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CONCERNING THE EXISTENCE OF A NONDEGENERATE CONNECTED, COUNTABLE DENSE HOMOGENEOUS SUBSET OF THE PLANE WHICH HAS A RIGID OPEN SUBSET

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ABSTRACT. In this paper, we describe how the construction of a previous paper may be enriched in order to obtain a space as in the title. The construction mentioned above was carried out assuming the continuum hypothesis. Therefore, the results here are obtained under $ZFC + CH$, as well.

A topological space X is countable dense homogeneous (CDH) provided that X is separable and that if A and B are countable dense subsets of X then there is an $h \in H(X)$, the autohomeomorphism group on X, such that $h(A) = B$. The space X is rigid if its only autohomeomorphism is the identity.

The question as to whether open subsets of CDH spaces are CDH was raised by G. S. Ungar [2]. Sufficient conditions for open subsets to inherit the CDH property have been given in [3]. The two known examples of non-CDH open subsets of CDH spaces ([4], [5]) both involve an open connected subset of a CDH space which is not homogeneous, therefore, not CDH. In this paper, we show that the continuum hypothesis implies the existence of a nondegenerate connected, CDH subset of the plane with a rigid open set. Left unanswered are the following. Is there, in ZFC, an example of a connected metrizable CDH space with a non-CDH open subset? If Y is a homogeneous open subset of a connected CDH space, must Y be CDH?

Our construction is a modification of that used in [1], and our notations, conventions, and definitions are as those of [1].

Theorem ($ZFC + CH$). *There is a nondegenerate connected, CDH subset of* $\mathbb{R}^2$ *which has a rigid open subset.*

Proof. The construction in [1] was by transfinite induction. We will list the twelve conditions of the transfinite induction hypothesis for that construction as the first twelve conditions of the new transfinite induction hypothesis ($NTIH$). In [1], we let S be the set of all plane separators where a plane separator is any closed subset of the plane which separates the plane. Let S' be the set of all plane separators and of every Cantor set in the plane which intersects uncountably many vertical lines. In this proof, S' will play the same role that S played in [1]. Let S' be minimally well-ordered by ω_1. Let U denote the interior of the square in the plane with vertices $(0,0) = 0$, $(1,0) = 1$, $(1,1)$, and $(0,1)$. Let T be the set of all ordered triples (X, Y, h) such that X and Y are both $g - \delta$ subsets of U, X and Y both intersect every Cantor set in U which intersects uncountably many vertical lines, h is a homeomorphism from X onto Y, and h is not the identity on X. Let T be minimally well-ordered by ω_1. Let D be the set of all countable dense subsets of $\mathbb{R}^2$, as in [1], and let $D \times D$ be minimally well-ordered by ω_1. We will now state the $NTIH$: Let $\alpha < \omega_1$ be an ordinal number. Then for every ordinal number β such that $\beta < \alpha$ there are sets A_β and G_β satisfying

(1) A_β is a countable subset of $\mathbb{R}^2$,
(2) A_β intersects no vertical line more than once,
(3) A_β and the β-th element of S' have a nonempty intersection,
(4) if γ is an ordinal number such that $\gamma < \beta$ then $A_\gamma \subseteq A_\beta$,
(5) G_β is a countable collection of autohomeomorphisms of $\mathbb{R}^2$,
(6) G_β is a group under composition,
(7) every element of G_β sends all vertical lines onto vertical lines and A_β onto A_β,

(8) if g is any element of G_β, except the identity, and g sends a vertical line onto itself, then that vertical line and A_β have a nonempty intersection,

(9) if γ is an ordinal number such that $\gamma < \beta$ then $G_\gamma \subseteq G_\beta$,

(10) if $\cup_{\theta<\beta} A_\theta$ is dense in $\mathbb{R}^2$ and (B, C) is the first element of $D \times D$ such that $B \cup C \subseteq \cup_{\theta<\beta} A_\theta$, $\cup_{\theta<\beta} A_\theta \backslash (B \cup C)$ is dense in $\mathbb{R}^2$, and no element of $\cup_{\theta<\beta} G_\theta$ sends B onto C, then there is an element $g \in G_\beta$ such that $g(B) = C$,

(11) if g is any element of G_β, $\mathcal{L}_1$ and $\mathcal{L}_2$ are any pair of vertical lines in $\mathbb{R}^2$ such that $g(\mathcal{L}_1) = \mathcal{L}_2$, the first coordinate of every element of $\mathcal{L}_1$ is a and $\mathcal{L}_2$ is b, then the homeomorphism $\overline{g}$ of $\mathbb{R}$ onto itself defined by $\overline{g}(a) = b$ is order-preserving (i.e. g is order-preserving in the x-axis direction),

(12) if g is any element of G_β, $\mathcal{L}_1$ and $\mathcal{L}_2$ are any pair of vertical lines in $\mathbb{R}^2$ such that $g(\mathcal{L}_1) = \mathcal{L}_2$, y_1 is the second coordinate of an element p_1 of $\mathcal{L}_1$, and y_2 is the second coordinate of an element p_2 of $\mathcal{L}_2$ such that $g(p_1) = p_2$, then the homeomorphism f of $\mathbb{R}$ onto itself defined by $f(y_1) = y_2$ is order-preserving (i.e. g is order preserving in the y-axis direction),

(13) if g is any element of G_β, except the identity, then g sends a dense open subset of [0,1] into $\mathbb{R}^2 \backslash [0, 1]$, and

(14) if (X, Y, h) is the β-th element of T, then there are real numbers a_1, a_2, b_1, b_2, and b_3 such that $(a_1, b_1) \in X \cap A_\beta$, $(a_2, b_2) \in A_\beta$, $h(a_1, b_1) = (a_2, b_3)$, and $b_2 \neq b_3$.

The $NTIH$ is vacuously true for $\alpha = 0$.

In order to establish the beginning step of this induction, we must describe sets A_0 and G_0 satisfying conditions 1 through 14 of the $NTIH$. Let A_0 be a countable dense subset of $\mathbb{R}^2$ which intersects no vertical line more than once, and let (B_0, C_0) be the first element of $D \times D$ such that $B_0 \cup C_0 \subseteq A_0$, and $A_0 \backslash (B_0 \cup C_0)$ is dense in $\mathbb{R}^2$. Let g_0 be an autohomeomorphism of $\mathbb{R}^2$ which sends vertical lines onto vertical lines, A_0 onto itself, and B_0 onto C_0, such that g_0 is order-preserving in the

x-axis and y-axis directions, g_0 sends no vertical line onto itself, and [0,1] does not intersect its image under g_0. Let G_0 be the group generated by g_0 under composition. We observe that A_0 and G_0 satisfy all conditions of the $NTIH$, except perhaps conditions 3,8,13, and 14. Since g_0 sends no vertical line onto itself, and since g_0 is order-preserving in the x-axis direction, then G_0 satisfies condition 8 vacuously. Since g_0 sends [0,1] into $\mathbb{R}^2 \setminus [0,1]$, and since g_0 is order-preserving in the x-axis and y-axis directions, then every element of G_0, except the identity, sends [0,1] into $\mathbb{R}^2 \setminus [0,1]$. Therefore, G_0 satisfies condition 13. Let S_0 be the first element of S'. If A_0 does not intersect S_0, then pick an element of S_0 which lies on a vertical line that does not intersect A_0. Put that element, and its images under the elements of G_0, into A_0. We have now that A_0 and G_0 satisfy conditions 1 through 13 of the $NTIH$.

Let M be a $g-\delta$ subset of U that intersects every Cantor set in U which intersects uncountably many vertical lines. Suppose $U \setminus M$ intersects uncountably many vertical lines. Since $U \setminus M$ is the union of countably many closed subsets of U, then one of these sets intersects uncountably many vertical lines. Any closed subset of U, that intersects uncountably many vertical lines, contains a Cantor set that intersects uncountably many vertical lines. This is a contradiction. Therefore, $\mathcal{L} \cap U \neq \mathcal{L} \cap M$ for at most countably many vertical lines, $\mathcal{L}$.

Let $L = \{x \in [0,1]$: every element of G_0, except the identity, sends x into $\mathbb{R}^2 \setminus [0,1]\}$. Then L is a dense $g-\delta$ subset of [0,1]. Let $(X_0,\ Y_0,\ h_0)$ be the first element of T. Letting $X_0 = M$ in the previous paragraph, we see that $\mathcal{L} \cap U \neq \mathcal{L} \cap X_0$ for at most countably many vertical lines, $\mathcal{L}$. The same is true for Y_0. Therefore, $\mathcal{L} \cap U = \mathcal{L} \cap X_0 = \mathcal{L} \cap Y_0$ for all vertical lines, $\mathcal{L}$, except at most countably many. Let $L' = \{\mathcal{L}' : \mathcal{L}' = \mathcal{L} \cap U$, where $\mathcal{L}$ is a vertical line, $\mathcal{L}$ intersects $L, \mathcal{L}$ does not intersect A_0, and $\mathcal{L} \cap U = \mathcal{L} \cap X_0 = \mathcal{L} \cap Y_0\}$.

Suppose A_0 and G_0 do not satisfy condition 14 of the $NTIH$. If there is a vertical line segment in X_0 whose image, under h_0, is not a vertical line segment, then there is $\mathcal{L}' \in L'$ such that

$h_0(\mathcal{L}')$ is not a vertical line segment. Then, $h_0(\mathcal{L}')$ intersects uncountably many vertical lines. Let $\mathcal{L}$ be the vertical line for which $\mathcal{L}' \subseteq \mathcal{L}$, and let $\mathcal{L}''$ be a vertical line such that $\mathcal{L}''$ intersects $h_0(\mathcal{L}'), \mathcal{L}''$ does not intersect A_0, and $g(\mathcal{L})$ is different from $\mathcal{L}''$ for every $g \in G_0$. Let $x \in \mathcal{L}'$ such that $h_0(x) \in \mathcal{L}''$, and let $y \in \mathcal{L}''$ different from $h_0(x)$. Put x, y, and all of their images, under the elements of G_0, into A_0. It follows now that A_0 and G_0 satisfy all of the conditions of the $NTIH$. Suppose that the image of every vertical line segment in X_0, under h_0, is a vertical line segment. Since h_0 is not the identity on X_0, since $\cup L' \subseteq X_0$, and since $\cup L'$ is a dense $g - \delta$ subset of U, then there exist $\mathcal{L}', \mathcal{L}'' \in L'$ such that h_0 is not the identity on $\mathcal{L}'$, and $h_0(\mathcal{L}') \subseteq \mathcal{L}''$. Suppose $g(\mathcal{L}')$ and $\mathcal{L}''$ are subsets of different vertical lines for every $g \in G_0$. Let $x \in \mathcal{L}'$, and let $y \in \mathcal{L}''$ different from $h_0(x)$. Put x, y, and their images under the elements of G_0, into A_0. We now have that A_0 and G_0 satisfy all of the conditions of the $NTIH$. Let $\mathcal{L}$ be the vertical line for which $\mathcal{L}' \subseteq \mathcal{L}$. Suppose there exist $h,\ g \in G_0$ such that $h(\mathcal{L}'), g(\mathcal{L}')$, and $\mathcal{L}''$ are subsets of the same vertical lines. Then $h(\mathcal{L}) = g(\mathcal{L})$, which implies that $(g^{-1} \circ h)(\mathcal{L}) = \mathcal{L}$. But, $\mathcal{L}$ does not intersect A_0, therefore by condition 8 of the $NTIH, g^{-1} \circ h$ is the identity. This gives us that $g = h$, and that there is at most one element of G_0 which sends $\mathcal{L}'$ into the vertical line containing $\mathcal{L}''$. If $\mathcal{L}' = \mathcal{L}''$, then the identity is the only element of G_0 which sends $\mathcal{L}'$ into the vertical line containing $\mathcal{L}''$. Let $x \in \mathcal{L}'$ where $h_0(x) \neq x$. Then, h_0 disagrees with the identity on x. Put x, and its images under G_0, into A_0. Then, A_0 and G_0 satisfy conditions 1 through 14 of the $NTIH$. Suppose $\mathcal{L}' \neq \mathcal{L}''$, and that there is a $g \in G_0$ where $\mathcal{L}''$ and $g(\mathcal{L}')$ are subsets of the same vertical lines. Since $\mathcal{L}'$ is a line segment with no endpoints, which is closed in X_0, then $h_0(\mathcal{L}')$ is a line segment with no endpoints, which is closed in Y_0. But, $\mathcal{L}'' \subseteq Y_0$, and so the only subset of $\mathcal{L}''$, satisfying those properties, is $\mathcal{L}''$ itself. Therefore, $h_0(\mathcal{L}') = \mathcal{L}''$. Since $\mathcal{L}' \in L'$, and $\mathcal{L}' \subseteq \mathcal{L}$, then $\mathcal{L}$ intersects L. If we combine this with the knowledge that g is order-preserving in the y-axis direction, we have that $g(\mathcal{L}') \neq \mathcal{L}''$. Hence, there is an $x \in \mathcal{L}'$ where $g(x) \neq h_0(x)$. Put x, and its images under G_0, into A_0. We have now that A_0 and G_0 satisfy all conditions of the *NTIH*. Since all possible cases have been discussed, we consider this part of the argument to be complete.

Let us proceed to the preliminary section of [1], to the place where small changes have just been made to $g_{n,0}$ and $g_{n,0}^{-1}$, so that $g_{n,0}$ now sends the (appropriately selected) element of B into C and its element of $A_{n,0}$ into $A_{n,0}$; and $g_{n,0}^{-1}$ now sends the (appropriately selected) element of C into B and its element of $A_{n,0}$ into $A_{n,0}$. Let E be the n-th element of a fixed, ordered basis of open subsets of [0,1]. Let $x \in E$ such that if $\mathcal{L}_1$ is the vertical line containing x; if $f_1(\mathcal{L}_1) = \mathcal{L}_2$, $f_2(\mathcal{L}_2) = \mathcal{L}_3, \dots, f_p(\mathcal{L}_p) = \mathcal{L}_{p+1} = w(\mathcal{L}_1)$; and if the letters for w come from $G \cup \{g_{n,0}, g_{n,0}^{-1}\}$; then there exists a positive integer, j, where $1 \leq j \leq p$, and either $f_j = g_{n,0}$ and $g_{n,0}$ has not been determined on $\mathcal{L}_j$, or $f_j = g_{n,0}^{-1}$ and $g_{n,0}$ has not been determined on $\mathcal{L}_{j+1}$. We know that such an x exists since the set of vertical lines , on which $g_{n,0}$ has been determined, is countable, and its union is closed in $\mathbb{R}^2$. If there was a positive integer, i, where $1 \leq i \leq p$, $i \neq j, f_i \in \{g_{n,0}, g_{n,0}^{-1}\}$, and altering f_j on $\mathcal{L}_j$ alters f_i on $\mathcal{L}_i$, then it follows that either $f_j = g_{n,0}$ and $g_{n,0}$ has been determined on $\mathcal{L}_j$, or $f_j = g_{n,0}^{-1}$ and $g_{n,0}$ has been determined on $\mathcal{L}_{j+1}$. Therefore, no such positive integer i exists, giving us that w can be altered predictably on $\mathcal{L}_1$ by altering f_j on $\mathcal{L}_j$. Make a small alteration of f_j on $\mathcal{L}_j$, in the y-axis direction, so that w sends x directly above or directly below where it was sent before, and so that $w(x) \notin [0,1]$. Let future alterations be small enough so that $w(x) \notin [0,1]$, even when the letters of w come from $G \cup \{g_\alpha, g_\alpha^{-1}\}$. Now, we let the letters of the finitely many words preceding w come from $G \cup \{g_{n,0}, g_{n,0}^{-1}\}$, we select appropriate elements of E for each of these words, and then accomplish what was done for w. Continuing in this way, we will be able to obtain A_α and G_α satisfying conditions 1 through 13 of the *NTIH*. Then, appealing to the argument we used earlier,we get A_α and G_α satisfying conditions 1 through 14 of the $NTIH$. From the preliminary section of [1], we have that $\bigcup_{\alpha<\omega_1} A_\alpha$ is a dense, connected subset of $\mathbb{R}^2$ which intersects vertical lines only once, and $\bigcup_{\alpha<\omega_1} A_\alpha$ is CDH due to $\bigcup_{\alpha<\omega_1} G_\alpha$. We can see that $\bigcup_{\alpha<\omega_1} A_\alpha$ intersects every Cantor set in $\mathbb{R}^2$ which intersects uncountably many vertical lines.

What we wish to do now is to show that $U \cap (\bigcup_{\alpha<\omega_1} A_\alpha)$ is rigid. To accomplish this purpose, suppose there is an autohomeomorphism h, of $U \cap (\bigcup_{\alpha<\omega_1} A_\alpha)$, which is not the identity on $U \cap (\bigcup_{\alpha<\omega_1} A_\alpha)$. By Lavrentieff's Theorem (see [6]), there exist $g-\delta$ subsets X and Y of U, each containing $U \cap (\bigcup_{\alpha<\omega_1} A_\alpha)$ such that h may be extended to a homeomorphism from X onto Y. We will abuse notation and denote the extension of h as h. The ordered triple (X, Y, h) is an element of T, therefore, due to condition 14 of the $NTIH$, there are real numbers a_1 and b_1 such that $(a_1, b_1) \in \bigcup_{\alpha<\omega_1} A_\alpha$ but $h(a_1, b_1) \notin \bigcup_{\alpha<\omega_1} A_\alpha$. This is a contradiction and so we now have that $U \cap (\bigcup_{\alpha<\omega_1} A_\alpha)$ is rigid.

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