k\backslash D^k)$. Approximate f_1 by mapping $f_1' : D^k \to X\backslash\{x\}$. There is a small homotopy form $f_1|bd\ D^k$ to $f_1'|bd\ D^k$ with values in $X\backslash\{x\}$. By the small homotopy extension property, the mapping f_2 is approximated by a mapping $f_2' : cl(S^k\backslash D^k) \to X\backslash\{x\}$ which extends $f_1'|bd\ D^k$. The mapping $g' : S^k \to X\backslash\{x\}$ defined by

$g'|D^k = f'_1$ and $g'|cl(S^k \backslash D^k) = f'_2$ approximates g and this completes the proof of the Lemma.

We shall prove $(C_n) \Longrightarrow (A_n)$ by induction. The property $D(0,1)$ follows from $D(1,1)$ which is satisfied in virtue of Theorem 2.3. Assume now $(C_{n-1}) \Longrightarrow (A_{n-1})$. The properties (A_{n-1}) and (B_{n-1}) being equivalent, we will employ the latter one. Let $\{x\}$ be LCC^{n-1} and let $f : B^n \to X$ be given. Triangulate B^n into small n-cells $B_1, B_2, \dots, B_k$ and put $S_i = \partial B_i$, $S = \cup_i S_i$. We will first construct an approximation $g : S_1 \to X \backslash \{x\}$ of $f|S$. By the Lemma, there exists a mapping $g_1 : S_1 \to X \backslash \{x\}$ that approximates $f|S_1$ and g_1 extends to an approximation $f_1 : S \to X$ of f. Now, $f_1|S_2$ is approximated by a mapping $g_2 : S_2 \to X \backslash \{x\}$ and g_2 extends to an approximation $f_2 : S \to X$ of f_1. The mapping f_2 can be chosen to be so close to f_1 that $f_2(S_1) \subset X \backslash \{x\}$. We continue this process : given $f_i : S \to X$ such that $f_i(S_i \cup \dots \cup S_i) \subset X \backslash \{x\}$, we construct an approximation $f_{i+1} : S \to X$ of f_i such that $f_{i+1}(S_1 \cup \dots \cup S_{i+1}) \subset X \backslash \{x\}$. Finally, the mapping $g = f_k : S \to X \backslash \{x\}$ is an approximation of $f|S$.

Now, the property LCC^{n-1} allows us to extend each $g|S_i$ to a mapping $\overline{g}_i : B_i \to X \backslash \{x\}$. The mapping $\overline{g} : B^n \to X \backslash \{x\}$ defined by $\overline{g}|B_i = \overline{g}_i$ approximates f. This proves $(C_n) \Longrightarrow (B_n) \Longleftrightarrow (A_n)$.

Problem. Suppose X is an infinite-dimensional (homogeneous) compact ANR and $\varepsilon > 0$. Does there exist a non-surjective mapping $f : X \to X$ such That f is ε-close to the identity mapping of X?

If the answer is positive, then by Theorem 2.4, the inclusion $X \backslash \{x\} \hookrightarrow X$ is a homotopy equivalence and the property $H(n)$ holds for every n.

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