Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Vol 16, 1991

NOTE ON PARACOMPACTNESS IN PRODUCT SPACES

AKIHIRO OKUYAMA

1. Introduction.

In the theory of product spaces one of the most interesting parts seems to investigate topological properties $(P), (Q)$ and (R) in theorems of the following type:

Let $\mathcal{C}$ be a class of all spaces with the property (R). The X is a space with the property (P) if and only if $X \times Y$ has the property (Q) for any space Y in $\mathcal{C}$.

In view of this point, there are two excellent theorems, as follows:

Theorem T (H. Tamano [5]). *Let X be a completely regular Hausdorff space. Then X is paracompact if and only if $X \times Y$ is normal for any compact Hausdorff space Y.*

Theorem M (K. Morita [1]). *Let X be a topological space. Then X is a normal (Morita) P-space if and only if $X \times Y$ is normal for any metrizable space Y.*

For a cardinal number m, X is called a *$P(m)$-space,* if for a set Ω of cardinality m and for any family

$$\{G(\alpha_1, \dots, \alpha_i) : \alpha_1, \dots, \alpha_i \in \Omega : \ i \in N\}$$

of open subsets of X such that

(i) $G(\alpha_1, \dots, \alpha_i) \subset G(\alpha_1, \dots, \alpha_i, \alpha_{i+1})$ for $\alpha_1 \dots, \alpha_i, \alpha_{i+1} \in \Omega$; $i \in N$, there exists a family

$$\{F(\alpha_1, \dots, \alpha_i) : \alpha_1, \dots, \alpha_i \in \Omega; \ i \in N\}$$

of closed subsets of X satisfying two conditions below

(ii) $F(\alpha_1, \ldots, \alpha_i) \subset G(\alpha_1, \ldots, \alpha_i)$ for $\alpha_1, \ldots, \alpha_i \in \Omega$; $i \in N$;

(iii) $X = \cup_{i=1}^{\infty} F(\alpha_1, \ldots, \alpha_i)$ if $X = \cup_{i=1}^{\infty} G(\alpha_1, \ldots, \alpha_i)$ for any sequence $\{\alpha_i\}$, where N denotes the set of all natural numbers.

X is said to be a (Morita) *P-space*, if X is a $P(m)$-space for any m.

The purpose of this note is to give another theorem of similar type as above theorems: that is, for the class $\mathcal{C}$ of all K-analytic spaces.

In this note all spaces are assumed to be completely regular Hausdorff.

2. Theorem.

Definition 1. A map F from a space Y to the power set of a space X is said to be *upper semi-continuous*, if for each point y of Y and each open subset G of X containing $F(y)$, there is a neighborhood U of y with $\cup\{F(y') : y' \in U\} \subset G$.

A space X is called a *K-analytic space*, if there exists an upper semi-continuous map F from N^N to the non-empty compact subsets of X with $\cup\{F(\xi) : \xi \in N^N\} = X$.

Compact spaces and complete separable metric spaces are all K-analytic (cf. [3]).

Definition 2. We say that a space X is a *weak* $P(\aleph_0)$*-space*, if for any family

$$\{G(n_1, \ldots, n_i) : n_1, \ldots, n_i \in N;\ i \in N\}$$

of open subsets of X such that

(1) $G(n_1, \ldots, n_i) \subset G(n_1, \ldots, n_i, n_{i+1})$ for $n_1, \ldots, n_i, n_{i+1} \in N$; $i \in N$;

(2) $\cup_{i=1}^{\infty} G(n_1, \ldots, n_i) = X$ for any sequence $\{n_i\}$ in N, there exists a family

$$\{F(n_1, \ldots, n_i) : n_1, \ldots, n_i \in N;\ i \in N\}$$

of closed subsets of X satisfying two conditions below:

(3) $F(n_1, \dots, n_i) \subset G(n_1, \dots, n_i)$ for $n_1, \dots, n_i \in N$; $i \in N$
(4) $\cup_{i=1}^{\infty} F(n_1, \dots, n_i) = X$ for any sequence $\{n_i\}$ in N.

Clearly, every $P(\aleph_0)$-space is a weak $P(\aleph_0)$-space; however, we do not know whether they coincide or not.[1]

Proposition 1. *Let X be a paracompact, weak $P(\aleph_0)$-space and Y a K-analytic space. Then the product space $X \times Y$ is paracompact.*

Proof. By the assumption for Y, there exists an upper semi-continuous map φ from N^N to the set of non-empty compact subsets of Y with $\cup\{\varphi(\xi) : \xi \in N^N\} = Y$.

Let $\mathcal{U}$ be an arbitrary open covering of $X \times Y$. We are going to find a σ-locally finite open refinement of $\mathcal{U}$.

Fix a point $\xi = (n_1, n_2, \dots)$ of N^N. Since $\varphi(\xi)$ is compact and X is paracompact, there exists a locally finite open covering $\{V_{\xi,\alpha} : \alpha \in A_\xi\}$ of X and for each $\alpha \in A_\xi$ there exists a finite family $\mathcal{W}_{\xi,\alpha}$ of open subsets of Y satisfying the following conditions:

(1) $V_{\xi,\alpha} \times \varphi(\xi) \subset V_{\xi,\alpha} \times (\cup \mathcal{W}_\xi)$ for each $\alpha \in A_\xi$,
(2) $\{V_{\xi,\alpha} \times W : W \in \mathcal{W}_{\xi.\alpha}\} < \mathcal{U}$.

Since φ is upper semi-continuous, for each $\alpha \in A_\xi$ there exists $k_{\xi,\alpha} \in N$ such that $\cup\varphi(T(\xi|k_{\xi,\alpha})) \subset \cup\mathcal{W}_{\xi,\alpha}$, where for $\xi = (n_1, n_2, \dots) \in N^N$ $\xi|k$ denotes the finite sequence $(n_1, n_2, \dots, n_k)$ and $T(\xi|k) = \{\eta \in N^N : \eta|k = \xi|k\}$.

Let N^i be the i-fold product of N. For each $(n_1, \dots, n_i) \in N^i$ and a natural number j with $j \leq i$, put

$$G'(n_1, \dots n_j) = \cup\{V_{\xi,\alpha} : k_{\xi,\alpha} = j,\ \xi|j = (n_1, \dots, n_j),\ \alpha \in A_\xi\}$$

and

$$G(n_1, \dots, n_i) = \cup\{G'(n_1, \dots, n_j) : j = 1, 2, \dots, i\}.$$

Then we can easily see that for each $\xi = (n_1, n_2, \dots) \in N^N$ the family $\{G(n_1, \dots n_i) : i \in N\}$ is an open covering of X. Furthermore, the family $\{G(n_1, \dots, n_i) : n_1, \dots, n_i \in N,\ i \in$

[1]Professor S. Watson pointed out that they are distinct; that is, there exists a weak $P(\aleph_0)$−, non-$P(\aleph_0)$−space.

$N\}$ satisfies the assumption (1) for X being a weak $P(\aleph_0)$-space.

Since X is a normal, weak $P(\aleph_0)$-space, there exists a family $\{H(n_1, \dots, n_i) : n_1, \dots, n_i \in N, i \in N\}$ of open subsets of X satisfying the following conditions:

(3) $\overline{H(n_1, \dots, n_i)} \subset G(n_1, \dots, n_i)$ for each $n_1, \dots, n_i \in N$; $i \in N$,

(4) $\cup\{H(n_1, \dots, n_i) : i \in N\} = X$ for each $\xi = (n_1, n_2, \dots) \in N^N$.

Since X is paracompact, for each $(n_1, \dots, n_i) \in N^i$ there exists a locally finite family $\mathcal{V}(n_1, \dots, n_i)$ of open subsets of X which satisfies the following conditions:

(5) $\cup\{V : V \in \mathcal{V}(n_1, \dots, n_i)\} = H(n_1, \dots, n_i)$,

(6) $\mathcal{V}(n_1, \dots, n_i)$ refines $\{V_{\xi,\alpha} : k_{\xi,\alpha} \le i,\ \xi | k_{\xi,\alpha} = (n_1, \dots, n_{k_{\xi,\alpha}})\alpha \in A_\xi\}$.

According to (6), correspond each $V \in \mathcal{V}(n_1, \dots, n_i)$ to V_{ξ_V,α_V} with $V \subset V_{\xi_V,\alpha_V}$.

Now, we show that for each $(n_1, \dots, n_i) \in N^i$ the family $\mathcal{G}(n_1, \dots, n_i) = \{V \times W : V \in \mathcal{V}(n_1, \dots, n_i),\ W \in \mathcal{W}_{\xi_V,\alpha_V}\}$ is locally finite in $X \times Y$ and it covers $H(n_1, \dots, n_i) \times (\cup\varphi(T(n_1, \dots, n_i)))$. Since $\mathcal{V}(n_1, \dots, n_i)$ is locally finite and each $\mathcal{W}_{\xi_V,\alpha_V}$ is finite, $\mathcal{G}(n_1, \dots, n_i)$ is locally finite in $X \times Y$. It remains to show that $\mathcal{G}(n_1, \dots, n_i)$ covers $H(n_1, \dots, n_i) \times (\cup\varphi(T(n_1, \dots, n_i)))$.

Let $< x, y >$ be an arbitrary point of $H(n_1, \dots, n_i) \times (\cup\varphi(T(n_1, \dots, n_i)))$. Since $y \in \cup\varphi(T(n_1, \dots, n_i))$, there exists $\eta \in N^N$ such that $y \in (\eta)$ and $\eta | i = (n_1, \dots, n_i)$. By (5) there exists $V \in \mathcal{V}(n_1, \dots, n_i)$ with $x \in V$ and by choice of V_{ξ_V,α_V} we have $V \subset V_{\xi_V,\alpha_V}$. Let $j = k_{\xi_V,\alpha_V}$. Then by (6) we have $j \le i$ and $\xi_V | j = (n_1, \dots, n_j)$. By the choice of k_{ξ_V,α_V}, we have $\varphi(\eta) \subset \cup\varphi(T(n_1, \dots, n_j)) \subset \cup\mathcal{W}_{\xi_V,\alpha_V}$ and hence, there exists $W \in \mathcal{W}_{\xi_V,\alpha_V}$ with $y \in W$. As a consequence, we have $< x, y > \in V \times W$, which shows the remaining part.

Let $\mathcal{G} = \cup\{\mathcal{G}(n_1, \dots, n_i) : (n_1, \dots, n_i) \in N^i,\ i \in N\}$. Then by (1), (2) and (6) $\mathcal{G}$ refines $\mathcal{U}$ and, hence, $\mathcal{G}$ is a σ-locally finite open refinement of $\mathcal{U}$. This shows that $X \times Y$ is paracompact.

Proposition 2. *Let $X \times N^N$ is paracompact, then X is a weak $P(\aleph_0)$-space.*

Proof. Let $\{G(n_1, \dots, n_i) : n_1, \dots, n_i \in N;\ i \in N\}$ be a family of open subsets of X, which satisfies conditions (1) and (2) in Definition 2. Then

$$\mathcal{U} = \{G(n_1, \dots, n_i) \times T(n_1, \dots, n_i) : n_1, \dots, n_i \in N;\ i \in N\}$$

is an open covering of $X \times N^N$. By the assumption, there exists a locally finite open covering $\{L_\lambda : \lambda \in \Lambda\}$ of $X \times N^N$ such that $\{\overline{L}_\lambda : \lambda \in \Lambda\}$ refines $\mathcal{U}$. For each $(n_1, \dots, n_i) \in N^i$ and $\lambda \in \Lambda$ put $L(n_1, \dots, n_i; \lambda) = \cup\{W : W$ is open in X and $W \times T(n_1, \dots, n_i) \subset L_\lambda\}$. Then $\{L(n_1, \dots, n_i; \lambda) \times T(n_1, \dots, n_i) : n_1, \dots, n_i \in N,\ i \in N,\ \lambda \in \Lambda\}$ is an open covering of $X \times N^N$.

Again, for each $(n_1, \dots, n_i) \in N^i$ and $\lambda \in \Lambda$ there exists $(m_1, \dots m_j) \in N^j$ such that $\overline{L(n_1, \dots, n_i; \lambda)} \times T(n_1, \dots, n_i) \subset \overline{L}_\lambda \subset G(m_1, \dots, m_j) \times T(m_1, \dots, m_j)$ holds. In this case, we have $j \leq i$ and $n_k = m_k$ for $k = 1, \dots, j$ and, hence, $G(m_1, \dots, m_j) = G(n_1, \dots, n_j) \subset G(n_1, \dots, n_j, \dots n_i)$. Put
$M(n_1, \dots, n_i; \lambda) = \cup\{L(n_1, \dots, n_j : \lambda) : j \leq i,\ \overline{L(n_1, \dots, n_j; \lambda)} \subset G(n_1, \dots, n_j, \dots, n_i)\}$. Then $\{M(n_1, \dots, n_i; \lambda) \times T(n_1, \dots, n_i) : n_1, \dots, n_i \in N,\ \lambda \in \Lambda,\ i \in N\}$ is an open covering of $X \times N^N$ such that $M(n_1, \dots, n_i;\ \lambda) \times T(n_1, \dots, n_i) \subset L_\lambda$ and $\overline{L(n_1, \dots, n_i;\ \lambda)} \subset G(n_1, \dots, n_i)$ hold.

Finally, for each $(n_1, \dots, n_i) \in N^i$ put $F(n_1, \dots, n_i) = \cup\{\overline{M(n_1, \dots, n_i; \lambda)} : \lambda \in \Lambda\}$. Then $F(n_1, \dots, n_i)$ is a closed subset of X, because $\{L_\lambda; \lambda \in \Lambda\}$ is locally finite and $\{\overline{M(n_1, \dots, n_i; \lambda)} \times T(n_1, \dots, n_i) : \lambda \in \Lambda\}$ refines $\{\overline{L}_\lambda : \lambda \in \Lambda\}$.

To complete the proof we show that $\{F(n_1, \dots, n_i) : n_1, \dots, n_i \in N, i \in N\}$ satisfies conditions (3) and (4) in Definition 2 for $X \times N^N$. By the construction, (3) is clearly satisfied. It remains to show (4). Let $< x, \xi >$ be an arbitrary point of $X \times N^N$, where $\xi = (n_1, n_2, \dots)$. Since $\{L(n_1, \dots, n_i : \lambda) \times T(n_1, \dots, n_i) : n_1, \dots, n_i \in N,\ i \in N,\ \lambda \in \Lambda\}$ covers $X \times N^N$, there exist $\lambda \in \Lambda$ and $m_1, \dots m_k \in N$ such that $< x, \xi > \in$

$L(m_1, \dots m_k; \lambda) \times T(m_1, \dots, m_k)$ holds. By $\xi \in T(m_1, \dots m_k)$ we have $n_j = m_j$ for $j \leq k$ and, hence, $x \in L(n_1, \dots, n_k; \lambda)$, which is contained in $M(n_1, \dots, n_k; \lambda)$. As a consequence, we have $x \in F(n_1, \dots, n_k)$, which shows $\cup\{F(n_1, \dots, n_i) : i \in N\} = X$.

As a consequence, we have the following theorem.

Theorem. *For a space X, X is a paracompact, weak $P(\aleph_0)$-space if and only if $X \times Y$ is paracompact for any K-analytic space Y.*

3. Comment.

As for Theorem T in Introduction $X \times Y$ is normal for any compact Hausdorff space Y if and only if $X \times Y$ is paracompact for any compact Hausdorff space Y. Also, as for Theorem M in Introduction, if X is paracompact, then $X \times Y$ is normal for any metrizable space Y if and only if $X \times Y$ is paracompact for any metrizable space Y, because for a paracompact space and a metrizable space M the normality of $X \times M$ is equivalent to the paracompactness of $X \times M$ (cf. [4]). Relating to these fact, here is one question:

If $X \times Y$ is normal for any K-analytic space Y, then is $X \times Y$ paracompact for any K-analytic space Y?

References

1. K. Morita, ***Products of normal spaces with metric spaces***, Math. Annalen **154** (1964), 365-382.
2. C. Przymusinski, ***Products of normal spaces***, Handbook of Set Theoretic Topology, (North-Holland 1984) Chapter 18, 781-826.
3. C. A. Rogers and J. E. Jayne, ***K-analytic sets***, Analytic sets Part I, Academic Press (1980), 1-181.
4. M. E. Rudin and M. Starbird, ***Products with a metric factor***, Gen. Topology Appl., **5** (1975), 235-248.
5. H. Tamano, ***On paracompactness***, Pacific J. Math., **10** (1960), 1043-1047.

Kobe University
Nada, Kobe 657, Japan