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A REMARK CONCERNING PERFECTLY NORMAL SPACES WITH DISTINCT LOCAL AND GLOBAL DIMENSION

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ABSTRACT. We show that there exists a perfectly normal space which has small transfinite dimension but is not countable dimensional and that there exist perfectly normal spaces of arbitrarily large finite dimension (resp. not strongly countable-dimensional) which are locally homeomorphic to the irrationals.

1. INTRODUCTION

The aim of this note is to give the following two examples, modifying constructions form [7] and [5]. The terminology is explained in the next section.

1.1 Example. *A perfectly normal space X with small transfinite dimension defined which is not a countable union of zero-dimensional subspaces.*

This example provides an answer to Problem 8.11 in [4]. An example (even locally compact) of that kind constructed under the continuum hypothesis can be found in [4], Example 3.17.

1.2 Example. *Perfectly normal spaces Y_n, $n = 1, 2, \ldots$ and Y such that each Y_n and Y is locally homeomorphic to the irrationals, but dim $Y_n = n$ and Y is not a countable union*

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of closed finite-dimensional subspaces. In particular the spaces Y_n and Y are homogeneous.

The spaces Y_n and Y are of weight $\aleph_1$ and each separable subset of Y_n or Y is contained in an open-and-closed subspace of this space homeomorphic to the irrationals.

Belnov [1] gave examples of homogeneous hereditarily normal spaces with different local and global dimensions. Example 1.2 provides some stronger results in this direction.

2. Terminology and notation

Our terminology follows [2], [3] and [4]; I denotes the closed unit interval and P is the set of irrational numbers from I.

2.1. The covering dimension of a space X is denoted by $\dim X$. The small transfinite dimension trind is the extension by the transfinite induction of the Menger-Urysohn inductive dimension ind. A space X is countable-dimensional if $X = \bigcup_{i=1}^{\infty} X_i$, where $\dim X_i < \infty$, and is strongly countable-dimensional if $X = \bigcup_{i=1}^{\infty} F_i$. where each F_i is closed in X and $\dim F_i < \infty$.

2.2. For an ordinal α we denote by $D(\alpha)$ the set of all ordinals less than α with the discrete topology and by $W(\alpha)$ the same set with the order topology. A cardinal is an initial ordinal of a given cardinality; ω_1 is the first uncountable cardinal, and $\mathfrak{c}^+$ denotes the first cardinal after the continuum $\mathfrak{c}$. A set $S \subset W(\alpha)$ is stationary if S meets every closed unbounded subset of $W(\alpha)$.

2.3. For an ordinal α let $B(\alpha) = D(\alpha)^\omega$ with the Cartesian product topology (topologically, $B(\alpha)$ is the Baire space of weight equal to cardinality of α if $\alpha \geq \omega_0$; see Example 4.1.23 of [2]). As shown in [6] (cf[7]), if λ is regular cardinal, then the family $\{B(\beta) : \beta < \lambda\}$ consists of closed subsets of $B(\lambda)$ and satisfies the following conditions:

$B(1) \subset B(2) \subset \ldots \subset B(\beta) \subset \ldots \subset B(\lambda)$,

$B(\beta) = cl(\cup_{\gamma<\beta} B(\gamma))$ for every limit ordinal $\beta < \lambda$, and
$B(\lambda) = \cup_{\beta<\lambda} B(\beta)$.

In the sequel by $\tilde{B}(\lambda)$ we will denote the set $B(\lambda)$ with the topology τ defined by taking the family

$$\{U \cap B(\beta) : U \text{ is open in } B(\lambda) \text{ and } \beta \leq \lambda\}$$

as a base of τ. As was proved in [6] for $\lambda = \omega_1$ and in [5] for any regular cardinal λ, the space $\tilde{B}(\lambda)$ is perfectly normal and collectionwise normal.

For $\alpha < \lambda$ put $B_\alpha = B(\alpha) \setminus \cup_{\beta<\alpha} B(\beta)$ and for any set $S \subset W(\lambda)$ put $B(S) = \cup_{\alpha \in S} B_\alpha$. Note that B_α is closed subset of $\tilde{B}(\lambda)$ on which the topology of the subspace of $\tilde{B}(\lambda)$ coincides with the topology of subspace of $B(\lambda)$ and that for any $\alpha < \lambda$ the set $\cup_{\beta<\alpha} B_\beta$ is open in $\tilde{B}(\lambda)$ and the set $\cup_{\beta\leq\alpha} B_\beta$ is closed in $\tilde{B}(\lambda)$.

3. Auxiliary lemmas

We will need two simple lemmas to prove some properties of our examples.

3.1 Lemma. *If a perfectly normal space X can be represented as the union of a transfiite sequence $X_1, X_2, \ldots, X_\alpha, \ldots, \alpha < \lambda$ of pairwise disjoint closed subspaces such that $\operatorname{ind} X_\alpha = 0$, the union $\cup_{\beta<\alpha} X_\beta$ is open and the union $\cup_{\beta\leq\alpha} X_\beta$ is closed for every $\alpha < \lambda$, then the space X has small transfinite dimension.*

Proof. We will show by induction with respect to α that for each $\alpha \leq \lambda$ the space $X'_\alpha = \cup_{\beta<\alpha} X_\beta$ has trind.

For $\alpha = 1$ this is true. Assume that for every $\beta < \alpha$ trind X'_β exists. To show that trind X'_α exists consider an arbitrary point $x \in X'_\alpha$ and an arbitrary closed set $F \subset X'_\alpha$ such that $x \notin F$; we will find a partition L between x and F such that trind L exists.

Suppose first that α is a limit ordinal. Let $\beta < \alpha$ be such that $x \in X_\beta$. Then $X'_{\beta+1}$ is an open-and-closed neighbourhood of a point x in X'_α, and by the inductive assumption there exists a partition L between x and $F \cap X'_{\beta+1}$ in $X'_{\beta+1}$ (hence in X'_α) such that trind L exists.

Suppose now that $\alpha = \beta + 1$; then $X'_\alpha = X'_\beta \cup X_\beta$. If $x \in X'_\beta$ and β is a limit ordinal, then we can proceed as in the first case. If $x \in X'_\beta$ and $\beta = \gamma + 1$, then X'_β is open-and-closed in X'_α and we can use the inductive assumption in a similar way.

If $x \in X_\beta$, then the zero-dimensionality of X_β yields that the empty set is a partition between x and $F \cap X_\beta$ in X_β. Thus there exists a partition L in X'_α between x and F such that $L \cap X_\beta = \emptyset$ (see Lemma 1.2.9 and Remark 1.2.10 of [2]). Hence $L \subset X'_\beta$ and trind L exists because by the inductive assumption trind X'_β exists (and trind $L \leq$ trind X'_β).

3.2 Lemma. *Let λ be a countable ordinal and suppose that a metrizable space X can be represented as the union of a transfinite sequence $X_1, X_2, \dots, X_\alpha, \dots, \alpha < \lambda$ of pairwise disjoint subspaces such that X_α is completely metrizable, the union $\bigcup_{\beta<\alpha} X_\beta$ is open and the union $\bigcup_{\beta\leq\alpha} X_\beta$ is closed for every $\alpha < \lambda$. Then X is completely metrizable.*

Proof. We will show by induction with respect to α that the subspace $X'_\alpha = \bigcup_{\beta<\alpha} X_\beta$ of X is completely metrizable for every $\alpha \leq \lambda$. Suppose that for every $\beta < \alpha$ the space X'_β is completely metrizable (obviously this is true for $\beta = 1$).

If α is a limit ordinal, then let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of non-limit ordinals such that $\alpha = lim\alpha_n$ and $\alpha_n < \alpha_{n+1}$ for every $n = 1, 2, \dots$. Then for every $n = 1, 2, \dots$ the subspace $X'_{\alpha_n} \setminus X'_{\alpha_{n-1}}$ is open-and-closed in X'_α and is completely metrizable by the inductive assumption. Thus X'_α is the union of a discrete family $\{X'_{\alpha_n} \setminus X'_{\alpha_{n-1}}\}_{n=1}^\infty$ of completely metrizable subspaces, hence it is completely metrizable.

If $\alpha = \beta + 1$, then $X'_\alpha = X'_\beta \cup X_\beta$, where both X'_β and X_β are completely metrizable, hence X'_α is completely metrizable.

4. Constructions of the examples

4.1. Construction of the space X from Example 1.1
As the space X we can take a perfectly normal not countable-

dimensional space Y constructed in Example 1 of [5] or its subspace X_1 constructed in Example 2 of [5]. Recall that the space Y was obtained by taking the decomposition $\{S_\alpha : \alpha < \mathfrak{c}\}$ of $W(\mathfrak{c}^+)$ into $\mathfrak{c}$ disjoint stationary sets, arranging all points of the Hilbert cube I^ω into a transfinite sequence $I^\omega = \{x_\alpha : \alpha < \mathfrak{c}\}$ and putting

$$Y = \cup_{\beta<\mathfrak{c}^+} Y_\beta, \text{ where } Y_\beta = B_\beta \times \{x_\alpha\} \text{ if } \beta \in S_\alpha.$$

It was also shown in [5], Example 2, that for some ordinal $\alpha_1 \leq \mathfrak{c}^+$ the space $X_1 = \cup_{\beta<\alpha_1} Y_\beta$ is locally countable-dimensional, but not countable-dimensional (both spaces Y and X_1 are taken with the topology of the subspace of the Cartesian product $\tilde{B}(\mathfrak{c}^+) \times I^\omega$).

It is easy to see that the decomposition of Y into the sets $Y_\beta, \beta < \mathfrak{c}^+$, satisfies the conditions of Lemma 3.1. Indeed, each Y_β is homeomorphic with the subspace of the Baire space $B(\mathfrak{c}^+)$, hence ind $Y_\beta = 0$. Moreover, for every $\alpha < \mathfrak{c}^+$ the set $\cup_{\beta<\alpha} Y_\beta = Y \cap (\cup_{\beta<\alpha} B(\beta) \times I^\omega)$ is open in Y and the set $\cup_{\beta\leq\alpha} Y_\beta = Y \cap (B(\alpha+1) \times I^\omega)$ is closed in Y (see sec. 2.3). Thus by Lemma 3.1 trind Y exists. The same is true for the subspace X_1 of Y.

Note that by the same argument the spaces X_0 and X_2 constructed in Example 2 of [5] have trind. It follows that there exists a perfectly normal locally weakly infinite-dimensional (respectively, locally 0-countable-dimensional) space having trind which is not weakly infinite-dimensional (respectively, which is not 0-countable-dimensional) (see [4] or [5] for these notions).

4.2 **Constructions of spaces Y_n and Y from Example 1.2** (A) The space Y_n is obtained by a minor modification of an n-dimensional perfectly normal space X_n which is locally 0-dimensional and locally second-countable, constructed in [7], Theorem 2. Let us recall that the space X_n was obtained by splitting $W(\omega_1)$ into $n+1$ disjoint stationary sets $S_0, S_1, \ldots, S_n$ and taking the subspace of the space $\tilde{B}(\omega_1) \times I^n$ defined by

$$X_n = \cup_{m=0}^{n} B(S_m) \times R_n^m,$$

where R_n^m is the set of points in I^n, exactly m of whose coordinates are rational.

Now, since dim $R_n^m = 0$ by the enlargement theorem (see [2], Theorem 1.5.11) there exists a G_σ-subset $\tilde{R}_n^m$ of I^n such that $R_n^m \subset \tilde{R}_n^m$ and dim$\tilde{R}_n^m = 0$. Let

$$Y_n = \cup_{m=0}^{n} B(S_m) \times \tilde{R}_n^m$$

be the subspace of $\tilde{B}(\omega_1) \times I^n$. Since Y_n contains X_n as a subspace, dim $Y_n = n$ (cf [2], Corollary 3.1.20). Represent Y_n as

$$Y_n = \cup_{\alpha<\omega_1} Y_\alpha \text{ where } Y_\alpha = B_\alpha \times \tilde{R}_n^m \text{ if } \alpha \in S_m.$$

Then every Y_α is 0-dimensional and completely metrizable and the decomposition of the space $Y'_\alpha = \cup_{\beta<\alpha} Y_\beta$ into the closed subsets $Y_1, Y_2, \ldots, Y_\beta, \ldots, \beta < \alpha$ satisfies the conditions of Lemma 3.2. Thus Y'_α is completely metrizable. Moreover, Y'_α is separable, 0-dimensional (by the sum theorem) and obviously does not contain any non-empty compact open subspace, hence it is homeomorphic to the space of irrational numbers by a theorem of Alexandroff and Urysohn (see [2], Problem 1.3.E).

Finally, observe that the spaces Y_n are homogeneous. Indeed, for every two points x and y of Y_n there exists an open-and-closed subspace U of Y_n homeomorphic with the irrationals and containing both x and y. Namely, one can take as U the subspace X'_α, where α is a non-limit ordinal and $x, y \in X'_\alpha$.

(B) The space Y is obtained by a modification of the space X constructed in Example 3 of [5]. Recall that the space X was obtained in the following way. Let Z be a compact countable-dimensional but not strongly countable-dimensional space which is the union of a family $\{I^i\}_{i=1}^\infty$ of disjoint subsets homeomorphic to i-dimensional cubes and of a subset $P = Z \setminus \cup_{i=1}^\infty I^i$ homeomorphic to the space of irrationals (see [4], Example 1.12). For each $i = 1, 2, \ldots$ and $m = 0, 1, \ldots, i$ let R_i^m denote the set of points in I^i exactly m of whose coordinates are rational and let $\tilde{R}_i^m$, be a 0-dimensional G_σ-subset of I^i containing R_i^m. Let us split $W(\omega_1)$ into countably many disjoint stationary sets S_i, $i = 1, 2, \ldots$ and let $S_i = \cup_{m=0}^{i} S_i^m$, where S_i^m are also disjoint and stationary in $W(\omega_1)$. As was proved in [5], Example 3, the subspace

$$X = \cup_{\alpha<\omega_1} X_\alpha, \text{ where } X_\alpha = B_\alpha \times (R_i^m \cup P) \text{ if } \alpha \in S_i^m,$$

of the space $\tilde{B}(\omega_1) \times Z$ is a perfectly normal, locally 0-dimensional and locally second-countable space which is not strongly countable-dimensional. Thus the subspace

$$Y = \cup_{\alpha<\omega_1} Y_\alpha, \text{ where } Y_\alpha = B_\alpha \times (\tilde{R}_i^m \cup P) \text{ if } \alpha \in S_i^m.$$

of $\tilde{B}(\omega_1) \times Z$ is also not strongly countable-dimensional, since it contains X (cf [4], Proposition 2.2). Moreover, similarly as in (A) one shows that the space Y is locally homeomorphic to the irrationals (since every space $Y'_\alpha = \cup_{\beta<\alpha} Y_\beta$, where $\alpha < \omega_1$, is an open-and -closed subspace of Y homeomorphic with P) and thus it is homogeneous.

Note that the spaces Y_n constructed above are not Čech complete. This follows from the fact that every C_σ-subset of $\tilde{B}(\omega_1) \times I^n$ containing Y_n contains a subset homeomorphic to I^n(see [7], sec. 2.4, Lemma).

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