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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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CONCERNING THE EXISTENCE OF A CONNECTED, COUNTABLE DENSE HOMOGENEOUS SUBSET OF THE PLANE WHICH IS NOT STRONGLY LOCALLY HOMOGENEOUS

W. L. SALTSMAN

ABSTRACT. Under $ZFC + CH$, a space as in the title is constructed.

A topological space X is countable dense homogeneous (CDH) provided that X is separable and that if A and B are countable dense subsets of X then there is an $h \in H(X)$, the autohomeomorphism group on X, such that $h(A) = B$. The space X is strongly locally homogeneous (SLH) provided that X has a basis of open sets U such that if $x, y \in U$ then there is $h \in H(X)$ such that $h(x) = y$ and $h(z) = z\ \forall z \in X \backslash U$.

It is known that SLH Polish spaces are CDH (first proved by Bennett [1] in the locally compact case, later by Fletcher and McCoy [2] in the general case). van Mill [3] showed that noncomplete separable metric SLH spaces need not be CDH and asked for examples of CDH connected spaces that are not SLH. Nonmetric examples have been provided: an Hausdorff example by Fitzpatrick and Zhou [4], and a regular Hausdorff example by Watson and Simon[5]. A natural place to look is in the class of Jones spaces. If $f : \mathbb{R} \to \mathbb{R}$ is a discontinuous, additive function ($f(x + y) = f(x) + f(y)\ \forall x, y \in \mathbb{R}$) and the graph, J, of f is a connected subspace of $\mathbb{R}^2$, then J is called a Jones space [6]. Every Jones space is a topological group and is, therefore, homogeneous. R. W. Heath noted that no

Jones space is SLH. To date, it is not known whether there is a CDH Jones space (see [7]). Extending Heath's observation, it is proved in Theorem 1 of reference [7] that no subset of the plane which is connected, dense in the plane, and which intersects vertical lines only once is SLH. The purpose of this paper is to find such a space which is CDH, assuming the continuum hypothesis.

Theorem (ZFC + CH). *There is a connected, CDH subset of $\mathbb{R}^2$ which is not SLH.*

Proof. Any subset of $\mathbb{R}^2$ which is dense in $\mathbb{R}^2$, connected and intersects vertical lines only once is not SLH. We will construct such a set which is CDH.

In this preliminary section, we give a version of the proof which does not have the technical detail of the second proof. Afterward, we will proceed with the second proof, filling in the technical details. Suppose A is a countable dense subset of $\mathbb{R}^2$ which intersects no vertical line more than once; and G is a countable collection of autohomeomorphisms of $\mathbb{R}^2$ such that G is a group under composition, each element of G sends vertical lines onto vertical lines and A onto itself; if $g \in G, g$ is not the identity, and g sends a vertical line onto itself, then that vertical line intersects A; and if $g \in G, \mathcal{L}_1$ and $\mathcal{L}_2$ are any pair of vertical lines in $\mathbb{R}^2$ such that $g(\mathcal{L}_1) = \mathcal{L}_2$, $\pi_1(\mathcal{L}_1) = a$ and $\pi_1(\mathcal{L}_2) = b$, then the homeomorphism $\overline{g}$ of $\mathbb{R}$ onto itself defined by $\overline{g}(a) = b$ is order-preserving (i.e. g is order-preserving in the x-axis direction). Also, suppose that if $g \in G$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are any pair of vertical lines in $\mathbb{R}^2$ such that $g(\mathcal{L}_1) = \mathcal{L}_2$, y_1 is the second coordinate of an element p_1 of $\mathcal{L}_1$, and y_2 is the second coordinate of an element p_2 of $\mathcal{L}_2$ such that $g(p_1) = p_2$, then the homeomorphism f of $\mathbb{R}$ onto itself defined by $f(y_1) = y_2$ is order-preserving (i.e. g is order-preserving in the y-axis direction). The preceding is what we have at the beginning of the α-th step of the construction where $\alpha < \omega_1$ is an ordinal number.

To see why the properties of A and G mentioned above are desirable, let $\mathcal{L}$ be any vertical line which does not intersect A and let x be any element of $\mathcal{L}$. Suppose $g_1, g_2 \in G$ such that $g_1(x)$ and $g_2(x)$ lie on the same vertical line. It follows then that $g_2^{-1} \circ g_1$ is the identity and $g_1 = g_2$. This enables us to put other points into A so that A together with these points and their images under the elements of G intersect no vertical line more than once and to ensure that the construction will result in a connected set. The purposes of the order-preserving properties will be seen a little later.

In order that the construction yield a CDH subset of $\mathbb{R}^2$, let B and C be appropriately selected dense subsets of A (by the end of the construction, we are assuming that all pairs of dense subsets of $\mathbb{R}^2$ relevant to the argument will have been treated). We wish to obtain a successor, A_α, of A and an autohomeomorphism, g_α, of $\mathbb{R}^2$ which sends B onto C such that the group, G_α, generated by $G \cup \{g_\alpha\}$ and A_α enjoy the same properties given for A and G. We will form A_α from A by putting a succession of points and their images under the elements of G into A. This guarantees that every element of G will send A_α onto itself. It might not be essential, but to help make it so that g_α sends A_α onto itself, we let the complement of $B \cup C$ in A be dense in $\mathbb{R}^2$.

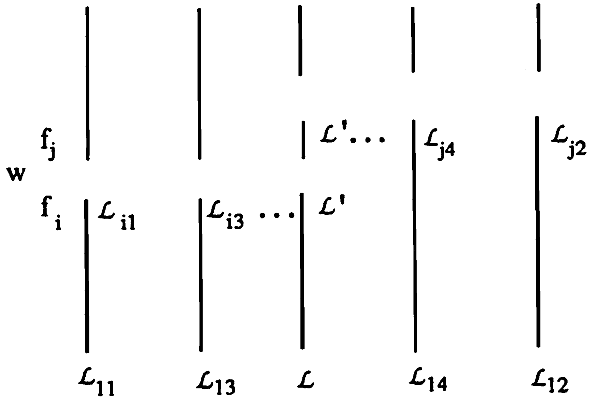
Continuing the discussion of obtaining a set which is CDH from the construction, suppose $A_{n-1,m-1}$, n and m positive integers, is a countable subset of $\mathbb{R}^2$ such that $A_{n-1,m-1}$ intersects no vertical line more than once, $A \subseteq A_{n-1,m-1}$, and each element of G sends $A_{n-1,m-1}$ onto itself. With an autohomeomorphism, $g_{0,0}$, of $\mathbb{R}^2$ as a starting point, where $g_{0,0}$ sends vertical lines onto vertical lines, $g_{0,0}$ sends no vertical line onto itself, $g_{0,0}$ is order-preserving in the x-axis and y-axis directions, and $g_{0,0} \notin G$, we will obtain g_α by a progression of alterations and convergences of homeomorphisms. These alterations will be carried out so that all homeomorphisms obtained along the way, and g_α itself, will have the properties given for $g_{0,0}$. Suppose $g_{n-1,m-1}$ is an autohomeomorphism of

$\mathbb{R}^2$ such that $g_{n-1,m-1}$ takes finitely many appropriately selected elements of B into C and $A_{n-1,m-1}$ into itself, $g_{n-1,m-1}^{-1}$ takes finitely many appropriately selected elements of C into B and $A_{n-1,m-1}$ into itself, and $g_{n-1,m-1}$ has the properties given for $g_{0,0}$.

In order that G_α and A_α have the property that if $g \in G_\alpha$, g is not the identity, and g sends a vertical line onto itself, then that vertical line intersects A_α; suppose that $w = f_p \circ \ldots \circ f_2 \circ f_1, p$ a positive integer, is a word where the letters come from the alphabet $G \cup \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$. If w consists of only one letter, that letter comes from G and is not the identity, then any vertical line which w sends onto itself would intersect A and hence A_α. So we need not be concerned with such words. In fact, we will not alter any of the elements of G at any time. If w consists of only one letter, and that letter comes from $\{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$, then w sends no vertical line onto itself (because $g_{n-1,m-1}$ is one of the homeomorphisms obtained along the way from $g_{0,0}$ to g_α). In this case, w satisfies the desired condition vacuously. From what we have seen so far, we may assume that w has more than one letter. Let us also assume that w is in reduced form (i.e. the identity is not in the spelling of w; and if f and g are two consecutive letters in the spelling of w, then either $f = g = g_{n-1,m-1}$, $f = g = g_{n-1,m-1}^{-1}$, or $\{f, g\}$ has a nonempty intersection with both G and $\{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$). Suppose that $sw = f_q \circ \ldots \circ f_{r+1} \circ f_r$, q and r positive integers with $p \geq q \geq r \geq 1$, is a sub-word of w. So that we may make statements about proper sub-words of w, we will assume sw represents a proper sub-word of w. Also, for reasons already discussed (in the context of w), we assume sw has more than one letter. The statement we now make about sw is that if any sw sends a vertical line onto itself, then that vertical line intersects $A_{n-1,m-1}$. Suppose $I_0, I_1, \ldots, I_{m-1}$ are nondegenerate, compact, non-overlapping intervals on the x-axis. We wish to make a statement about w which is similar to the statement about sw. That statement is that there are at most finitely many vertical lines, $\mathcal{L}$, intersecting $I_0 \cup I_1 \cup \ldots \cup I_{m-1}$ such that $w(\mathcal{L}) = \mathcal{L}$, that each of these $\mathcal{L}$'s intersects $A_{n-1,m-1}$, and none of these $\mathcal{L}$'s contains an endpoint of $I_0, I_1, \ldots$, or I_{m-1}. In order to make a statement about vertical lines which do not intersect the interiors of $I_0, I_1, \ldots$, or I_{m-1}, suppose $\mathcal{L}_1$ is such a vertical line and let $f_1(\mathcal{L}_1) = \mathcal{L}_2, f_2(\mathcal{L}_2) = \mathcal{L}_3, \ldots, f_p(\mathcal{L}_p) = \mathcal{L}_{p+1}$. The statement we make about $\mathcal{L}_1$ is that if i is any positive integer, $1 \leq i \leq p$, such that $f_i \in \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$, and for each such i, if f_i may be altered to send $\mathcal{L}_i$ to a different vertical line from $\mathcal{L}_{i+1}$ without disturbing any sw's on vertical lines which they send onto themselves, without altering any of the images, under $g_{n-1,m-1}$, of the finitely many appropriately selected elements of B or $A_{n-1,m-1}$ mentioned earlier, and without altering any of the images, under $g_{n-1,m-1}^{-1}$, of the finitely many appropriately selected elements of C or $A_{n-1,m-1}$ mentioned earlier; then there is a positive integer j, $1 \leq j \leq p$, such that $f_j \in \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$, for which f_j may be altered to send $\mathcal{L}_j$ to a different vertical line from $\mathcal{L}_{j+1}$ without disturbing w on any vertical lines which intersect $I_0 \cup I_1 \cup \ldots \cup I_{m-1}$ and which w sends onto themselves. If $z = h_k \circ \ldots \circ h_2 \circ h_1$, k a positive integer, is a word whose letters come from $G \cup \{g_{n-1,m-1},\ g_{n-1,m-1}^{-1}\}$, and $\mathcal{L}_1$ is a vertical line, then we say that z is disturbed on $\mathcal{L}_1$ if $h_1(\mathcal{L}_1) = \mathcal{L}_2,\ h_2(\mathcal{L}_2) = \mathcal{L}_3, \ldots, h_k(\mathcal{L}_k) = \mathcal{L}_{k+1}$; and if there is a positive integer, j, where $1 \leq j \leq k$, $h_j \in \{g_{n-1,m-1},\ g_{n-1,m-1}^{-1}\}$, such that some (or all) of the images of $\mathcal{L}_j$, under h_j, are altered. The preceding is what we have at the beginning of the m-th step of the n-th step of the α-th step of the construction.

To begin the m-th step of the n-th step of the α-th step of the construction, suppose I_m is a nondegenerate, compact interval on the x-axis such that I_m does not overlap any of the intervals $I_0, I_1, \ldots, I_{m-1}$; if $\mathcal{L}$ is any vertical line which intersects I_m, then $\mathcal{L}$ satisfies the hypotheses (and therefore the conclusion) of the statement about vertical lines which do not intersect the interiors of $I_0, I_1, \ldots, or I_{m-1}$ at the end of the last paragraph; and if j, k are nonnegative integers, $0 \leq j,\ k \leq m-1$, there is an element of $\{I_0, I_1, \ldots, I_{m-1}\}$ to the left of I_m of which I_j is the closest, and there is an element of $\{I_0, I_1, \ldots, I_{m-1}\}$ to the right of I_m of which I_k is the closest, then I_m extends from I_j to I_k provided the previous condition is not violated.

We wish to assert the existence of a positive number which has a certain property in conjunction with I_m. Before we can do this in the way we desire, we must make another assumption about w: we assume that $\{f_p, f_1\} \neq \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$. The case where those two sets are equal is trivial and will be discussed briefly after we are finished with this case. By the way I_m was chosen, and by the assumption on w, there is a $\delta > 0$ such that if $x, y \in I_m$, $x \neq y$, $|x - y| < \delta$, $\mathcal{L}_1$ is the vertical line containing x, $f_1(\mathcal{L}_1) = \mathcal{L}_2, f_2(\mathcal{L}_2) = \mathcal{L}_3, \ldots, f_p(\mathcal{L}_p) = \mathcal{L}_{p+1}, \mathcal{L}$ is the vertical line containing y, and i is a positive integer, $1 \leq i \leq p$, where $f_i \in \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$, then f_i may be altered to send $\mathcal{L}_i$ to a different vertical line from $\mathcal{L}_{i+1}$ without disturbing w on $\mathcal{L}$. The denial of the existence of such a δ would result in a contradiction to the next to the last condition under which I_m was chosen (see Figure 1).



If $i < j$; $f_i = f_j$; $\mathcal{L}_{i1} = \mathcal{L}_{j2}, \mathcal{L}_{i3} = \mathcal{L}_{j4}, \ldots$; $\mathcal{L}_{11}, \mathcal{L}_{12}, \mathcal{L}_{13}, \mathcal{L}_{14}, \ldots$ intersect I_m; and $sw = f_{j-1} \circ \ldots \circ f_i$; then $\mathcal{L}$ intersects I_m; and altering f_i on $\mathcal{L}'$ disturbs sw on $\mathcal{L}'$ where $sw(\mathcal{L}') = \mathcal{L}'$.

Figure 1

Let U be a nondegenerate closed subinterval of I_m of length less that δ which has the same left endpoint as I_m, and for which there is at most one $x \in U$ with the following property: there are a vertical line $\mathcal{L}$ and a positive integer i, $1 \leq i \leq p$, such that $\mathcal{L}$ intersects $I_0 \cup I_1 \cup \ldots \cup I_{m-1}, w(\mathcal{L}) = \mathcal{L}$, $f_i \in \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$, and if $\mathcal{L}_1$ is the vertical line containing x where $f_1(\mathcal{L}_1) = \mathcal{L}_2$, $f_2(\mathcal{L}_2) = \mathcal{L}_3, \ldots$, $f_p(\mathcal{L}_p) = \mathcal{L}_{p+1}$, then f_i cannot be altered to send $\mathcal{L}_i$ to a different vertical line from $\mathcal{L}_{i+1}$ without disturbing w on $\mathcal{L}$. There are at most finitely many vertical lines, $\mathcal{L}$, intersecting $I_0 \cup I_1 \cup \ldots \cup I_{m-1}$ for which $w(\mathcal{L}) = \mathcal{L}$, and this clearly makes the above possible. Let j be a positive integer,

$$1 \leq j \leq p, \quad f_j \in \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\},$$

such that if $\mathcal{L}$ is a vertical line intersecting $I_0 \cup I_1 \cup \ldots \cup I_{m-1}$ where $w(\mathcal{L}) = \mathcal{L}$ and $\mathcal{L}_1$ is a vertical line interesecting U where $f_1(\mathcal{L}_1) = \mathcal{L}_2$, $f_2(\mathcal{L}_2) = \mathcal{L}_3, \ldots, f_p(\mathcal{L}_p) = \mathcal{L}_{p+1}$, then f_j may be altered to send $\mathcal{L}_j$ to a different vertical line from $\mathcal{L}_{j+1}$ without disturbing w on $\mathcal{L}$. Such a positive integer as j exists for U due to the last condition under which U was chosen, and the next to the last condition under which I_m was chosen. Let $V_1 = \{x \in \mathbb{R}^2 : x \text{ is on a vertical line that intersects } U\}$ and let $f_1(V_1) = V_2, \ldots, f_p(V_p) = V_{p+1}$.

What we wish to do is to alter f_j on V_j so that it still sends vertical lines onto vertical lines and V_j onto V_{j+1}. Since the length of U is less than δ, then we can predict how this change will affect w on V_1. In order to preserve the property which the positive number δ, defined above, has in conjunction with I_m; to ensure that each sw will send no more vertical lines onto themselves; to ensure that w will send no more vertical lines which intersect $I_0 \cup I_1 \cup \ldots \cup I_{m-1}$ onto themselves; to ensure that I_m continues to satisfy the next to the last condition under which it was chosen, until we progress to I_{m+1}; and to ensure that the desired convergences of homeomorphisms take place; the second proof asserts the existence of a positive number, δ', having certain properties, after which the division of that proof's equivalent to the interval, U, into finitely many nondegenerate, closed, non-overlapping subintervals, each of length less than δ', takes place. Alterations then take place with respect to these subintervals. Let $U = [a, b]$. Let $\mathcal{L}_a$ and $\mathcal{L}_b$ be the vertical lines containing a and b, respectively. Assume that w does not send $\mathcal{L}_a$ onto itself nor $\mathcal{L}_b$ onto itself, since small changes in f_j would produce this result, if it was not already true.

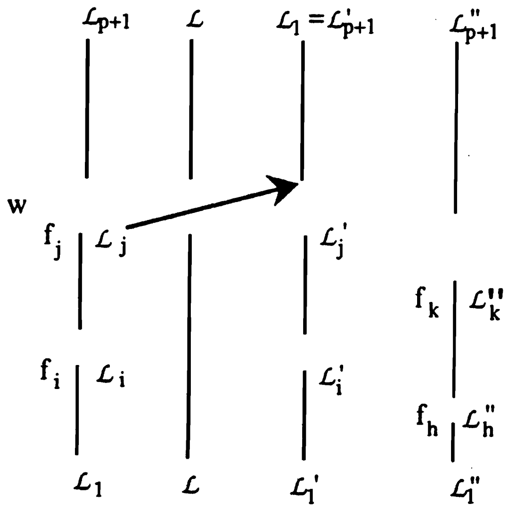
Now, assume that f_j has been altered so that f_j still sends vertical lines onto vertical lines and V_j onto V_{j+1}, but so that there is now only one vertical line intersecting U which w sends onto itself, and that vertical line intersects the interior of U. Let $\mathcal{L}_1$ be this vertical line, let

$$f_1(\mathcal{L}_1) = \mathcal{L}_2, f_2(\mathcal{L}_2) = \mathcal{L}_3, \ldots , f_p(\mathcal{L}_p) = \mathcal{L}_{p+1} = \mathcal{L}_1,$$

and assume that $\mathcal{L}_1$ has been selected so that $\mathcal{L}_i$ does not intersect $A_{n-1,m-1}$, $1 \leq i \leq p+1$. Assume, also, that $\mathcal{L}_1$ has been selected so that if $\mathcal{L}_1'$ is a vertical line which does not intersect the interior of $I_0, I_1, \ldots , I_{m-1}$, or

$$U; \; f_1(\mathcal{L}_1') = \mathcal{L}_2', \; f_2(\mathcal{L}_2') = \mathcal{L}_3', \ldots , f_p(\mathcal{L}_p') = \mathcal{L}_{p+1}';$$

and if there is a positive integer, i, where $1 \leq i \leq p$ and $f_i \in \{g_{n-1,m-1}, \; g_{n-1,m-1}^{-1}\}$, for which f_i cannot be altered to send $\mathcal{L}_i'$ to a different vertical line from $\mathcal{L}_{i+1}'$ without disturbing w on $\mathcal{L}_1$; then either f_k cannot be altered to send $\mathcal{L}_k'$ to a different vertical line from $\mathcal{L}_{k+1}'$ without disturbing w on $\mathcal{L}_1$ for any positive integer, k, where $1 \leq k \leq p$ and $f_k \in \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$, or there is a positive integer, k, where $1 \leq k \leq p$ and $f_k \in \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$, such that f_k may be altered to send $\mathcal{L}_k'$ to a different vertical line from $\mathcal{L}_{k+1}'$ without any of the disturbances or alterations mentioned in the hypothesis or conclusion of the statement on vertical lines which do not intersect the interiors of $I_0, I_1, \ldots$, or I_{m-1}, and without disturbing w on $\mathcal{L}_1$ (see Figure 2).



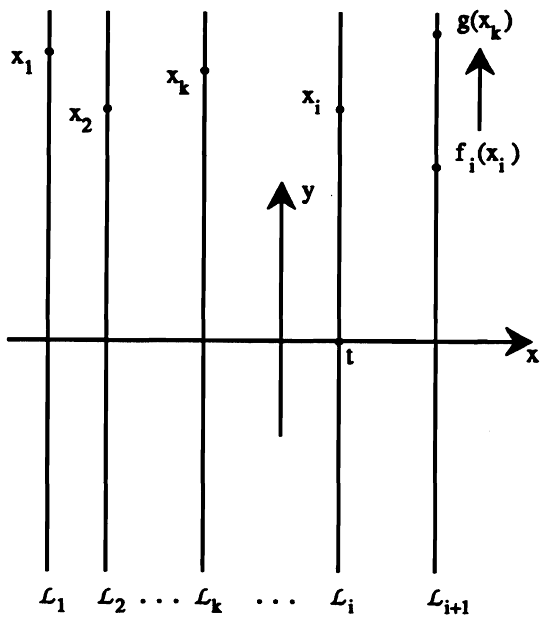
(1) Assume there is an $\mathcal{L}$ such that $\mathcal{L}$ intersects U, and $w(\mathcal{L}) = \mathcal{L}$. Then, there are uncountably many ways to choose $\mathcal{L}_1$ and $\mathcal{L}'_1$ so that $\mathcal{L}_1$ and $\mathcal{L}'_1$ intersect U, and $w(\mathcal{L}'_1) = \mathcal{L}_1$. After choosing, alter f_j so that $w(\mathcal{L}_1) = \mathcal{L}_1$.

(2) For simplicity, assume all f's shown equal $g_{n-1,m-1}$. Let $\mathcal{L}''_k$ be a vertical line on which f_k should not be altered. From our choice of $\mathcal{L}_1$ and $\mathcal{L}'_1$, assume that $\{\mathcal{L}_i, \mathcal{L}'_i, \mathcal{L}_j, \mathcal{L}'_j\} \cap \{\mathcal{L}''_h, \mathcal{L}''_k\} = \emptyset$.

(3) The pairs, f_k and $\mathcal{L}''_k$, are countable; and the associated pairs, f_h and $\mathcal{L}''_h$, are countable.

Figure 2

Now, select a point x_1, on $\mathcal{L}_1$. Suppose that x_2 on $\mathcal{L}_2, x_3$ on $\mathcal{L}_3, \ldots, x_i$ on $\mathcal{L}_i$ where i is a positive integer, $1 \leq i \leq p+1$, have been subsequently appropriately obtained. If there are a positive integer $k, 1 \leq k \leq i$, and a $g \in G$ such that g sends $\mathcal{L}_k$ onto $\mathcal{L}_{i+1}$, then either $g(x_k) = f_i(x_i)$ or $g(x_k) \neq f_i(x_i)$. If $g(x_k) = f_i(x_i)$, then no alteration of f_i is needed. If $g(x_k) \neq f_i(x_i)$, then it follows that $f_i \in \{g_{n-1,m-1}, g^{-1}_{n-1,m-1}\}$. In this case, alter f_i so that $g(x_k) = f_i(x_i)$. Now, let $x_{i+1} = f_i(x_i)$. When this process is complete, take the points obtained together with their images under the elements of G, and put them into $A_{n-1,m-1}$. We find that $A_{n-1,m-1}$ still intersects no vertical line more than once. If x_1 is chosen far enough from the origin, then, since $g_{n-1,m-1}$ and all elements of G are order-preserving in the y-axis direction, those points whose images had to be changed under the process are far enough from the origin so that the desired convergences of homeomorphisms will still take place (see Figure 3).



(1) Alter the image of x_i, under f_i, so that $f_i(x_i) = g(x_k)$.

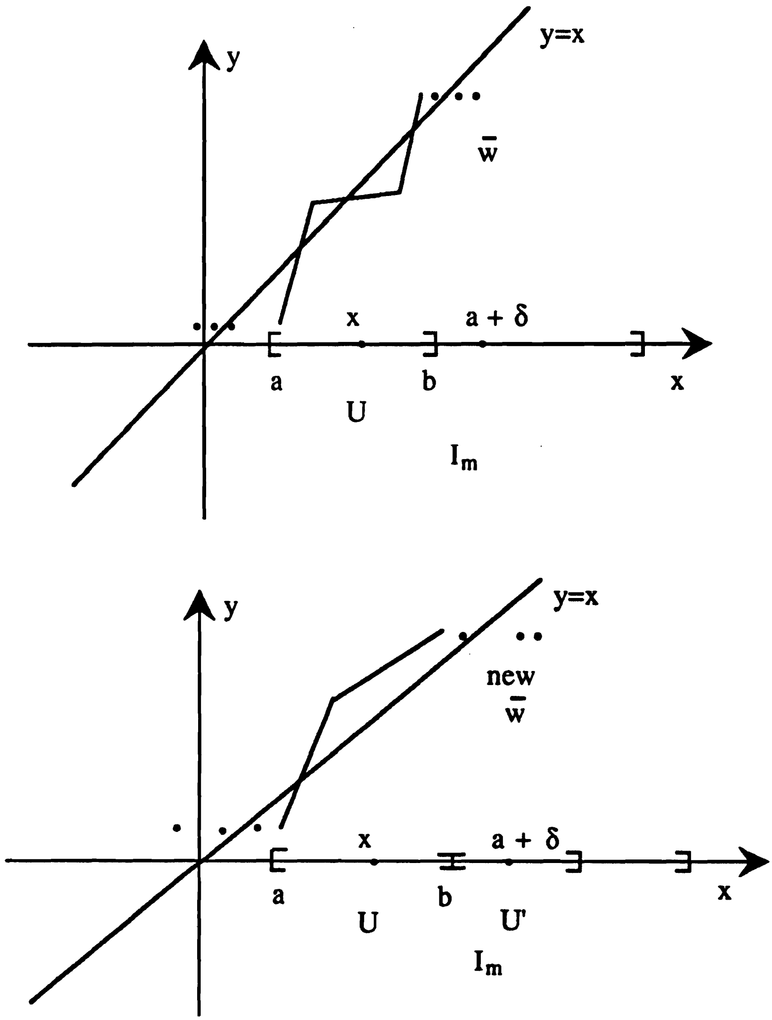
(2) Images under f_i, of points further down $\mathcal{L}_i$ from x_i, need not be altered.

(3) But, if f_i was not order-preserving in the y-axis direction, and if $g(x_k)$ lies above $f_i(t)$, then making $f_i(x_i) = g(x_k)$ would alter the image, under f_i, of each element of $\mathcal{L}_i$ from t to x_i.

Figure 3

Let i, k be positive integers, $1 \leq i \leq k \leq p$, such that i is the least positive integer where $f_i \in \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$ and k is the greatest positive integer where $f_k \in \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$. Suppose there is a vertical line, $\mathcal{L}_1'$, which does not intersect the interior of $I_0, I_1, \ldots, I_{m-1}$, or U, and which satisfies the following: if $f_1(\mathcal{L}_1') = \mathcal{L}_2'$, $f_2(\mathcal{L}_2') = \mathcal{L}_3', \ldots, f_p(\mathcal{L}_p') = \mathcal{L}_{p+1}$; and c is any positive integer, $i \leq c \leq k$, where $f_c \in \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$; then f_c cannot be altered to send $\mathcal{L}_c'$ to a different vertical line from $\mathcal{L}_{c+1}'$ without disturbing w on $\mathcal{L}_1$. Since w is in reduced form, this gives us that the set $\{\mathcal{L}_i', \mathcal{L}_{i+1}', \ldots, \mathcal{L}_{k+1}'\}$ is a subset of the set $\{\mathcal{L}_i, \mathcal{L}_{i+1}, \ldots, \mathcal{L}_{k+1}\}$. We know that $\mathcal{L}_1 \neq \mathcal{L}_1'$ since $\mathcal{L}_1$ intersects the interior of U and $\mathcal{L}_1'$ does not. Assume, for the sake of specificity, that $\pi_1(\mathcal{L}_1) < \pi_1(\mathcal{L}_1')$. Since $f_1, f_2, \ldots, f_p$ are all order-preserving in the x-axis direction, then $\pi_1(\mathcal{L}_2) < \pi_1(\mathcal{L}_2'), \ldots, \pi_1(\mathcal{L}_{p+1}) < \pi_1(\mathcal{L}_{p+1}')$. This would give us that the largest element of the set $\{\pi_1(\mathcal{L}_i), \pi_1(\mathcal{L}_{i+1}), \ldots, \pi_1(\mathcal{L}_{k+1})\}$ is less than an element of the set $\{\pi_1(\mathcal{L}_i'), \pi_1(\mathcal{L}_{i+1}'), \ldots, \pi_1(\mathcal{L}_{k+1}')\}$, which is a contradiction. Therefore, $\mathcal{L}_1'$ does not exist as supposed.

Combining the results of the previous paragraph with the last condition under which $\mathcal{L}_1$ was selected implies that the statement on vertical lines which do not intersect the interiors of $I_0, I_1, \ldots$, or I_{m-1} is true if "$I_0, I_1, \ldots$, or I_{m-1}" is replaced by "$I_0, I_1, \ldots, I_{m-1}$, or U" and "$I_0 \cup I_1 \cup \ldots \cup I_{m-1}$" is replaced by "$I_0 \cup I_1 \cup \ldots \cup I_{m-1} \cup U$". The statement on vertical lines which do not intersect the interiors of $I_0, I_1, \ldots$, or I_{m-1}, the next to the last condition under which I_m was chosen, and the last condition under which U was chosen furnished the positive integer, j, for U. By continuing to obtain modifications on the statement on vertical lines which do not intersect the interiors of $I_0, I_1, \ldots$, or I_{m-1} as we did above, maintaining the next to the last condition under which I_m was chosen, until we go to I_{m+1}, and choosing succeeding subintervals of I_m in the way that U was chosen, we will continue to obtain equivalents to the positive integer j for these intervals. We now assume that finitely many subintervals of I_m, covering I_m, have been chosen and each one treated as U was treated (see Figure 4).



(1) The problem with elements of the nature of x, between a and $a+\delta$, has not increased in going from $\overline{w}$ to the new $\overline{w}$.

(2) The next interval to be treated, following U, is U'.

Figure 4

Let the versions of $g_{n-1,m-1}$ and $A_{n-1,m-1}$ attained on the final subinterval of I_m mentioned above be called $g_{n-1,m}$ and $A_{n-1,m}$, respectively. There are countably many vertical lines for which the homeomorphism $g_{n-1,m}$ has been determined and will not change (i.e. if the image of a certain vertical line, under $g_{n-1,m}$, could not be changed to a different vertical line without disturbing some sw on a vertical line which it sends onto itself, without altering one of the images, under $g_{n-1,m}$, of the finitely many appropriately selected elements of B or $A_{n-1,m-1}$ mentioned above, without altering one of the images, under $g_{n-1,m}^{-1}$, of the finitely many appropriately selected elements of C or $A_{n-1,m-1}$ mentioned above, or without disturbing w on one of the vertical lines intersecting $I_0 \cup I_1 \cup \ldots \cup I_m$ which w sends onto itself, then the image of that vertical line will never change under $g_{n-1,m}$ nor under any successor of $g_{n-1,m}$). Let $\mathcal{L}$ be an appropriately selected vertical line for which $g_{n-1,m}$ has been determined. We assume that finitely many of these vertical lines have already been appropriately selected and treated as $\mathcal{L}$ is about to be treated. Suppose $\mathcal{L}_1$ is a vertical line, where $w(\mathcal{L}_1) = \mathcal{L}_1$, such that if $f_1(\mathcal{L}_1) = \mathcal{L}_2$, $f_2(\mathcal{L}_2) = \mathcal{L}_3, \ldots, f_p(\mathcal{L}_p) = \mathcal{L}_{p+1}$; then there exist positive integers i, j where $1 \leq i, j \leq p$; $f_i, f_j \in \{g_{n-1,m}, g_{n-1,m}^{-1}\}$; either $\mathcal{L}_i = \mathcal{L}$ and $f_i = g_{n-1,m}$, or $\mathcal{L}_{i+1} = \mathcal{L}$ and $f_i = g_{n-1,m}^{-1}$; and either $f_j = g_{n-1,m}$ and $g_{n-1,m}$ has not been determined on $\mathcal{L}_j$, or $f_j = g_{n-1,m}^{-1}$ and $g_{n-1,m}$ has not been determined on $\mathcal{L}_{j+1}$. There are at most $p+1$ many such $\mathcal{L}_1$'s, and so, by finitely many small alterations of the corresponding f_j's on the corresponding $\mathcal{L}_j$'s, we eliminate all such $\mathcal{L}_1$'s. Assuming this has been done, what we have is that if $\mathcal{L}_1$ is a vertical line such that $f_1(\mathcal{L}_1) = \mathcal{L}_2$, $f_2(\mathcal{L}_2) = \mathcal{L}_3, \ldots, f_p(\mathcal{L}_p) = \mathcal{L}_{p+1}$; and if there exists a positive integer, i, where $1 \leq i \leq p$; $f_i \in \{g_{n-1,m}, g_{n-1,m}^{-1}\}$; and either $\mathcal{L}_i = \mathcal{L}$ and $f_i = g_{n-1,m}$, or $\mathcal{L}_{i+1} = \mathcal{L}$ and $f_i = g_{n-1,m}^{-1}$; then either $w(\mathcal{L}_1) \neq \mathcal{L}_1$, or if j is any positive integer, $1 \leq j \leq p$, where $f_j \in \{g_{n-1,m}, g_{n-1,m}^{-1}\}$, then either $f_j = g_{n-1,m}$ and $g_{n-1,m}$ has been determined on $\mathcal{L}_j$, or $f_j = g_{n-1,m}^{-1}$ and $g_{n-1,m}$ has been determined on $\mathcal{L}_{j+1}$. In order for the above argument to be sound, we need to know whether a vertical line, $\mathcal{L}_1$, can exist under the following circumstances:

$$f_1(\mathcal{L}_1) = \mathcal{L}_2,\ f_2(\mathcal{L}_2) = \mathcal{L}_3, \dots, f_p(\mathcal{L}_p) = \mathcal{L}_{p+1};\ i, j$$

are positive integers,

$$1 \le i < j \le p,\ f_i, f_j \in \{g_{n-1,m}, g_{n-1,m}^{-1}\}, f_i = f_j,\ \ \mathcal{L}_i = \mathcal{L}_j,$$

and either $f_i = g_{n-1,m}$ and $g_{n-1,m}$ has not been determined on $\mathcal{L}_i$, or $f_i = g_{n-1,m}^{-1}$ and $g_{n-1,m}$ has not been determined on $\mathcal{L}_{i+1}$. The answer is that no such $\mathcal{L}_1$ can exist, because if it did, then it would provide us with an sw for which altering $g_{n-1,m}$ to send $\mathcal{L}_i$ to a different vertical line from $\mathcal{L}_{i+1}$ (or the reverse, as the case may be) would disturb sw on a vertical line that it sends onto itself. This is a contradiction since $g_{n-1,m}$ is determined on such vertical lines. We conclude, then, that the above argument is valid. Moreover, we assume that, through our choices of δ''s, the results of the argument are not undone for successors of $g_{n-1,m}$.

Let $\bigcup_{m=0}^{\infty} A_{n-1,m} = A_{n,0}$, and $\lim_{m\to\infty} g_{n-1,m} = g_{n,0}$. Suppose $\mathcal{L}_1$ is a vertical line for which $w(\mathcal{L}_1) = \mathcal{L}_1$, where all occurrences of $g_{n-1,m}$ and $g_{n-1,m}^{-1}$ have been replaced by $g_{n,0}$ and $g_{n,0}^{-1}$, respectively. We need to show that $\mathcal{L}_1$ intersects $A_{n,0}$. Let

$$f_1(\mathcal{L}_1) = \mathcal{L}_2, f_2(\mathcal{L}_2) = \mathcal{L}_3, \dots, f_p(\mathcal{L}_p) = \mathcal{L}_{p+1} = \mathcal{L}_1.$$

Suppose there is a positive integer, i, where $1 \le i \le p$, $f_i \in \{g_{n,0}, g_{n,0}^{-1}\}$, and either $f_i = g_{n,0}$ and $g_{n,0}$ has been determined on $\mathcal{L}_i$, or $f_i = g_{n,0}^{-1}$, and $g_{n,0}$ has been determined on $\mathcal{L}_{i+1}$ (for any vertical line, $\mathcal{L}, g_{n,0}$ has been determined on $\mathcal{L}$ provided that there is a positive integer, m, such that $g_{n-1,m}$ has been determined on $\mathcal{L}$). Since $w(\mathcal{L}_1) = \mathcal{L}_1$, we see, by the results of the previous paragraph, that if j is any positive integer, $1 \le j \le p$, where $f_j \in \{g_{n,0}, g_{n,0}^{-1}\}$, then either $f_j = g_{n,0}$ and $g_{n,0}$ has been determined on $\mathcal{L}_j$, or $f_j = g_{n,0}^{-1}$, and $g_{n,0}$ has been determined on $\mathcal{L}_{j+1}$. Therefore, for each positive integer, j, where $1 \le j \le p$ and $f_j \in \{g_{n,0}, g_{n,0}^{-1}\}$, we have that $\mathcal{L}_j$ and $\mathcal{L}_{j+1}$ intersect $A_{n,0}$. Since every element of G sends $A_{n,0}$ onto itself, it follows that

$\mathcal{L}_1$ intersects $A_{n,0}$. Suppose, now, that there is no positive integer, i, where $1 \leq i \leq p$, $f_i \in \{g_{n,0}, g_{n,0}^{-1}\}$, and either $f_i = g_{n,0}$ and $g_{n,0}$ has been determined on $\mathcal{L}_i$, or $f_i = g_{n,0}^{-1}$ and $g_{n,0}$ has been determined on $\mathcal{L}_{i+1}$. Let m be a positive integer. The collection of vertical lines on which $g_{n-1,m}$ has been determined is a subcollection of the collection of vertical lines on which $g_{n,0}$ has been determined. There are four conditions, given above, such that if a vertical line satisfies any one of those four conditions, then $g_{n-1,m}$ is determined on that vertical line. The collection of vertical lines which satisfy any of the first three conditions does not change, no matter what positive integer m is taken to be. Since there are only finitely many sw's, then these vertical lines form a closed subset of $\mathbb{R}^2$. Since $g_{n,0}$ has been determined on these vertical lines, then there must be a positive number, x, such that for all sufficiently large positive integers, k: if $\mathcal{L}'_1$ is a vertical line whose distance from $\mathcal{L}_1$ is less than x; and if $f_1(\mathcal{L}'_1) = \mathcal{L}'_2$, $f_2(\mathcal{L}'_2) = \mathcal{L}'_3, \ldots, f_p(\mathcal{L}'_p) = \mathcal{L}'_{p+1}$, where all occurrences of $g_{n,0}$ and $g_{n,0}^{-1}$ have been replaced by $g_{n-1,k}$ and $g_{n-1,k}^{-1}$; then there is no positive integer, i, where $1 \leq i \leq p$, $f_i \in \{g_{n-1,k}, g_{n-1,k}^{-1}\}$, and either $f_i = g_{n-1,k}$, and $g_{n-1,k}$ has been determined on $\mathcal{L}'_i$ because $\mathcal{L}'_i$ satisfies one of the first three conditions; or $f_i = g_{n-1,k}^{-1}$, and $g_{n-1,k}$ has been determined on $\mathcal{L}'_{i+1}$ because $\mathcal{L}'_{i+1}$ satisfies one of the first three conditions. We are going to assume that the I_m's form a dense subset of the x-axis. Then, there must be an I_m to the left of $\mathcal{L}_1$ and an I_m to the right of $\mathcal{L}_1$ such that both I_m's have their distances from $\mathcal{L}_1$ being less than x. Every element of the x-axis, which lies between these two I_m's, satisfies the next to the last condition for choosing I_m's, for sufficiently large positive integers, m. Therefore, if neither of these two I_m's intersects $\mathcal{L}_1$, then the last condition for choosing I_m's implies that a succeeding I_m will extend from one I_m to the other. We see, then, that there is an I_m which intersects $\mathcal{L}_1$. Let k be the positive integer for which I_k intersects $\mathcal{L}_1$. If $w(\mathcal{L}_1) = \mathcal{L}_1$, where the letters of w come from $G \cup \{g_{n-1,k}, g_{n-1,k}^{-1}\}$; then $\mathcal{L}_1$ intersects $A_{n-1,k}$, and $\mathcal{L}_1$ intersects $A_{n,0}$ since $A_{n-1,k} \subseteq A_{n,0}$. If $w(\mathcal{L}_1) \neq \mathcal{L}_1$, where the letters of w come from $G \cup \{g_{n-1,k}, g_{n-1,k}^{-1}\}$, then, through our choices of δ''s, $w(\mathcal{L}_1) \neq \mathcal{L}_1$, where the letters of w come from $G \cup \{g_{n,0}, g_{n,0}^{-1}\}$. This contradicts the hypothesis on $\mathcal{L}_1$ made at the beginning of this paragraph. So, we have shown what we needed to show.

Let us now make appropriate selections of an element of B, an element of C, and a pair of elements from $A_{n,0}$. Next, let us make small changes to $g_{n,0}$ and $g_{n,0}^{-1}$ so that $g_{n,0}$ now sends the element of B into C and its element of $A_{n,0}$ into $A_{n,0}$; and $g_{n,0}^{-1}$ now sends the element of C into B and its element of $A_{n,0}$ into $A_{n,0}$.

Let $A_\alpha = \cup_{n=0}^{\infty} A_{n,0}$, $g_\alpha = \lim_{n\to\infty} g_{n,0}$, and G_α be the group generated by $G \cup \{g_\alpha\}$. We need to show that A_α and G_α enjoy the same properties given for A and G. For each positive integer, n, we assume that $A_{n-1,0}$ is a countable dense subset of $\mathbb{R}^2$ which intersects no vertical line more than once, that $A \subseteq A_{n-1,0}$, and that the $A_{n-1,0}$'s are nested. Then, A_α has these properties, as well. By its definition, G_α is a group under composition. Since G_α is generated by $G \cup \{g_\alpha\}$, then G_α is countable. For each positive integer, n, we assume that $g_{n-1,0}$ is an autohomeomorphism of $\mathbb{R}^2$ which sends vertical lines onto vertical lines so that $g_{n-1,0}$ is order-preserving in the x-axis and y-axis directions. Then, g_α has these properties, as well. By hypothesis, every element $g \in G$ is an autohomeomorphism of $\mathbb{R}^2$ which sends vertical lines onto vertical lines so that g is order-preserving in the x-axis and y-axis directions. All elements of G, and g_α, have these properties; therefore every element of G_α has these properties. We assume that every element of G sends A_α onto itself. For every positive integer, $n, g_{n,0}$ is altered so that it sends an appropriately chosen element of $A_{n,0}$ into $A_{n,0}$, and the same thing is done for $g_{n,0}^{-1}$ and the element of $A_{n,0}$ chosen for it. Furthermore, successors of $g_{n,0}$ and $g_{n,0}^{-1}$ maintain the results of the previous sentence for those elements of $A_{n,0}$. Since $A_\alpha = \cup_{n=1}^{\infty} A_{n-1,0}$, we assume that every element of A_α is chosen at some point along the way and treated as required. We conclude, then, that g_α sends A_α onto itself. The same type of argument involving B and C shows that g_α sends B onto C. Suppose $z = h_k \circ \ldots \circ h_2 \circ h_1$, k a positive integer, is a word whose letters come from $G \cup \{g_\alpha, g_\alpha^{-1}\}$, and $\mathcal{L}_1$ is a vertical line for which $z(\mathcal{L}_1) = \mathcal{L}_1$. We need to show that $\mathcal{L}_1$ intersects A_α. We assume that there is a positive integer, j, such that z has the same relationship with j as w has with n. So, if $\mathcal{L}$ is any vertical line in $\mathbb{R}^2$ such that $z(\mathcal{L}) = \mathcal{L}$, with the letters for z coming from $G \cup \{g_{j,0}, g_{j,0}^{-1}\}$, then $\mathcal{L}$ intersects $A_{j,0}$. Therefore, if $z(\mathcal{L}_1) = \mathcal{L}_1$, where the letters of z are from $G \cup \{g_{j,0}, g_{j,0}^{-1}\}$; then $\mathcal{L}_1$ intersects $A_{j,0}$, and, since $A_{j,0} \subseteq A_\alpha$, $\mathcal{L}_1$ intersects A_α. If, however, $z(\mathcal{L}_1) \neq \mathcal{L}_1$, where the letters are from $G \cup \{g_{j,0}, g_{j,0}^{-1}\}$; then, since the choice of δ''s does not let this condition change, $z(\mathcal{L}_1) \neq \mathcal{L}_1$ with the letters coming from $G \cup \{g_\alpha, g_\alpha^{-1}\}$. This is a contradiction to the hypothesis on $\mathcal{L}_1$. It follows that $\mathcal{L}_1$ intersects A_α. We have shown that A_α and G_α enjoy the same properties given for A and G, and that g_α sends B onto C. Let S be an appropriately chosen plane separator which does not intersect A_α; let $\mathcal{L}$ be a vertical line which does not intersect A_α, but which does intersect S; and let x be in the intersection S and $\mathcal{L}$. Put x, along with its images under the elements of G_α, into A_α. We now have that A_α and G_α enjoy the properties given for A and G, that g_α sends B onto C, and that A_α intersects S.

Let $D = \cup_{\alpha<\omega_1} A_\alpha$, and $J = \cup_{\alpha<\omega_1} G_\alpha$. For each $a < \omega_1$, we assume that A_α intersects no vertical line more than once; and if β is any ordinal number such that $\beta < \alpha$, we assume that $A_\beta \subseteq A_\alpha$ (i.e. the A_α's are nested). Since D is the union of these sets, it follows that D intersects no vertical line more than once. Since the A_α's are dense in $\mathbb{R}^2$, then D is, too. For every plane separator, S, we assume there is an A_α which intersects S. Therefore, D intersects S, and it follows that D is a connected subset of $\mathbb{R}^2$. Let B' and C' be any pair of countable dense subsets of D. We assume that there is a $\beta < \omega_1$ such that $g_\beta \in G_\beta$; that g_β sends B' onto C'; that every element of G_α sends A_α onto itself where α is any ordinal number with $\alpha < \omega_1$; and that the G_α's are nested. So, for each ordinal number, α, where $\beta \leq \alpha < \omega_1$; g_β sends A_α onto itself. From this, we have that g_β sends D onto itself. So, D is a CDH subset of $\mathbb{R}^2$ which is dense in $\mathbb{R}^2$, connected, and intersects vertical lines only once; which is what we needed to show.

We recall that $w = f_p \circ \ldots \circ f_2 \circ f_1$, and that the case we have considered above is the case where $\{f_p, f_1\} \neq \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$. Let us now consider the other case. Therefore, $\{f_p, f_1\} = \{g_{n-1,m-1}, g_{n-1,m-1}^{-1}\}$ is now the case. We will now show that w sends no more than countably many vertical lines onto themselves. Suppose w has only three letters. Then, it must be true that $f_2 \in G$. Suppose $\mathcal{L}_1$ is a vertical line which w sends onto itself, and let $f_1(\mathcal{L}_1) = \mathcal{L}_2, f_2(\mathcal{L}_2) = \mathcal{L}_3$ and $f_3(\mathcal{L}_3) = \mathcal{L}_4 = \mathcal{L}_1$. Since f_1 and f_3 are inverses, and $\mathcal{L}_1 = \mathcal{L}_4$, it follows that $\mathcal{L}_2 = \mathcal{L}_3$. Therefore, f_2 sends $\mathcal{L}_2$ onto itself. Since $f_2(\mathcal{L}_2) = \mathcal{L}_2$, $f_2 \in G$, and f_2 is not the identity, then there are no more than countably many vertical lines which could equal $\mathcal{L}_2$. It follows that no more than countably many vertical lines could equal $\mathcal{L}_1$. Suppose w has more than three letters, and suppose $\mathcal{L}_1$ is a vertical line which w sends onto itself. Let $f_1(\mathcal{L}_1) = \mathcal{L}_2$, $f_2(\mathcal{L}_2) = \mathcal{L}_3, \ldots, f_p(\mathcal{L}_p) = \mathcal{L}_{p+1} = \mathcal{L}_1$. Since f_1 and f_p are inverses, and $\mathcal{L}_1 = \mathcal{L}_{p+1}$, it follows that $\mathcal{L}_2 = \mathcal{L}_p$; and since $f_{p-1} \circ \ldots \circ f_3 \circ f_2$ has more than one letter, then it is one of the sw's. We have now that $f_{p-1} \circ \ldots \circ f_3 \circ f_2$ sends $\mathcal{L}_2$ onto itself, and $f_{p-1} \circ \ldots \circ f_3 \circ f_2$ is an sw. No sw sends more than countably many vertical lines onto themselves, and so there are only countably many vertical lines which could equal $\mathcal{L}_2$. It follows that no more than countably many vertical lines can equal $\mathcal{L}_1$. So, we have shown that w sends no more than countably many vertical lines onto themselves. If w sends no vertical line onto itself, then nothing needs to be done. Suppose there are vertical lines which w sends onto themselves. Pick one such vertical line and call it $\mathcal{L}_1$. Let $f_1(\mathcal{L}_1) = \mathcal{L}_2$, $f_2(\mathcal{L}_2) = \mathcal{L}_3, \ldots, f_p(\mathcal{L}_p) = \mathcal{L}_{p+1}$, and let the letters for w come from $G \cup \{g_{n-1,0}, g_{n-1,0}^{-1}\}$. If $f_1 = g_{n-1,0}$ and

$g_{n-1,0}$ is determined on $\mathcal{L}_1$, or if $f_1 = g_{n-1,0}^{-1}$ and $g_{n-1,0}$ is determined on $\mathcal{L}_2$; then $\mathcal{L}_1$ intersects $A_{n-1,0}$, and no changes need to be considered. If $f_1 = g_{n-1,0}$ and $g_{n-1,0}$ is not determined on $\mathcal{L}_1$, or if $f_1 = g_{n-1,0}^{-1}$ and $g_{n-1,0}$ is not determined on $\mathcal{L}_2$; then make a small alteration of $g_{n-1,0}$ so that $f_1(\mathcal{L}_1') = \mathcal{L}_2$. If $\mathcal{L}_2$ intersects either B or C, then it may be necessary to require that $\mathcal{L}_1'$ intersect either B or C, and that f_1 take the appropriate element of $\mathcal{L}_1'$ to the appropriate element of $\mathcal{L}_2$. If this is not necessary, then let $\mathcal{L}_1'$ be chosen so that it does not intersect $A_{n-1,0}$. We recall, from above, that if w has only three letters, then $f_2 \in G$, f_2 is not the identity, and f_2 sends $\mathcal{L}_2$ onto itself; and if w has more than three letters, then $f_{p-1} \circ \ldots \circ f_3 \circ f_2$ sends $\mathcal{L}_2$ onto itself, and $f_{p-1} \circ \ldots \circ f_3 \circ f_2$ is an sw. We assume that if some sw sends a vertical line onto itself, then that vertical line intersects $A_{n-1,0}$. If an element of G, different from the identity, sends a vertical line onto itself, then that vertical line intersects A, and intersects $A_{n-1,0}$, since $A \subseteq A_{n-1,0}$. In our present situation, we have, then, that $\mathcal{L}_2$ intersects $A_{n-1,0}$. Let x be the element of $A_{n-1,0}$ on $\mathcal{L}_2$, and let the altered version of $g_{n-1,0}$ be called $g_{n-1,1}$. Now, we put $f_p(x)$, which lies on $\mathcal{L}_1'$, and its images under the elements of G, into $A_{n-1,0}$. Let this enlarged version of $A_{n-1,0}$ be called $A_{n-1,1}$. If there is another vertical line which w sends onto itself, then we repeat the above process, letting $A_{n-1,1}$ take the place of $A_{n-1,0}$, and not undoing the previous process. When we are done, $A_{n,0}$ and $g_{n,0}$ will satisfy what they did in the first case; and so the construction can proceed as it did before.

We will now proceed with the second (more technical) proof of the theorem.

Let S be the set of all plane separators (a plane separator is a closed subset of the plane which separates the plane) and D the collection of all countable dense subsets of $\mathbb{R}^2$. Let S and $D \times D$ be minimally well-ordered by ω_1. The construction will be by transfinite induction and so we will now state the transfinite induction hypothesis (TIH) : Let $\alpha < \omega_1$ be an ordinal number. Then for every ordinal number β such that

$\beta < \alpha$ there are sets A_β and G_β satisfying

(1) A_β is a countable subset of $\mathbb{R}^2$,
(2) A_β intersects no vertical line more than once,
(3) A_β and the β-th element of S have a nonempty intersection,
(4) if γ is an ordinal number such that $\gamma < \beta$ then $A_\gamma \subseteq A_\beta$,
(5) G_β is a collection of autohomeomorphisms of $\mathbb{R}^2$,
(6) G_β is a group under composition,
(7) every element of G_β sends all vertical lines onto vertical lines and A_β onto A_β,
(8) if g is any element of G_β except the identity and g sends a vertical line onto itself, then that vertical line and A_β have a nonempty intersection,
(9) if γ is an ordinal number such that $\gamma < \beta$ then $G_\gamma \subseteq G_\beta$,
(10) if $\cup_{\theta<\beta} A_\theta$ is dense in $\mathbb{R}^2$ and (B, C) is the first element of $D \times D$ such that $B \cup C \subseteq \cup_{\theta<\beta} A_\theta, \cup_{\theta<\beta} A_\theta \backslash (B \cup C)$ is dense in $\mathbb{R}^2$, and no element of $\cup_{\theta<\beta} G_\theta$ sends B onto C, then there is an element $g \in G_\beta$ such that $g(B) = C$,
(11) if g is any element of $G_\beta, \mathcal{L}_1$ and $\mathcal{L}_2$ are any pair of vertical lines in $\mathbb{R}^2$ such that $g(\mathcal{L}_1) = \mathcal{L}_2$, the first coordinate of every element of $\mathcal{L}_1$ is a and $\mathcal{L}_2$ is b, then the homeomorphism $\overline{g}$ of $\mathbb{R}$ onto itself defined by $\overline{g}(a) = b$ is order-preserving (i.e. g is order-preserving in the x-axis direction), and
(12) if g is any element of $G_\beta, \mathcal{L}_1$ and $\mathcal{L}_2$ are any pair of vertical lines in $\mathbb{R}^2$ such that $g(\mathcal{L}_1) = \mathcal{L}_2$, y_1 is the second coordinate of an element p_1 of $\mathcal{L}_1$, and y_2 is the second coordinate of an element p_2 of $\mathcal{L}_2$ such that $g(p_1) = p_2$, then the homeomorphism f of $\mathbb{R}$ onto itself defined by $f(y_1) = y_2$ is order-preserving (i.e. g is order-preserving in the y-axis direction).

The TIH is vacuously true for $\alpha = 0$.

Now assume that $0 \leq \alpha < \omega_1$, α is either a limit or successor ordinal and the TIH is true for α. To complete the transfinite induction, we must prove that there exist sets A_α

and G_α satisfying conditions 1 through 12 of the TIH. For $\alpha > 0$, the construction of A_α and G_α will be by induction (actually, two inductions) and so we will state the first induction hypothesis ($IH1$) after making some remarks. The sets A_0 and G_0 result from knowledge we already have and so we won't use induction here to obtain them. When we discuss A_0 and G_0 later on, A_0 will be selected to be dense in $\mathbb{R}^2$. In stating $IH1$, then, we assume $\alpha > 0$ and $\cup_{\theta<\alpha}A_\theta$ is dense in $\mathbb{R}^2$ and we let (B, C) be the first element of $D \times D$ such that $B \cup C \subseteq \cup_{\theta<\alpha}A_\theta$, $\cup_{\theta<\alpha}A_\theta \backslash (B \cup C)$ is dense in $\mathbb{R}^2$ and no element of $\cup_{\theta<\alpha}G_\theta$ sends B onto C. Let 1 and -1 denote autohomeomorphisms of $\mathbb{R}^2$ such that 1 sends the y-axis onto the line $y = 1$ and -1 sends the y-axis onto the line $y = -1$. Let W be the set of all words obtainable from $\cup_{\theta<\alpha}G_\theta \cup \{1, -1\}$ such that if $w \in W$ and w is spelled with only one letter then $w = 1$ or $w = -1$, the identity is not in the spelling of w, and if f and g are two consecutive letters in the spelling of w, then either $f = g = 1$, $f = g = -1$, or $\{f, g\}$ has a nonempty intersection with both $\cup_{\theta<\alpha}G_\theta$ and $\{1, -1\}$. Let W be minimally well-ordered by ω_0 such that if $w \in W$ and $x \in W$ is a proper sub-word of w, then x precedes w. Suppose $f_1, f_2, \ldots, f_k$ are each functions from $\mathbb{R}$ onto $\mathbb{R}$ and k is some positive integer. Let u be the word $u = f_k \circ \ldots \circ f_2 \circ f_1$, x_0 any real number, $x_1 = f_1(x_0)$, $x_2 = f_2(x_1), \ldots$, and $x_k = f_k(x_{k-1})$. Then we say that (x_0, x_k) on u is related to (x_0, x_1) by f_1, to (x_1, x_2) by $f_2, \ldots$, and to (x_{k-1}, x_k) by f_k. Let B and C each be minimally well-ordered by ω_0.

We will now state $IH1$. Let $n < \omega_0$ be a nonnegative integer. Then for every nonnegative integer k such that $k < n$ there are a set A'_k and functions f_k and h_k satisfying

(1) $\cup_{\theta<\alpha}A_\theta \subseteq A'_k$,
(2) A'_k is a countable subset of $\mathbb{R}^2$ which intersects no vertical line more than once,
(3) h_k is a function from ω_0 onto A'_k,
(4) if j is a nonnegative integer such that $j < k$ then $A'_j \subseteq A'_k$,

(5) f_k is a homeomorphism from $\mathbb{R}^2$ onto itself which sends vertical lines onto vertical lines,

(6) f_k is order-preserving in the x-axis and y-axis directions,

(7) if $j \le k$ and j is a nonnegative integer then $f_k(\{h_j(0), h_j(1), \dots, h_j(k)\})$ and $f_k^{-1}(\{h_j(0), h_j(1), \dots, h_j(k)\})$ are both subsets of A'_k,

(8) if i and j are nonnegative integers such that $i \le j < k$ then f_k and f_j agree at each element of $\{h_i(0), h_i(1), \dots, h_i(j)\}$ and f_k^{-1} and f_j^{-1} also agree at each element of $\{h_i(0), h_i(1), \dots, h_i(j)\}$,

(9) f_k sends the first $k+1$ elements of B into C and f_k^{-1} sends the first $k+1$ elements of C into B,

(10) if j is a nonnegative integer such that $j < k$ then f_k and f_j agree at each of the first $j+1$ elements of B and f_k^{-1} and f_j^{-1} agree at each of the first $j+1$ elements of C,

(11) $f_k \notin \cup_{\theta<\alpha} G_\theta$,

(12) if $g \in \cup_{\theta<\alpha} G_\theta$ then g sends A'_k onto itself,

(13) if j is a nonnegative integer such that $j \le k, w = v_t \circ \dots \circ v_2 \circ v_1$ is the $(j+1)$-th element of W and t is a positive integer, $w' = v'_t \circ \dots \circ v'_2 \circ v'_1$ is obtained from w by replacing each occurrence of 1 by f_j and of -1 by $f_j^{-1}, w'' = v''_t \circ \dots \circ v''_2 \circ v''_1$ is obtained from w by replacing each occurrence of 1 by f_k and of -1 by f_k^{-1}, x is a real number, $\overline{w'}(x) = (\overline{v'_t} \circ \dots \circ \overline{v'_2} \circ \overline{v'_1})(x)$ and $\overline{w''}(x) = (\overline{v''_t} \circ \dots \circ \overline{v''_2} \circ \overline{v''_1})(x)$, and $\overline{w'}(x) \ne x$, then $| \overline{w'}(x) - \overline{w''}(x) | < s_{k+1} | \overline{w'}(x) - x |$ where $s_0, s_1, s_2, \dots$ is an increasing sequence of positive real numbers which converges to $1/2$,

(14) if j is a nonnegative integer such that $j \le k, w = v_t \circ \dots \circ v_2 \circ v_1$ is the $(j+1)$-th element of W and t is a positive integer, $w' = v'_t \circ \dots \circ v'_2 \circ v'_1$ is obtained from w by replacing each occurrence of 1 by f_j and of -1 by f_j^{-1}, x is a real number such that $\overline{w'}(x) = x$, a and b are real numbers such that $\overline{f_j}(a) = b$, and (x,x) on $\overline{w'}$ is either related to (a,b) by $\overline{f_j}$ or to (b,a) by $\overline{f_j^{-1}}$, then the lines $x = a$ and $x = b$ each contain an element of

A'_j so that if the line $x = a$ intersects B, then the line $x = b$ intersects C, and both f_k and f_j send the element of A'_j on the line $x = a$ to the element of A'_j on the line $x = b$,

(15) if x is a real number then $\overline{f_k}(x) \neq x$, and

(16) if $y \in \mathbb{R}^2$ such that $|y| < k$, then $| f_{k-1}(y) - f_k(y) |< \frac{1}{2^k}$ and $| f_{k-1}^{-1}(y) - f_k^{-1}(y) |< \frac{1}{2^k}$.

The $IH1$ is vacuously true for $n = 0$.

Now assume that $0 \leq n < \omega_0$ and $IH1$ is true for n. To complete the induction, we must prove that there are a set A'_n and functions f_n and h_n satisfying conditions 1 through 16 of $IH1$. The construction of A'_n, f_n, and h_n will be by induction and so we will state the second induction hypothesis ($IH2$). Let $m < \omega_0$ be a nonnegative integer and the rationals be minimally well-ordered by ω_0. Then for every nonnegative integer k such that $k < m$ there are a set A''_k and a function f'_k satisfying

(1) if j is a nonnegative integer such that $j < n$ then $A'_j \subseteq A''_k$,

(2) $\cup_{\theta<\alpha} A_\theta \subseteq A''_k$,

(3) if j is a nonnegative integer such that $j < k$ then $A''_j \subseteq A''_k$,

(4) A''_k is a countable subset of $\mathbb{R}^2$ which intersects no vertical line more than once,

(5) f'_k is a homeomorphism from $\mathbb{R}^2$ onto itself which sends vertical lines onto vertical lines,

(6) f'_k is order-preserving in the x-axis and y-axis directions,

(7) if i and j are nonnegative integers such that $i \leq j < n$ then f'_k agrees with f_j at each element of $\{h_i(0), h_i(1), \dots, h_i(j)\}$ and f'^{-1}_k and f_j^{-1} also agree at each element of $\{h_i(0), h_i(1), \dots, h_i(j)\}$,

(8) if j is a nonnegative integer such that $j < n$ then f_j and f'_k agree at each of the first $j+1$ elements of B and f_j^{-1} and f'^{-1}_k agree at each of the first $j+1$ elements of C,

(9) $f'_k \notin \cup_{\theta<\alpha} G_\theta$,

(10) if $g \in \cup_{\theta<\alpha} G_\theta$ then g sends A''_k onto itself,

(11) if j is a nonnegative integer such that $j < n, w = v_t \circ \ldots \circ v_2 \circ v_1$ is the $(j+1)$-th element of W and t is a positive integer, $w' = v'_t \circ \ldots \circ v'_2 \circ v'_1$ is obtained from w by replacing each occurrence of 1 by f_j and of -1 by f_j^{-1}, $w'' = v''_t \circ \ldots \circ v''_2 \circ v''_1$ is obtained from w by replacing each occurrence of 1 by f'_k and of -1 by $f_k'^{-1}$, and x is a real number such that $\overline{w'}(x) \neq x$, then $| \overline{w'}(x) - \overline{w''}(x) | < s_n | \overline{w'}(x) - x |$,

(12) if j is a nonnegative integer such that $j < n$, $w = v_t \circ \ldots \circ v_2 \circ v_1$ is the $(j+1)$-th element of W and t is a positive integer, $w' = v'_t \circ \ldots \circ v'_2 \circ v'_1$ is obtained from w by replacing each occurrence of 1 by f_j and of -1 by f_j^{-1}, x is a real number such that $\overline{w'}(x) = x$, a and b are real numbers such that $\overline{f_j}(a) = b$, and (x, x) on $\overline{w'}$ is either related to (a, b) by $\overline{f_j}$ or to (b, a) by $\overline{f_j^{-1}}$, then the lines $x = a$ and $x = b$ each contain an element of A'_j so that if the line $x = a$ intersects B, then the line $x = b$ intersects C, and both f'_k and f_j send the element of A'_j on the line $x = a$ to the element of A'_j on the line $x = b$,

(13) if x is a real number then $\overline{f'_k}(x) \neq x$,

(14) if $y \in \mathbb{R}^2$ such that $|y| < n$ then $| f_{n-1}(y) - f'_k(y) | < \frac{1}{2^{n+1}}$ and $| f_{n-1}^{-1}(y) - f_k'^{-1}(y) | < \frac{1}{2^{n+1}}$,

(15) if $y \in \mathbb{R}^2$ such that $|y| < k$ then $| f'_{k-1}(y) - f'_k(y) | < \frac{1}{2^k}$ and $| f_{k-1}'^{-1}(y) - f_k'^{-1}(y) | < \frac{1}{2^k}$,

(16) if j is a nonnegative integer such that $j \leq k$, $z = u_p \circ \ldots \circ u_2 \circ u_1$ is the $(n+1)$-th element of W and p is a positive integer, and $z' = u'_p \circ \ldots \circ u'_2 \circ u'_1$ is obtained from z by replacing each occurrence of 1 by f'_j and of -1 by $f_j'^{-1}$, then there is a nondegenerate, compact interval I_j of real numbers such that $I_0, I_1, \ldots, I_j$ are non-overlapping for $j > 0$, $x \in I_0 \cup I_1 \cup \ldots \cup I_j$ and $\overline{z'}(x) = x$ are both true for no more than finitely many real numbers x, if y is a real number such that y is an

endpoint of either $I_0, I_1, \ldots,$ or I_j then $\overline{z'}(y) \neq y$, if x is a real number such that $x \in I_0 \cup I_1 \cup \ldots \cup I_j$ and $\overline{z'}(x) \neq x$ and if $z'' = u''_p \circ \ldots \circ u''_2 \circ u''_1$ is obtained from z by replacing each occurrence of 1 by f'_k and of -1 by f'^{-1}_k then $| \overline{z'}(x) - \overline{z''}(x) | < \frac{1}{2} | \overline{z'}(x) - x |$, and if $x, a,$ and b are real numbers such that $x \in I_0 \cup I_1 \cup \ldots \cup I_j$, $\overline{z'}(x) = x$, $\overline{f'_j}(a) = b$, and (x,x) on $\overline{z'}$ is either related to (a,b) by $\overline{f'_j}$ or to (b,a) by $\overline{f'^{-1}_j}$, then the lines $x = a$ and $x = b$ each contain an element of A''_j so that if the line $x = a$ intersects B, then the line $x = b$ intersects C, and both f'_k and f'_j send the element of A''_j on the line $x = a$ to the element of A''_j on the line $x = b$,

(17) let $M_1 = \{(a,b) \in \mathbb{R}^2 :$ there exist nonnegative integers i and j for which $i \leq j < n$, (a,b) is on the graph of $\overline{f_j}$, and an element of $\{h_i(0), h_i(1), \ldots, h_i(j)\}$ either lies on the line $x = a$ or on the line $x = b\}$, let $M_2 = \{(a,b) \in \mathbb{R}^2 :$ there exists a nonnegative integer j for which $j < n$, (a,b) is on the graph of $\overline{f_j}$, and one of the first $j+1$ elements of B lies on the line $x = a$ or one of the first $j+1$ elements of C lies on the line $x = b\}$, let $M_3 = \{(a,b) \in \mathbb{R}^2 :$ there exist a nonnegative integer j and a real number x for which $j < n$, (a,b) is on the graph of $\overline{f_j}$, $w = v_t \circ \ldots \circ v_2 \circ v_1$ is the $(j+1)$-th element of W and t is a positive integer, $w' = v'_t \circ \ldots \circ v'_2 \circ v'_1$ is obtained from w by replacing each occurrence of 1 by f_j and of -1 by f_j^{-1}, $\overline{w'}(x) = x$, and (x,x) on $\overline{w'}$ is either related to (a,b) by $\overline{f_j}$, or to (b,a) by $\overline{f_j^{-1}}\}$, and let $M = M_1 \cup M_2 \cup M_3$; then every element of M is on the graph of $\overline{f'_k}$, and if $y \in \mathbb{R}$ such that $y \in I_0 \cup I_1 \cup \ldots \cup I_k$, $z = u_p \circ \ldots \circ u_2 \circ u_1$ is the $(n+1)$-th element of W and p is a positive integer, $z' = u'_p \circ \ldots \circ u'_2 \circ u'_1$ is obtained from z by replacing each occurrence of 1 by f'_k and of -1 by f'^{-1}_k, and a and b are real numbers such that $(a,b) \in M$, then $(y, \overline{z'}(y))$ on $\overline{z'}$ is not related to (a,b) by $\overline{f'_k}$ nor to (b,a) by $\overline{f'^{-1}_k}$,

(18) let M be minimally well-ordered by ω_0; if j is a non-negative integer such that $j \leq k$, a and b are real numbers such that (a,b) is the $(j+1)$-th element of M, $z = u_p \circ \ldots \circ u_2 \circ u_1$ is the $(n+1)$-th element of W and p is a positive integer, $z' = u'_p \circ \ldots \circ u'_2 \circ u'_1$ is obtained from z by replacing each occurrence of 1 by f'_j and of -1 by $f'^{-1}_j, z'' = u''_p \circ \ldots \circ u''_2 \circ u''_1$ is obtained from z by replacing each occurrence of 1 by f'_k and of -1 by f'^{-1}_k and if $x_0 \in \mathbb{R}$ such that $x_1 = \overline{u'_1}(x_0)$, $x_2 = \overline{u'_2}(x_1), \ldots$, $x_q = \overline{u'_q}(x_{q-1}), \ldots$, and $x_p = \overline{u'_p}(x_{p-1})$ where $1 \leq q \leq p$, q is a positive integer, $u'_q = f'_j$, $x_{q-1} = a$ and $x_q = b$ or $u'_q = f'^{-1}_j$, $x_{q-1} = b$ and $x_q = a$, then if c and d are real numbers such that (c,d) is on the graph of $\overline{f'_j}$ and if (x_0, x_p) on $\overline{z'}$ is either related to (c,d) by $\overline{f'_j}$ or to (d,c) by $\overline{f'^{-1}_j}$, we have $(c,d) \in M$ or else we have that $x_0 \neq x_p$ and if $y_0 \in \mathbb{R}$ such that $y_1 = \overline{u''_1}(y_0)$, $y_2 = \overline{u''_2}(y_1), \ldots$, $y_q = \overline{u''_q}(y_{q-1}), \ldots$, and $y_p = \overline{u''_p}(y_{p-1})$ where $u''_q = f'_k$, $y_{q-1} = a$ and $y_q = b$ or $u''_q = f'^{-1}_k$, $y_{q-1} = b$ and $y_q = a$, then $| y_0 - y_p | > \frac{1}{2} | x_0 - x_p |$,

(19) let $M'_k = \{(a,b) \in \mathbb{R}^2 : (a,b)$ is on the graph of $\overline{f'_k}$, and if $z = u_p \circ \ldots \circ u_2 \circ u_1$ is the $(n+1)$-th element of W and p is a positive integer, and $z' = u'_p \circ \ldots \circ u'_2 \circ u'_1$ is obtained from z by replacing each occurrence of 1 by f'_k and of -1 by f'^{-1}_k, then there exists a real number x such that $x \in I_0 \cup I_1 \cup \ldots \cup I_k$, $\overline{z'}(x) = x$ and (x,x) on $\overline{z'}$ is either related to (a,b) by $\overline{f'_k}$ or to (b,a) by $\overline{f'^{-1}_k}\}$; if $y \in \mathbb{R}$ such that y is not in the interior of $I_0, I_1, \ldots$, or I_k, a and b are real numbers such that $(a,b) \in M'_k$, and $(y, \overline{z'}(y))$ on $\overline{z'}$ is either related to (a,b) by $\overline{f'_k}$ or to (b,a) by $\overline{f'^{-1}_k}$, then there is a pair of real numbers c and d such that (c,d) is on the graph of $\overline{f'_k}$, $(c,d) \notin M \cup M'_k$ and $(y, \overline{z'}(y))$ is either related to (c,d) by $\overline{f'_k}$ or to (d,c) by $\overline{f'^{-1}_k}$,

(20) if k is a positive integer, $I_0 = [a_0, b_0]$, $I_1 = [a_1, b_1], \ldots$,

and $I_{k-1} = [a_{k-1}, b_{k-1}]$, $z = u_p \circ \ldots \circ u_2 \circ u_1$ is the $(n+1)$-th element of W, $z' = u'_p \circ \ldots \circ u'_2 \circ u'_1$ is obtained from z by replacing each occurrence of 1 by f'_{k-1} and of -1 by f'^{-1}_{k-1}, r is the first rational number such that $r \notin I_0 \cup I_1 \cup \ldots \cup I_{k-1}$ and if c and d are real numbers such that $(c, d) \in M$ then $(r, \overline{z'}(r))$ is not related to (c, d) by $\overline{f'_{k-1}}$ nor to (d, c) by $\overline{f'^{-1}_{k-1}}$, there are an element of $\{b_0, b_1, \ldots, b_{k-1}\}$ less than r and an element of $\{a_0, a_1, \ldots, a_{k-1}\}$ greater than r, b denotes the largest element of $\{b_0, b_1, \ldots, b_{k-1}\}$ less than r and a denotes the smallest element of $\{a_0, a_1, \ldots, a_{k-1}\}$ greater than r, and if for every real number e such that $b \leq e \leq a$ and every pair of real numbers x and y such that $(x, y) \in M$ we have that $(e, \overline{z'}(e))$ is not related (x, y) by $\overline{f'_{k-1}}$ nor to (y, x) by $\overline{f'^{-1}_{k-1}}$, then $I_k = [b, a]$.

The $IH2$ is vacuously true for $m = 0$.

We will now establish the beginning step of each induction which takes place. There is only one induction process taking place which the TIH applies to directly. What we must do then is to describe sets A_0 and G_0 satisfying conditions 1 through 12 of the TIH. Let A_0 be a countable dense subset of the plane which contains a point of the first plane separator and intersects no vertical line more than once and select the first element of $D \times D$ which meets the requirements of condition 10 relative to A_0. To form G_0, let g_0 be a homeomorphism of $\mathbb{R}^2$ onto itself such that g_0 sends all vertical lines onto vertical lines, g_0 is order-preserving in the x-axis and y-axis directions, if x is any real number then $\overline{g_0}(x) \neq x$, and g_0 sends A_0 onto itself and the first coordinate of the selected element of $D \times D$ onto the second. Let G_0 be the group generated by g_0 under composition. Then G_0 and A_0 satisfy conditions 1 through 12 of the TIH. For every ordinal α, $0 < \alpha < \omega_1$, there is an induction process taking place which must satisfy the conditions of $IH1$. These inductions result in the construction of A_α and G_α, $0 < \alpha < \omega_1$. Therefore let α be an ordinal number such that $0 < \alpha < \omega_1$. For this α we must describe a set A'_0 and functions f_0 and h_0 satisfying conditions 1 through 16 of $IH1$. Let $A'_0 = \bigcup_{\theta<\alpha} A_\theta$, let h_0 be a function from ω_0 onto A'_0 and let f_0 be a homeomorphism from $\mathbb{R}^2$ onto itself which satisfies conditions 5,6,7,11, and 15 of $IH1$. Condition 12 of $IH1$ is satisfied due to conditions 7 and 9 of the TIH. Condition 14 of $IH1$ is satisfied since the 0-th element of W must be either 1 or -1 meaning that $w' = f_0$ or $w' = f_0^{-1}$. Because f_0 satisfies condition 15 of $IH1$, $w' = f_0$ or $w' = f_0^{-1}$ means that if x is any real number then $\overline{w'}(x) \neq x$ and so condition 14 of $IH1$ is satisfied vacuously. It is clear that the remaining conditions of $IH1$ are satisfied. For every ordinal $\alpha, 0 < \alpha < \omega_1$, and positive integer n, there is an induction process which must satisfy the conditions of $IH2$. These inductions result in the construction of the set A'_n and functions f_n and h_n, $0 < \alpha < \omega_1$, $0 < n < \omega_0$. Therefore let α be an ordinal number such that $0 < \alpha < \omega_1$ and n a positive integer. For this α and n we must describe a set A''_0 and a function f'_0 satisfying conditions 1 through 20 of $IH2$. If we let $A''_0 = A'_{n-1}$ and $f'_0 = f_{n-1}$ then conditions 1 through 15 of $IH2$ are satisfied due to the fact that A'_{n-1} and f_{n-1} satisfy conditions 1 through 16 of $IH1$. In order to satisfy conditions 16 through 20 of $IH2$ as well, A'_{n-1} must be expanded and f_{n-1} altered. This can be accomplished using the exact same procedure which will be used later to obtain A''_m and f'_m, m a positive integer. Therefore we assert that there do exist a set A''_0 and a function f'_0 satisfying conditions 1 through 20 of $IH2$.

Suppose α is an ordinal, $0 < \alpha < \omega_1$ such that the TIH is true for α, n is a positive integer such that $IH1$ is true for α and n, and m is a positive integer such that $IH2$ is true for α, n, and m. Let f'_{m-1} be the function whose existence is asserted at the beginning of $IH2$ with $k = m - 1$. Suppose $z = u_p \circ \ldots \circ u_2 \circ u_1$ is the $(n+1)$-th element of W, p is a positive integer, $z' = u'_p \circ \ldots \circ u'_2 \circ u'_1$ is obtained from z by replacing each occurrence of 1 by f'_{m-1} and of -1 by f'^{-1}_{m-1}, and $\{u'_p, u'_1\}$ does not contain both f'_{m-1} and f'^{-1}_{m-1} (later we shall consider the case where $\{u'_p, u'_1\}$ does contain both f'_{m-1} and f'^{-1}_{m-1}). Let $I_0, I_1, \dots, I_{m-1}$ be the intervals of real numbers whose existence is asserted by condition 16 of $IH2$, let M be the set described in condition 17 of $IH2$, let M'_{m-1} be the set described in condition 19 of $IH2$ with $k = m-1$, and let I_m be selected in accordance with condition 20 of $IH2$ so that I_m is a non-degenerate, compact interval of real numbers which does not overlap $I_0, I_1, \dots,$ or I_{m-1}, and if $y \in \mathbb{R}$ such that $y \in I_m$, and a and b are real numbers such that $(a,b) \in M$, then $(y, \overline{z'}(y))$ on $\overline{z'}$ is not related to (a,b) by $\overline{f'_{m-1}}$ nor to (b,a) by $\overline{f'^{-1}_{m-1}}$ (I_m may be chosen to meet this last requirement because conditions 7 and 8 of $IH1$ and condition 7 of $IH2$ imply that M_1 is a finite subset of the graph of $\overline{f'_{m-1}}$, conditions 9 and 10 of $IH1$ and condition 8 of $IH2$ imply that M_2 is a finite subset of the graph of $\overline{f'_{m-1}}$, and condition 14 of $IH1$ and condition 12 of $IH2$ imply that M_3 is a countable, closed subset of the graph of $\overline{f'_{m-1}}$,therefore $M = M_1 \cup M_2 \cup M_3$ is a countable, closed subset of the graph of $\overline{f'_{m-1}}$).

Suppose $x_0 \in I_m$, $\overline{u'_1}(x_0) = x_1$, $\overline{u'_2}(x_1) = x_2, \dots, \overline{u'_p}(x_{p-1}) = x_p$, a and b are real numbers such that $(x_0, \overline{z'}(x_0)) = (x_0, x_p)$ on $\overline{z'}$ is related to (a,b) by $\overline{f'_{m-1}}$, and (x_0, x_p) on $\overline{z'}$ is related to (b,a) by $\overline{f'^{-1}_{m-1}}$. We are under the assumption now that $\{u'_p, u'_1\}$ does not contain both f'_{m-1} and f'^{-1}_{m-1}. Let us assume for the sake of specificity that $1 \le r < q < p$, r and q are positive integers, $u'_r = f'_{m-1}$, $u'_q = f'^{-1}_{m-1}$, $x_{r-1} = a$, $x_r = b$, $x_{q-1} = b$, and $x_q = a$. The proper sub-word $\overline{u'_q} \circ \dots \circ \overline{u'_r}$ of $\overline{z'}$ is the identity at $x_{r-1} = a$, $\overline{u'_r}(a) = b$, and $u'_r = f'_{m-1}$ and so (a,a) on $\overline{u'_q} \circ \dots \circ \overline{u'_r}$ is related to (a,b) by $\overline{f'_{m-1}}$. Since $u_q \circ \dots \circ u_r$ is a proper sub-word of z and $u_q \circ \dots \circ u_r$ is an element of W then $u_q \circ \dots \circ u_r$ precedes z by the way W is ordered. The $(n+1)$-th element of W is z and so there is a nonnegative integer j such that $j < n$ and $u_q \circ \dots \circ u_r$ is the $(j+1)$-th element of W. The above facts together with conditions 11 and 12 of $IH2$ tell us that $(a,b) \in M_3 \subseteq M$. This contradicts the way I_m was chosen and so we conclude that it is not possible that (x_0, x_p)

on $\overline{z'}$ is related both to (a,b) by $\overline{f'_{m-1}}$ and to (b,a) by $\overline{f'^{-1}_{m-1}}$. Now, instead of what we supposed before, let us suppose that $1 \leq r < q \leq p$, $u'_r = f'_{m-1}$, $u'_q = f'_{m-1}$, $x_{r-1} = a$, $x_r = b$, $x_{q-1} = a$, and $x_q = b$. What we have then is that (x_0, x_p) on $\overline{z'}$ is related to (a,b) by $\overline{f'_{m-1}}$ twice. It must be the case that $r < q-1$, for otherwise we would have that $a = b$, which would contradict condition 13 of $IH2$. The proper sub-word $\overline{u'_{q-1}} \circ \ldots \circ \overline{u'_r}$ of $\overline{z'}$ is the identity at $x_{r-1} = a$, $\overline{u'_r}(a) = b$, and $u'_r = f'_{m-1}$ and so (a,a) on $\overline{u'_{q-1}} \circ \ldots \circ \overline{u'_r}$ is related to (a,b) by $\overline{f'_{m-1}}$. Since $u_{q-1} \circ \ldots \circ u_r$ is a proper sub-word of z and $u_{q-1} \circ \ldots \circ u_r$ is an element of W then we can conclude as before that $(a,b) \in M_3 \subseteq M$. We then conclude that it is not possible that (x_0, x_p) on $\overline{z'}$ is related to (a,b) by $\overline{f'_{m-1}}$ twice nor to (b,a) by $\overline{f'^{-1}_{m-1}}$ twice.

Because I_m is compact, we can expand on what we have just seen and say that there is a $\delta > 0$ such that if x and y are elements of I_m, $|x-y| \leq \delta$, and a and b are real numbers where (a,b) is on the graph of $\overline{f'_{m-1}}$, then it is not possible for both $(x, \overline{z'}(x))$ and $(y, \overline{z'}(y))$ on $\overline{z'}$ to be related to (a,b) by $\overline{f'_{m-1}}$, to (b,a) by $\overline{f'^{-1}_{m-1}}$, nor is it possible for one to be related to (a,b) by $\overline{f'_{m-1}}$ while the other is related to (b,a) by $\overline{f'^{-1}_{m-1}}$. Let K be a compact subset of $\mathbb{R}$ such that if a and b are real numbers with $(x, \overline{z'}(x))$ on $\overline{z'}$ either being related to (a,b) by $\overline{f'_{m-1}}$ or to (b,a) by $\overline{f'^{-1}_{m-1}}$ where $x \in I_m$, then a is in the interior of K, and if a and b are real numbers such that $(a,b) \in M$, then $a \notin K$. Let h be a homeomorphism of $\mathbb{R}^2$ onto itself which sends all vertical lines onto vertical lines such that if $x \notin K$, then $\overline{h}(x) = \overline{f'_{m-1}}(x)$. Then there is a $\delta' > 0$ such that if $| \overline{h}(c) - \overline{f'_{m-1}}(c) | < \delta'$ and $| \overline{h^{-1}}(c) - \overline{f'^{-1}_{m-1}}(c) | < \delta'$ for every $c \in K$, and if $z'' = u''_p \circ \ldots \circ u''_2 \circ u''_1$ is obtained from z by replacing each occurrence of 1 by h and of -1 by h^{-1}, and if x and y are elements of I_m with $|x-y| \leq \delta$, and if a and b are real numbers where (a,b) is on the graph of $\overline{h}$, then it is not possible for both $(x, \overline{z''}(x))$ and $(y, \overline{z''}(y))$ on $\overline{z''}$ to be related to (a,b) by $\overline{h}$, to (b,a) by $\overline{h^{-1}}$, nor is it possible for one to be related to (a, b) by $\overline{h}$ while the other is related to (b, a) by $\overline{h^{-1}}$. Since K is compact, $\overline{h}$ agrees with $\overline{f'_{m-1}}$ off of K, first coordinates of elements of M are not in K, the inequality of condition 11 of $IH2$ is strict, and there are only finitely many elements of W under consideration, then we may assume that $\delta' > 0$ is small enough so that if $\mid \overline{h}(c) - \overline{f'_{m-1}}(c) \mid < \delta'$ and $\mid \overline{h^{-1}}(c) - \overline{f'^{-1}_{m-1}}(c) \mid < \delta'$ for every $c \in K$ then h satisfies condition 11 of $IH2$ just as f'_{m-1} does. We will require any h's that we employ to have the property that if $y \in \mathbb{R}^2$ such that $|y| < n$ or $|y| < m$, then the second coordinates of $f_{m-1}(y)$ and $h(y)$ are equal and the second coordinates of $f'^{-1}_{m-1}(y)$ and $h^{-1}(y)$ are equal. Therefore we may assume that $\delta' > 0$ is small enough so that h satisfies condition 14 of $IH2$ just as f'_{m-1} does provided that $\mid \overline{h}(c) - \overline{f'_{m-1}}(c) \mid < \delta'$ and $\mid \overline{h^{-1}}(c) - \overline{f'^{-1}_{m-1}}(c) \mid < \delta'$ for every $c \in K$. Regarding condition 15 of $IH2$, we may assume that $\delta' > 0$ is small enough so that if $\mid \overline{h}(c) - \overline{f'_{m-1}}(c) \mid < \delta'$ and $\mid \overline{h^{-1}}(c) - \overline{f'^{-1}_{m-1}}(c) \mid < \delta'$ for every $c \in K$, then $\mid f'_{m-1}(y) - h(y) \mid < \frac{1}{2^m}$ and $\mid f'^{-1}_{m-1}(y) - h^{-1}(y) \mid < \frac{1}{2^m}$ for every $y \in \mathbb{R}^2$ such that $|y| < m$. Since only finitely many elements of M are under consideration in condition 18 of $IH2$, and since we are only concerned with the composition of finitely many homeomorphisms, then we may assume that $\delta' > 0$ is small enough so that if $\mid \overline{h}(c) - \overline{f'_{m-1}}(c) \mid < \delta'$ and $\mid \overline{h^{-1}}(c) - \overline{f'^{-1}_{m-1}}(c) \mid < \delta'$ for every $c \in K$, then $\overline{h}$ satisfies the requirements of condition 18 of $IH2$ just as f'_{m-1} does. Let I be a subinterval of I_m with the same left endpoint as I_m such that the length of I is greater than 0 and less than δ, and such that there is at most one $d \in I$ for which there exists a pair of real numbers a and b such that $(a, b) \in M'_{m-1}$ and $(d, \overline{z'}(d))$ on $\overline{z'}$ is either related to (a, b) by $\overline{f'_{m-1}}$ or to (b, a) by $\overline{f'^{-1}_{m-1}}$. By condition 19 of $IH2$, there is a pair of real numbers a' and b' such that $(a', b') \notin M'_{m-1}$ and $(d, \overline{z'}(d))$ is either related to (a', b') by $\overline{f'_{m-1}}$ or to (b', a') by $\overline{f'^{-1}_{m-1}}$. We may assume, then, for the sake of specificity, that there is a positive integer q such that $1 \leq q < p$, $u'_{q+1} = f'_{m-1}$, and if $I'_q =$

$(\overline{u'_q} \circ \ldots \circ \overline{u'_1})(I)$, then $\{(x, \overline{f'_{m-1}}(x)) : x \in I'_q\}$ does not intersect M'_{m-1}. If we require $\overline{h}$ to agree with $\overline{f'_{m-1}}$ off of I'_q, then , since $\{(x, \overline{f'_{m-1}}(x)) : x \in I'_q\}$ doesn't intersect M'_{m-1}, we may assume that $\delta' > 0$ is small enough so that if $\mid \overline{h}(c) - \overline{f'_{m-1}}(c) \mid < \delta'$ and $\mid \overline{h^{-1}}(c) - \overline{f'^{-1}_{m-1}}(c) \mid < \delta'$ for every $c \in I'_q$, then h satisfies the part of condition 16 of $IH2$ related to the $\frac{1}{2}|\overline{z'}(x) - x|$ expression just as f'_{m-1} does. Let $c_0 < c_1 < \ldots < c_e$ be a subdivision of I such that c_0 is the left endpoint of I, c_e is the right endpoint of I, e is a positive integer, if k is a nonnegative integer such that $k < e$ and i is a positive integer such that $i \leq p$, then $(\overline{u'_i} \circ \ldots \circ \overline{u'_1})(c_{k+1}) - (\overline{u'_i} \circ \ldots \circ \overline{u'_1})(c_k) < \delta'$, $c_{k+1} - c_k < \delta'$, and if a, b, and x are real numbers such that $(a, b) \in M'_{m-1}$, $x \in I$, and $(x, \overline{z'}(x))$ on $\overline{z'}$ is either related to (a, b) by $\overline{f'_{m-1}}$ or to (b, a) by $\overline{f'^{-1}_{m-1}}$, then $x \in \{c_0, c_1, \ldots, c_e\}$. Since the length of I is less than δ, $\overline{z'}$ may be altered predictably on the subset, I, of its domain by altering $\overline{f'_{m-1}}$ on the subset, I'_q, of its domain. Since this is true, since K's interior is defined as it is, which would permit us to select δ' relative to a subset of the domain of $\overline{f'_{m-1}}$ which contains I'_q in its interior and does not contain the first coordinate of an element of M'_{m-1}, then we may assume that $\overline{z'}(c_k) \neq c_k$ for any integer k with $0 \leq k \leq e$ and that no alterations of $\overline{f'_{m-1}}$ have yet been made.

Let $(\overline{u'_q} \circ \ldots \circ \overline{u'_1})(c_0) = c'_0$ and $(\overline{u'_q} \circ \ldots \circ \overline{u'_1})(c_1) = c'_1$. What we wish to do now is to alter $\overline{f'_{m-1}}$ over the interval $[c'_0, c'_1]$ of its domain so that the correspondingly altered $\overline{z'}$ will be the identity at most once over the interval $[c_0, c_1]$ of its domain and condition 19 of $IH2$ is maintained. We will assume that there is an $x \in [c_0, c_1]$ such that $\overline{z'}(x) = x$. Since the set A''_{m-1} is countable, $M \cup M'_{m-1}$ is countable, and since we are only concerned with homeomorphisms, then we may select real numbers a_0 and b_0 such that $c_0 < a_0 < b_0 < c_1$, if $\overline{u'_1}(a_0) = a_1$, $\overline{u'_2}(a_1) = a_2, \ldots, \overline{u'_p}(a_{p-1}) = a_p$, and $b_1, \ldots, b_p$ are obtained from b_0 the same way, then $a_0 = b_p$ and none of the lines $x = a_0$, $x = a_1, \ldots, x = a_p$, $x = b_0$, $x = b_1, \ldots, x = b_p$ intersect A''_{m-1}, and if t_1, t_2, t_3, t_4, and y are real numbers such that one of the points (a_0, a_p) or (b_0, b_p) on $\overline{z'}$ is either related to (t_1, t_2) by $\overline{f'_{m-1}}$ or to (t_2, t_1) by $\overline{f'^{-1}_{m-1}}$, y is not in the interior of $I_0, I_1, \ldots, I_{m-1}$, or $[c_0, c_1]$, and $(t_3, t_4) \in M \cup M'_{m-1}$ such that $(y, \overline{z'}(y))$ on $\overline{z'}$ is either related to (t_3, t_4) by $\overline{f'_{m-1}}$ or to (t_4, t_3) by $\overline{f'^{-1}_{m-1}}$, then $(y, \overline{z'}(y))$ on $\overline{z'}$ is not related to (t_1, t_2) by $\overline{f'_{m-1}}$ nor to (t_2, t_1) by $\overline{f'^{-1}_{m-1}}$. Let h' be a homeomorphism from $[c_0, c_1]$ onto $[\overline{z'}(c_0), \overline{z'}(c_1)]$ such that h' agrees with $\overline{z'}$ at c_0 and c_1, $h'(a_0) = a_0$, h' is not the identity anywhere except at a_0, and there is an element of $[c_0, c_1]$ such that h' disagrees with every $\overline{g}$ at that point where $g \in \cup_{\theta<\alpha} G_\theta$. Let h'' be a homeomorphism from $[c'_0, c'_1]$ onto $[\overline{f'_{m-1}}(c'_0), \overline{f'_{m-1}}(c'_1)]$ such that h'' agrees with $\overline{f'_{m-1}}$ at c'_0 and c'_1, and if the graph of $\overline{f'_{m-1}}$ over the interval $[c'_0, c'_1]$ of its domain is replaced by the graph of h'', the rest of the graph of $\overline{f'_{m-1}}$ is left unchanged, the homeomorphism so obtained is called $\overline{\phi}$, and $\overline{\phi'}$ is obtained from $\overline{z'}$ by replacing each occurrence of $\overline{f'_{m-1}}$ by $\overline{\phi}$ and of $\overline{f'^{-1}_{m-1}}$ by $\overline{\phi^{-1}}$, then $\overline{\phi'}$ agrees with h' over the the interval $[c_0, c_1]$ of its domain. What we wish to do now is to show that we have maintained condition 19 of $IH2$. Let $H = M'_{m-1} \cup \{(a, b) \in \mathbb{R}^2 : (a_0, a_0)$ on $\overline{\phi'}$ is either related to (a, b) by $\overline{\phi}$ or to (b, a) by $\overline{\phi^{-1}}\}$. Suppose r is a real number such that r is not in the interior of $I_0, I_1, \ldots, I_{m-1}$, or $[c_0, c_1]$, there are real numbers d_1 and d_2 such that $(d_1, d_2) \in H$ and $(r, \overline{\phi'}(r))$ on $\overline{\phi'}$ is either related to (d_1, d_2) by $\overline{\phi}$ or to (d_2, d_1) by $\overline{\phi^{-1}}$, and if e_1 and e_2 are any real numbers such that $(r, \overline{\phi'}(r))$ on $\overline{\phi'}$ is either related to (e_1, e_2) by $\overline{\phi}$ or to (e_2, e_1) by $\overline{\phi^{-1}}$, then $(e_1, e_2) \in M \cup H$. If we had supposed that all (e_1, e_2) 's were contained by $M \cup M'_{m-1}$ instead of $M \cup H$, then we would contradict condition 19 of $IH2$ with $k = m - 1$. If we had supposed that there was an (e_1, e_2) contained in $M \cup M'_{m-1}$ and one in $\{(a, b) \in \mathbb{R}^2 : (a_0, a_0)$ on $\overline{\phi'}$ is either related to (a, b) by $\overline{\phi}$ or to (b, a) by $\overline{\phi^{-1}}\}$, then we would contradict the way a_0 and b_0 were chosen. If we suppose that all (e_1, e_2)'s are contained in $\{(a, b) \in \mathbb{R}^2 : (a_0, a_0)$ on $\overline{\phi'}$ is either related to (a, b) by $\overline{\phi}$ or to (b, a) by $\overline{\phi^{-1}}\}$, then this provides us with finite subsequences $a'_{i_0}, a'_{i_1}, \ldots, a'_{i_j}$ and $r_{i_0}, r_{i_1}, \ldots, r_{i_j}$

of the finite sequences $a'_0 = a_0$, $a'_1 = a_1, \dots, a'_q = a_q$, $a'_{q+1} = b_{q+1}, \dots, a'_p = b_p$ and $r = r_0$, the image of r_0 under the first letter of the word $\overline{\phi'}$ is $r_1, \dots$, the image of r_{p-1} under the p-th letter of the word $\overline{\phi'}$ is r_p (the first letter means the letter indexed by $1, \dots$, and the p-th letter is indexed by p) such that $\{r_{i_0}, r_{i_1}, \dots, r_{i_j}\} \subseteq \{a'_{i_0}, a'_{i_1}, \dots, a'_{i_j}\}$. Since r is not in the interior of $I_0, I_1, \dots, I_{m-1}$, or $[c_0, c_1]$, then we may assume $a_0 < r$. Since all the letters of $\overline{\phi'}$ are order-preserving homeomorphisms, then we know that $a'_{i_0} < r_{i_0}$, $a'_{i_1} < r_{i_1}, \dots$, and $a'_{i_j} < r_{i_j}$. We can now apply the knowledge of the last two sentences to obtain a chain of inequalities which would yield the contradiction that one of the elements of $\{a'_{i_0}, a'_{i_1}, \dots, a'_{i_j}\}$ is less than itself. What we have shown is that if r is a real number such that r is not in the interior of $I_0, I_1, \dots, I_{m-1}$, or $[c_0, c_1]$, and such that there are real numbers d_1 and d_2 for which $(d_1, d_2) \in H$ and $(r, \overline{\phi'}(r))$ is either related to (d_1, d_2) by $\overline{\phi}$ or to (d_2, d_1) by $\overline{\phi^{-1}}$, then there exist real numbers e_1 and e_2 such that $(e_1, e_2) \notin M \cup H$ and $(r, \overline{\phi'}(r))$ on $\overline{\phi'}$ is either related to (e_1, e_2) by $\overline{\phi}$ or to (e_2, e_1) by $\overline{\phi^{-1}}$. This is condition 19 of $IH2$ with H playing the role of M'_{m-1}, $\overline{\phi}$ that of $\overline{f'_{m-1}}$, $I_0 \cup I_1 \cup \dots \cup I_{m-1} \cup [c_0, c_1]$ that of $I_0 \cup I_1 \cup \dots I_{m-1}$, and $I_0, I_1, \dots, I_{m-1}$, or $[c_0, c_1]$ that of $I_0, I_1, \dots$, or I_{m-1}.

We have changed $\overline{f'_{m-1}}$ into $\overline{\phi}$ and now we wish to obtain ϕ from $\overline{\phi}$ and to enlarge A''_{m-1}. We have a process which will, when a point $\mathcal{L}_0$ is selected on the line $x = a'_0$, select a point $\mathcal{L}_1$ on the line $x = a'_1$, a point $\mathcal{L}_2$ on the line $x = a'_2, \dots$, and a point $\mathcal{L}_p$ on the line $x = a'_p$. Since $a'_0 = a'_p$, we will have $\mathcal{L}_0 = \mathcal{L}_p$. The process will also result in the formation of ϕ. To describe the process, let us assume that we have chosen an $\mathcal{L}_0$ on the line $x = a'_0$ and that the process has then chosen $\mathcal{L}_1$ on $x = a'_1, \dots$, and $\mathcal{L}_k$ on $x = a'_k$ where k is a positive integer less than p. If the $(k+1)$-th letter (i.e. the letter indexed by $k+1$) of $\overline{\phi'}$ is $\overline{\phi}$, $c''_0 = (\overline{u'_k} \circ \dots \circ \overline{u'_1})(c_0)$, $c''_1 = (\overline{u'_k} \circ \dots \circ \overline{u'_1})(c_1)$, j is a nonnegative integer such that $j \leq k$, and there exists $g \in \cup_{\theta<\alpha} G_\theta$ such that $\overline{g}(a'_j) = a'_{k+1}$, then let ϕ be defined on the subset $\{(x, y) \in \mathbb{R}^2 : x \in [c''_0, c''_1]\}$ of its domain so that if p_1 and p_2 are real numbers such that $(p_1, p_2) \in \{(x, y) \in \mathbb{R}^2 : x \in [c_0'', c_1'']\}$, then the first coordinate of $\phi(p_1, p_2)$ is $\overline{\phi}(p_1)$, ϕ is a homeomorphism on $\{(x, y) \in \mathbb{R}^2 : x \in [c_0'', c_1'']\}$, ϕ agrees with f_{m-1}' on the lines $x = c_0''$ and $x = c_1''$, and $\phi(\mathcal{L}_k) = g(\mathcal{L}_j)$. If there was no such g, we would define ϕ on $\{(x, y) \in \mathbb{R}^2 : x \in [c_0'', c_1'']\}$ so that the first coordinate of $\phi(p_1, p_2)$ is $\overline{\phi}(p_1)$ and the second coordinate of $\phi(p_1, p_2)$ is equal to the second coordinate of $f_{m-1}'(p_1, p_2)$. Under the process, we let $\mathcal{L}_{k+1} = \phi(\mathcal{L}_k)$. If the $(k+1)$-th letter of $\overline{\phi'}$ was $\overline{\phi^{-1}}$, then we would proceed in an analogous way, obtaining $\phi^{-1}(\mathcal{L}_k) = g(\mathcal{L}_j)$ and letting $\mathcal{L}_{k+1} = \phi^{-1}(\mathcal{L}_k)$. If the $(k+1)$-th letter of $\overline{\phi'}$ was an element g', of $\cup_{\theta<\alpha} G_\theta$, then we would not alter g' and we would let $\mathcal{L}_{k+1} = g'(\mathcal{L}_k)$. Upon the completion of the process, ϕ will have been defined on a certain subset of $\mathbb{R}^2$. To define ϕ on all of $\mathbb{R}^2$ we require that if $x \in \mathbb{R}^2$ such that $\phi(x)$ hasn't been determined through the process, then $\phi(x) = f_{m-1}'(x)$. We note that, because of the way the subdivision $\{c_0, c_1, \ldots, c_e\}$ of I is defined, if a and b are real numbers such that $(a, b) \in M_{m-1}'$, then ϕ will agree with f_{m-1}' on the line $x = a$. We note, too, that ϕ, as defined above, is a homeomorphism of $\mathbb{R}^2$ onto itself which sends every vertical line onto a vertical line. Another benefit of the process is that $\{g(\mathcal{L}_i) : g \in \cup_{\theta<\alpha} G_\theta$ and i is a nonnegative integer such that $i \leq p\}$ intersects no vertical line more than once. We now let $H' = A_{m-1}'' \cup \{g(\mathcal{L}_i) : g \in \cup_{\theta<\alpha} G_\theta$ and i is a nonnegative integer such that $i \leq p\}$. Suppose we have that $|\mathcal{L}_i| > m$, $|\mathcal{L}_i| > n$, and the absolute value of the second coordinate of $\mathcal{L}_i$ is greater than the absolute values of the second coordinates of both $f_{m-1}'^{-1}(x)$ and $f_{m-1}'(x)$ for every $x \in \mathbb{R}^2$ such that $|x| \leq n$ or $|x| \leq m$, for every integer i such that $0 \leq i \leq p$. Then, in addition to the other conditions satisfied by ϕ, we could have that if $|\,(p_1, p_2)\,| < m$ or $|\,(p_1, p_2)\,| < n$, then the second coordinate of $\phi(p_1, p_2)$ equals the second coordinate of $f_{m-1}'(p_1, p_2)$ and the second coordinate of $\phi^{-1}(p_1, p_2)$ equals the second coordinate of $f_{m-1}'^{-1}(p_1, p_2)$. But, if $\mathcal{L}_0$ is chosen far enough up or far enough down the line $x = a_0'$, then we will have what we just described, since f_{m-1}' is order-preserving in the y-axis direction. For this reason, we assume that everything stated so far is true and, in addition, if $x \in \mathbb{R}^2$ such that $|x| < m$ or $|x| < n$, then the second coordinate of $\phi(x)$ equals the second coordinate of $f'_{m-1}(x)$ and the second coordinate of $\phi^{-1}(x)$ equals the second coordinate of $f'^{-1}_{m-1}(x)$. We recall that we wanted this property so that we could choose $\delta' > 0$ for conditions 14 and 15 of $IH2$.

What we wish to do now is discuss why the things stated above are true. Suppose $a'_0, a'_1, \ldots, a'_p$ are as above, i and j are nonnegative integers such that $i \leq j \leq p$, $g \in \cup_{\theta<\alpha} G_\theta$ such that $\overline{g}(a'_i) = a'_j$ and $g' \in \cup_{\theta<\alpha} G_\theta$ such that $\overline{g'}(a'_i) = a'_j$. We recall that none of the lines $x = a'_0$, $x = a'_1, \ldots, x = a'_p$ intersect A''_{m-1}. By condition 2 of $IH2$, $\cup_{\theta<\alpha} A_\theta \subseteq A''_{m-1}$. By condition 8 of the TIH, if any element of $\cup_{\theta<\alpha} G_\theta$, except the identity, sends a vertical line onto itself, then that vertical line and $\cup_{\theta<\alpha} A_\theta$ have a nonempty intersection. So what we have is that $(\overline{g'^{-1}} \circ \overline{g})(a'_i) = a'_i$ and the line $x = a'_i$ does not intersect $\cup_{\theta<\alpha} A_\theta$. But $(\overline{g'^{-1}} \circ \overline{g})(a'_i) = a'_i$ implies that $g'^{-1} \circ g$ sends the vertical line $x = a'_i$ onto itself. By condition 9 of the TIH, $g'^{-1} \circ g \in \cup_{\theta<\alpha} G_\theta$. Therefore, it is clear that $g'^{-1} \circ g$ is the identity which means that $g = g'$. The fact that at most one element of $\cup_{\theta<\alpha} G_\theta$ can send the line $x = a'_i$ onto the line $x = a'_j$ implies that the things we stated previously are true.

By the way ϕ is constructed, it satisfies all the conditions of $IH2$ just as its predecessor, f'_{m-1}, does whenever we let $m - 1$ equal j or k as appropriate in the statements of those conditions. The sets $I_0 \cup I_1 \cup \ldots \cup I_{m-1} \cup [c_0, c_1]$, H, and H' satisfy $IH2$ just as their predecessors do. We can now proceed to construct successors of ϕ and the above sets with respect to $[c_1, c_2]$ just as the construction went for $[c_0, c_1]$. This would be followed by $[c_2, c_3], \ldots, [c_{e-1}, c_e]$. After this, we would choose a new I the same way I was chosen. The new I would be accompanied by a new δ'. With appropriate choices of I's, this process will end in finitely many steps with the I's covering I_m exactly. The last successors of ϕ, H, and H' will be called f'_m, M'_m, and A''_m, respectively. There is a portion of condition

18 of $IH2$ which f'_m may not satisfy. Let $z = u_p \circ \ldots \circ u_2 \circ u_1$ be the $(n+1)$-th element of W where p is a positive integer. Let $z' = u'_p \circ \ldots \circ u'_2 \circ u'_1$ be obtained from z by replacing each occurrence of 1 by f'_m and of -1 by f'^{-1}_m. This portion of condition 18 is concerned with the relationship between $\overline{z'}, \overline{f'_m}$, and the $(m+1)$-th element of M. This problem can be eliminated by finitely many changes in $\overline{f'_m}$ which affect it over a length of its domain which is as small as desired. So we will consider that this has been done, that all conditions of $IH2$ are satisfied, that the new autohomeomorphism of $\mathbb{R}^2$ has been obtained using the same method as we have used before, and that the new homeomorphisms are still called $\overline{f'_m}$ and f'_m.

The following lemma is needed in order to continue with the proof.

Lemma. *Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of autohomeomorphisms of $\mathbb{R}^2$. If, for every positive integer n and every $x \in \mathbb{R}^2$ such that $|x| < n$, $|\,\phi_n(x) - \phi_{n+1}(x)\,| < \frac{1}{2^n}$ and $|\,\phi_n^{-1}(x) - \phi_{n+1}^{-1}(x)\,| < \frac{1}{2^n}$, then $\{\phi_n\}_{n=1}^{\infty}$ converges to an autohomeomorphism of $\mathbb{R}^2$.*

This lemma is similar to a lemma which is cited in [2].

By the lemma, and condition 15 of $IH2$, the sequence $\{f'_m\}_{m=1}^{\infty}$, converges to an autohomeomorphism of $\mathbb{R}^2$ which we shall denote as f_n. Let $\cup_{m=1}^{\infty} A''_m = A'_n$, and let h_n be a function from ω_0 onto A'_n. Let $z = u_p \circ \ldots \circ u_2 \circ u_1$ be the $(n+1)$-th element of W where p is a positive integer. Let $z' = u'_p \circ \ldots \circ u'_2 \circ u'_1$ be obtained from z by replacing each occurrence of 1 by f_n and of -1 by f_n^{-1}. Suppose x is a real number such that if a and b are any real numbers for which $(a,b) \in M$, then $(x, \overline{z'}(x))$ on $\overline{z'}$ is not related to (a,b) by $\overline{f_n}$ nor to (b,a) by $\overline{f_n^{-1}}$. Since M is a closed subset of the graph of $\overline{f_n}$, then there are real numbers c and d such that $c < x < d$ and every element of the open interval, (c,d), has the same property as x does with respect to $\overline{z'}$ and M. From the above, it follows that there exists a positive integer, k, such that if j is a positive integer larger than k and if $z'' = u''_p \circ \ldots \circ u''_2 \circ u''_1$ is obtained from z by replacing each occurrence of 1 by f'_j and of

-1 by $f_j'^{-1}$, then every element of (c, d) has the same property with respect to $\overline{z''}$ and M as they do with respect to $\overline{z'}$ and M. By the above, and since I_m is chosen in accordance with condition 20 of $IH2$ for every positive integer m, there is a positive integer v such that $x \in I_v$. From the above, the conditions of $IH2$, and since f_n is the limit of the f_m''s, we have that f_n, A_n', and h_n satisfy all the conditions of $IH1$ whenever we let n equal j or k as appropriate in the statements of those conditions. Actually, f_n might not satisfy conditions 7,9, and 11 of $IH1$, but by finitely many alterations of f_n where the changes in functional values are as small as desired, we can obtain a function which satisfies all of the conditions of $IH1$. We shall consider that this has been done, that all conditions of $IH1$ are satisfied, and that the new autohomeomorphism of $\mathbb{R}^2$ is still called f_n.

By the lemma, and by condition 16 of $IH1$, the sequence, $\{f_n\}_{n=1}^{\infty}$, converges to an autohomeomorphism of $\mathbb{R}^2$ which we shall denote as g_α. Let G_α be the group generated by $\cup_{\theta<\alpha} G_\theta \cup \{g_\alpha\}$. Let $\cup_{n=1}^{\infty} A_n = A_\alpha$. From the conditions of $IH1$, and since g_α is the limit of the f_n's, we have that G_α and A_α satisfy all the conditions of the TIH whenever we let α equal β in the statements of those conditions. Actually, A_α might not satisfy condition 3 of the TIH since it might not intersect the α-th element of S. If A_α does not intersect the α-th element of S, then let $y \in \mathbb{R}^2$ such that y is contained by the α-th element of S and the vertical line containing y does not intersect A_α. Then the set, $A_\alpha \cup \{g(y) : g \in G_\alpha\}$, contains y and intersects no vertical line more than once. We now have that G_α, and the new set, which we will also call A_α, satisfy all conditions of the TIH. We can now see that $\cup_{\alpha<\omega_1} A_\alpha$ is a dense, connected subset of $\mathbb{R}^2$ which intersects vertical lines only once. The group, $\cup_{\alpha<\omega_1} G_\alpha$, of autohomeomorphisms of $\mathbb{R}^2$, witnesses that $\cup_{\alpha<\omega_1} A_\alpha$ is CDH.

Suppose $z = u_p \circ \ldots \circ u_2 \circ u_1$ is the $(n+1)$-th element of W where p is a positive integer, and, for the sake of specificity, that $u_1 = 1$ and $u_p = -1$. Let $w = u_{p-1} \circ \ldots \circ u_2$.

Let z' be obtained from z by replacing each occurrence of 1 by f'_{m-1} and of -1 by f'^{-1}_{m-1}. Let w' be obtained from w the same way. Since w is either an element of W which precedes z or an element of $\cup_{\theta<\alpha} G_\theta$, we may assume that $\overline{w'}$ is the identity only on a countably infinite set of real numbers which we list as the sequence of distinct elements $c_0, c_1, c_2, \ldots$. Let $d_0, d_1, d_2, \ldots$ be the sequence of real numbers such that $\overline{f'_{m-1}}(d_0) = c_0$, $\overline{f'_{m-1}}(d_1) = c_1, \ldots$. We can see that $\overline{z'}$ is the identity only on $d_0, d_1, d_2, \ldots$. We observe, as an example, that (d_0, d_0) on $\overline{z'}$ is related to (d_0, c_0) by $\overline{f'_{m-1}}$ and to (c_0, d_0) by $\overline{f'^{-1}_{m-1}}$, yet it does not follow from this that $(d_0, c_0) \in M$. That is why we must handle this case separately. From what we have said about w, we know that the line $x = c_m$ intersects A'_{n-1} at a point which we shall call a. If $(d_m, c_m) \in M$, then do not alter f'_{m-1}. If $(d_m, c_m) \notin M$ and $a \in C$, then alter f'_{m-1} so that the resulting function sends an element of B to a. We shall call the resulting function f'_m and we will use d_m to denote the first coordinate of the element of B which f'_m sends to a even though this will be an abuse of notation. If $(d_m, c_m) \notin M$ and $a \notin C$, then alter f'_{m-1} so that the resulting function sends a vertical line onto the line $x = c_m$ which contains no element of A''_{m-1}. We will handle the notation as we did in the case where $(d_m, c_m) \notin M$ and $a \in C$. In the case where $(d_m, c_m) \in M$ and the case where $(d_m, c_m) \notin M$ and $a \in C$, we let $A''_{m-1} = A'_m$. In the case where $(d_m, c_m) \notin M$ and $a \notin C$, we let $A''_m = A''_{m-1} \cup \{g(f'^{-1}_m(a)) : g \in \cup_{\theta<\alpha} G_\theta\}$.

In all the cases above, we have that f'_m is an autohomeomorphism of $\mathbb{R}^2$ which sends every vertical line onto a vertical line. Instead of being concerned with the interval I_m as we were before, if $u_1 = 1$ and $u_p = -1$, then the concern for I_m is replaced by the simpler things discussed above. Since we can achieve our objectives above while, at the same time, changing the functional values of f'_{m-1} by as small an amount as desired, leaving un-altered the ones we do not want to change, and ensuring that $f'_m \notin \cup_{\theta<\alpha} G_\theta$, then we can consider this argument and the proof to be complete.

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Troy State University
Troy, AL 36082