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ALMOST DISJOINT FAMILIES OF PATHS IN LATTICE GRIDS

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ABSTRACT. Almost disjoint families with certain maximal properties are considered and their existence is shown to be dependent on set theoretic assumptions.

1. INTRODUCTION

The origins of the following definition can be found in a pair of papers by D. E. Cook dealing with compactness in spaces satisfying certain Moore axioms [2, 3].

Definition 1.1. *A monotone path in* $\mathbb{N}^k$ *is a function* $f : \omega \to \mathbb{N}^k$ *such that if* $f(m) = (f_1(m), f_2(m), \ldots, f_k(m))$ *then there exists a unique* j *such that*

- $1 \leq j \leq k$
- $f_j(m) + 1 = f_j(m+1)$
- $f_i(m) = f_i(m+1)$ *if* $i \neq j$

There is a temptation to confuse a monotone path f *with its image but this will be resisted and* $\overline{f}$ *will denote the image of the path* f. *Two monotone paths* f *and* g *will be said to be* separated *if* $\overline{f} \cap \overline{g}$ *is finite.*

The term *finite monotone path* will refer to the restriction of a monotone path to some initial segment of integers. This notion is only used in the proof of Theorem 3.1.

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Definition 1.2. *Let $\mathfrak{B}_n$ be the ideal on $\mathbb{N}^n$ generated by sets of the form*

$$\{(x_1, x_2, \ldots, k, \ldots, x_{n-1}) : (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{N}^{n-1}\}$$

where $k \in \omega$. A family, $\mathcal{F}$, of monotone paths in $\mathbb{N}^k$ is said to be maximal if and only if $\{\overline{f} : f \in \mathcal{F}\}$ is a maximal antichain in the Boolean algebra $\mathcal{P}(\mathbb{N}^n)/\mathfrak{B}_n$.

It is worth noting that if f and g are monotone paths not in $\mathfrak{B}_n$ then f and g are separated if and only if $\overline{f} \cap \overline{g} \in \mathfrak{B}_n$. Maximal antichains of monotone paths in $\mathcal{P}(\mathbb{N}^2)/\mathfrak{B}_2$ are known as Cook sets.

This notion has been used by I. J. Tree who constructed a pseudocompact, non-metrisable, Moore manifold, assuming the existence of a Cook set in $\mathbb{N}^2$. As is pointed out in Problem 12 in [4], there is currently no known example of a pseudocompact non-metrisable manifold which does not require some extra set theoretic assumptions. Consequently, there has been some interest in the following question which will be answered by Theorem 2.1.

Question 1.1. *Does there exist a maximal family of monotone paths in $\mathbb{N}^2$?*

It is shown in [2] that $2^{\aleph_0} = \aleph_1$ implies that there is a Cook set. Theorem 2.1 will show that the existence of Cook sets can not be proved without assuming some other axioms.

2. The 2-dimensional Case

In the case of $\mathbb{N}^2$ it is possible to order separated monotone paths in $\mathbb{N}^2$.

Definition 2.1. *If $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ belong to $\mathbb{N}^n$ then define $a \leq b$ if and only if $a_i \leq b_i$ for each i. If X and Y are infinite subsets of $\mathbb{N}^2$ then define $X \prec Y$ if and only if there exist infinitely many disjoint pairs $\{(x^n, y^n) \in X \times Y : n \in \omega\}$ such that $x_n \not\leq y_n$ for each $n \in \omega$.*

In other words, $X \prec Y$ if and only if there are infinitely many pairs in X which are above and to the left of some pair from Y The reader is left with the task of verifying the next lemma.

Lemma 2.1. *The relation $\prec$ is a linear order when restricted to any family of separated monotone paths in $\mathbb{N}^2$ — in other words, if $\mathcal{F}$ is a family of separated monotone paths in $\mathbb{N}^2$ then $(\{\overline{f} : f \in \mathcal{F}\}, \prec)$ is a linear order.*

Definition 2.2. *Let $\| x \| \leq M$ denote the taxicab*[1] *norm for $x \in \mathbb{N}^k$ — in other words, if $x = (x_1, x_2, \ldots, x_k)$ then $\| x \| = \sum i = 1^{i=k}|x_i|$. If X and Y are infinite subsets of $\mathbb{N}^k$ then define $X \equiv Y$ if and only if there is some $M \in \omega$ and infinitely many pairs $\{(x_n, y_n) \in X \times Y : n \in \omega\}$ such that $\| x_n - y_n \| \leq M$. Define $X \sim Y$ if and only if there is some $M \in \omega$ such that for each $x \in X$ and $y \in Y$ there is some $y' \in Y$ and $x' \in X$ such that $\| x - y' \| \leq M$ and $\| x' - y \| \leq M$.*

Notice that $\sim$ is an equivalence relation while $\equiv$ is not.

Definition 2.3. *A family, $\mathcal{F}$, of monotone paths in $\mathbb{N}^k$ is said to be strongly maximal if and only if*

- *if $X \subset \mathbb{N}^k$ and $X \notin \mathfrak{B}_k$ then there is $f \in \mathcal{F}$ such that $X \equiv \overline{f}$*
- *if $\{f, g\} \in [\mathcal{F}]^2$ then $\overline{f} \not\equiv \overline{g}$*[2]

Lemma 2.2. *There is a maximal family of monotone paths in $\mathbb{N}^2$ if and only if there is a strongly maximal family.*

Proof. If $\mathcal{F}$ is a strongly maximal family of monotone paths, $f \in \mathcal{F}$ and $k \in \mathbb{Z}$ then let $J(f, k)$ be the least integer such that $f(J(f, k)) + (k, -k) \in \mathbb{N}^2$ and define $f_k(i) = f(i + J(f, k)) + (k, -k)$ and notice that if $k \neq m$ then f_k and f_m are separated. Hence $\{f_k : f \in \mathcal{F} \text{ and } k \in \mathbb{Z}\}$ is a maximal family. To see this suppose that $X \subset \mathbb{N}^2$ and $X \notin \mathfrak{B}_2$. Then there is $f \in \mathcal{F}$ such that $X \equiv \overline{f}$. Hence, there is some $M \in \omega$ and infinitely

[1]This is, of course, also the l_1 norm restricted to $\mathbb{N}^k$.

[2]Hence f and g are separated.

many pairs $\{(x_n, f(y_n)) \in X \times \overline{f} : n \in \omega\}$ such that $\| x_n - f(y_n) \| \leq M$. This implies that either $f(y_n) + (M, -M) \in X$ or $f(y_n) + (-M, M) \in X$. Hence either $\overline{f_M} \cap X \notin \mathfrak{B}_2$ or $\overline{f_{-M}} \cap X \notin \mathfrak{B}_2$.

On the other hand, if $\mathcal{F}$ is a maximal family then let $\mathcal{H} \subset \mathcal{F}$ be such that for each $f \in \mathcal{F}$ there is exactly one $h \in \mathcal{H}$ such that $\overline{f} \sim \overline{h}$. It suffices to show that if $\{f, g\} \in [\mathcal{H}]^2$ then $\overline{f} \not\equiv \overline{g}$.

To see that this is so suppose not and let M be minimal such that there is some pair $\{f, g\} \in [\mathcal{F}]^2$ such that there are infinitely many pairs $\{(I_n, J_n) \in \omega \times \omega : n \in \omega\}$ such that $\| f(I_n) - g(J_n) \| \leq M$ and such that $f \not\sim g$.[3] If $M \geq 2$ then it is possible to choose a point K_n such that $\| K_n - g(J_n) \| \leq M - 1$ and $\| f(I_n) - K_n \| \leq M - 1$.[4] The maximality of $\mathcal{F}$ ensures that there is some $h \in \mathcal{F}$ such that $\overline{h} \cap \{K_n : n \in \omega\}$ is infinite; but then there are infinitely many pairs $\{(I_n^0, J_n^0) \in \omega \times \omega : n \in \omega\}$ such that $\| h(I_n^0) - f(J_n^0) \| \leq M - 1$ as well as infinitely many pairs $\{(I_n^1, J_n^1) \in \omega \times \omega : n \in \omega\}$ such that $\| h(I_n^1) - g(J_n^1) \| \leq M - 1$. Since $f \not\sim g$ and $\sim$ is an equivalence relation it must be the case that either $h \not\sim f$ or $h \not\sim g$. Hence either the pair $\{f, h\}$ or the pair $\{g, h\}$ contradicts the minimality of M.

If $M = 0$ then f and g are not separated so the only other possibility to consider is $M = 1$. then there is a single edge, e_n, connecting $f(I_n)$ and $g(J_n)$. Let C_n be the cycle formed by e_n, e_{n+1} and the paths from $f(I_n)$ to $f(I_{n+1})$ and from $g(J_n)$ to $g(J_{n+1})$. Because $f \not\sim g$ it is possible to choose K_n which is in the interior of the cycle C_n for infinitely many integers n.[5] Choosing $h \in \mathcal{F}$ such that $| \overline{h} \cap \{K_n : n \in \omega\} |$ is infinite now produces h such that either $| \overline{h} \cap \overline{f} |$ or $| \overline{h} \cap \overline{g} |$ is infinite. This contradicts that $\mathcal{F}$ is a Cook set.

The following question remains open.

[3]Such a pair must exist because any pair $\{f, g\} \in \mathcal{H}$ such that $\overline{f} \equiv \overline{g}$ will satisfy the requirement.

[4]K_n is chosen to be approximately the centre of mass of $f(I_n)$ and $g(J_n)$ with respect to the taxicab metric.

[5]The fact that these are paths in $\mathbb{N}^2$ is crucial at this point.

Question2.1. *Can the restriction to dimension 2 be in Lemma 2.2 be omitted?*

Lenna 2.3. *If Martin's Axiom holds and*

- $\mathcal{F} \cup \mathcal{G}$ *is a separated family of monotone paths in* $\mathbb{N}^2$
- $|\mathcal{F} \cup \mathcal{G}| < 2^{\aleph_0}$
- $\overline{f} \prec \overline{g}$ *for each* $f \in \mathcal{F}$ *and* $g \in \mathcal{G}$
- $\overline{f} \not\equiv \overline{g}$ *for each* $f \in \mathcal{F}$ *and* $g \in \mathcal{G}$

then there is some infinite $X \subset \mathbb{N}^2$ *such that* $\overline{f} \prec X \prec \overline{g}$ *for all* f *and* g *in* $\mathcal{F}$. *Moreover,* $X \not\equiv \overline{h}$ *for all* $h \in \mathcal{F} \cup \mathcal{G}$.

Proof. The obvious partial order of finite approximations and finite side conditions works. For more details the reader can consult [2]

It is worth noting that if it was demanded that X was actually a monotone path in the conclusion of Lemma 2.3 then the result would no longer be true unless one of $\mathcal{F}$ or $\mathcal{G}$ was countable.

The next definition will be used in formulating the following theorem which provides the answer to Question 1.1. The Open Colouring Axiom was first considered by Abraham, Rubin and Shelah in [1] and later strengthened by Todorcevic [5].

Definition 2.4. *The Open Colouring Axiom states that if* $X \subseteq \mathbb{R}$ *and* $\mathcal{V} \subset [X]^2$ *is an open set*[6] *then either there is* $Y \in [X]^{\aleph_1}$ *such that* $[Y]^2 \subset \mathcal{V}$ *or there exists a partition of* $X = \bigcup_{n\in\omega} X_n$ *such that* $[X_n]^2 \cap \mathcal{V} = \emptyset$ *for each* $n \in \omega$. $\mathbb{R}$ *can be replaced by any second countable space in the statement of the Open Colouring Axiom.*

Theorem 2.1. *The conjunction of Martin's Axiom and* $2^{\aleph_0} \geq \aleph_2$ *and the Open Colouring Axiom implies that there is no strongly maximal, and hence no maximal, family of monotone paths in* $\mathbb{N}^2$.

[6]Here $[X]^2$ can be thought of as the set of points in X^2 above the diagonal.

Proof. Let $\mathcal{M}$ be a strongly maximal family of monotone paths in $\mathbb{N}^2$. From Lemma 2.1 it follows that $(\mathcal{M}, \prec)$ is a linear order. Using Lemma 2.3 it is possible to show that there are families $\{f_\xi : \xi \in \mathfrak{c}\}$ and $\{g_\xi : \xi \in \mathfrak{c}\}$ such that

- $\overline{f_\xi} \prec \overline{g_\xi}$ for each $\xi \in \mathfrak{c}$
- $\overline{f_\xi} \prec \overline{f_\eta}$ for each $\xi \in \eta \in \mathfrak{c}$
- $\overline{g_\xi} \prec \overline{g_\eta}$ for each $\eta \in \xi \in \mathfrak{c}$
- there is no monotone path h in $\mathbb{N}^2$ such that $\overline{f_\xi} \prec \overline{h} \prec \overline{g_\xi}$ for each $\xi \in \mathfrak{c}$

Now apply the Open Colouring Axiom [5].

To be precise, it may be assumed that $\overline{f_\xi} \cap \overline{g_\xi} = \emptyset$ for each $\xi \in \mathfrak{c}$. Define a partition

$$P = \{\{\xi, \eta\} \in [\mathfrak{c}]^2 : \overline{f_\xi} \cap \overline{g_\eta} = \overline{f_\eta} \cap \overline{g_\xi} = \emptyset\}$$

and note that P corresponds to a closed set in $[\mathcal{P}(\mathbb{N}^2)]^4$ with the usual Cantor topology on each factor of the symmetric fourfold product — consider the mapping which takes $\xi \in \mathfrak{c}$ to the pair $\{f_\xi, g_\xi\}$ in $[\mathcal{P}(\mathbb{N}^2)]^2$ and use the topology induced on $\mathfrak{c}$ to get a second countable space. It is easy to check that there is no set $X \in [\mathfrak{c}]^{\aleph_1}$ such that $[X]^2 \cap P = \emptyset$ because otherwise, since $2^{\aleph_0} \geq \aleph_2$, there is some $\zeta \in \mathfrak{c}$ such that $\overline{f_\xi} \prec \overline{f_\zeta} \prec \overline{g_\xi}$ for each $\xi \in X$. It follows that there is $Y \in [X]^{\aleph_1}$ and $n \in \mathbb{N}$ such that $(\overline{f_\xi} \cup \overline{g_\xi}) \cap \overline{f_\zeta} \subset n \times n$ for all $\xi \in Y$. Choosing ξ and η in Y such that $\overline{f_\xi} \cap n \times n = \overline{f_\eta} \cap F$ and $\overline{g_\xi} \cap n \times n = \overline{g_\eta} \cap n \times n$ yields a pair, $\{\xi, \eta\} \in P$ because $\overline{f_\xi} \cap \overline{g_\eta} = \overline{g_\xi} \cap \overline{f_\eta} = \emptyset$.

It follows from the Open Colouring Axiom that $\mathfrak{c} = \cup\{D_n : n \in \omega\}$ such that $[D_n]^2 \subset P$ for each $n \in \omega$. Hence there is some $n \in \omega$ such that D_n is cofinal in $\mathfrak{c}$. This yields a contradiction to the fact that there is no monotone path h in $\mathbb{N}^2$ such that $\overline{f_\xi} \prec \overline{h} \prec \overline{g_\xi}$ for each $\xi \in \mathfrak{c}$. The monotone path h can be taken to be the enumerating function of the supremum of $\{f_\xi : \xi \in D_n\}$.

3. THE 3-DIMENSIONAL CASE

The use of the planarity of $\mathbb{N}^2$ in Theorem 2.1 was crucial. This follows from the fact that Martin's Axiom implies that there is a strongly maximal family of monotone paths in $\mathbb{N}^3$. The following simple lemmas will prove to be useful in establishing this.

Definition 3.1. *The symbols $\delta_{i,j}$ will denote the Dirac delta defined by*

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

where $\{i,j\} \subseteq 2$. If $j \in 3$ then the triple $(\delta_{0,j}, \delta_{1,j}, \delta_{2,j})$ will be denoted by δ_j.

Lemma 3.1. *If f is a monotone path in $\mathbb{N}^k$ and $i \leq j$ then $f(i) \leq f(j)$.*

Proof. Proceed by induction.

Lemma 3.2. *Suppose that*

- *x, y and z belong to $\mathbb{N}^3$*
- *$n \geq m$ and $n' \geq m$*
- *$\{i,j,k\} \in 3$*
- *$D_t(x, z + n \cdot \delta_j) < m/2$ and $D_t(y, z + n' \cdot \delta_i) < m/2$*

Then $x + K_x \cdot \delta_k \not\leq y + K_x \cdot \delta_k$ for any integers K_x and K_y.

Proof. Let $x = (x_0, x_1, x_2)$, $y = (y_0, y_1, y_2)$ and $z = (z_0, z_1, z_2)$. Then $x_j > z_j + n - m/2 \geq x_j + m/2$ and $y_i > z_i + n' - m/2 \geq z_i + m/2$. On the other hand $x_k < z_k + m/2$ if $k \neq j$ and $y_k < z_k + m/2$ if $k \neq i$. Since $i \neq j$ it follows that $y_i > z_i + m/2 > x_i$ but $y_j < z_j + m/2 < x_j$. Finally, note that the i^{th} and $j^{(}th)$ coordinates of x are the same as those of $x + K_x \cdot \delta_k$ and that the i^{th} and $j^{(}th)$ coordinates of y are the same as those of $y + K_x \cdot \delta_k$.

The next lemma exploits the freedom allowed by the extra dimension in $\mathbb{N}^3$.

Lemma 3.3. *If f is a monotone path in $\mathbb{N}^3$ and $x \in \mathbb{N}^3$ is such that $D_t(\{x\}, \overline{f}) \geq n$ then there is $j \in 3$ such that $D_t(\{x + i \cdot \delta_k\}, \overline{f}) \geq n/4$ for all $i \geq 0$ and $k \in 3 \setminus \{j\}$.*

Proof. First notice that if $j \in 3$ and $D_t(\{x + i \cdot \delta_j\}, \overline{f}) \leq n/2$ then $i \geq n/2$ because otherwise the triangle inequality gives that $D_t(\{x\}, \overline{f}) < n$. It now follows from Lemmas 3.1 and 3.2 that there is at most one $j \in 3$ such that

$$D_t(\{x + i \cdot \delta_j\}, \overline{f}) < n/4$$

for some integer i.

Lemma 3.4. *If f is a monotone path in $\mathbb{N}^3$ and $x \in \mathbb{N}^3$ is such that $D_t(\{x\}, \overline{f}) \geq n$ then there is $j \in 3$ such that*

- $D_t(\{x + 2kn \cdot \delta_j\}, \overline{f}) \geq kn$
- $D_t(\{x + i \cdot \delta_j\}, \overline{f}) \geq n/4$ *for all* $i \geq 0$

for every integer k.

Proof. Use Lemma 3.3 to find $j \in 3$ such that $D_t(\{x + i \cdot \delta_k\}, \overline{f}) \geq n/4$ for all $i \geq 0$ and $k \in 3 \setminus \{j\}$.

An application of Lemma 3.2 yields at most one $j' \in 3$ such that

$$D_t(\{x + 2kn \cdot \delta_{j'}\}, \overline{f}) < kn$$

This leaves at least one $j'' \in 3$ which satisfies the requirements of the lemma.

Lemma 3.5. *If f_0 and f_1 are monotone paths in $\mathbb{N}^3$ and $x \in \mathbb{N}^3$ is such that $D_t(\{x\}, \overline{f_0}) > 7n$ and $D_t(\{x\}, \overline{f_1}) > n$ then there is a finite monotone path p such that*

- $dom(p) = K \neq 0$
- $D_t(\{p(K)\}, \overline{f_i}) \geq n$ *for* $i \in 2$
- $K \leq 8n$
- $D_t(\overline{p}, \overline{f_i}) \geq n/4$ *for* $i \in 2$.

Proof. Use Lemma 3.4 to find $j \in 3$ such that

- $D_t(\{x + 6n \cdot \delta_j\}, \overline{f_0}) \geq 3n$

- $D_t(\{x + i \cdot \delta_j\}, \overline{f_0}) \geq n/4$ for all $i \geq 0$

and let p_0 be the path defined by $p_0(k) = x + k \cdot \delta_j$ for $k \leq M$ where M is the least integer such that $M \neq 0$ and $D_t(\{x + M \cdot \delta_j\}, \overline{f_0}) \geq 3n$. Let $y = p_0(M)$ and observe that the triangle inequality implies that $D_t(\{y\}, \overline{f_1}) \geq 7n - 6n \geq n$.

Hence it is possible to again use Lemma 3.4 to find $j' \in 3$ such that

- $D_t(\{y + 2n \cdot \delta_{j'}\}, \overline{f_1}) \geq n$
- $D_t(\{y + i \cdot \delta_{j'}\}, \overline{f_1}) \geq n/4$ for all $i \geq 0$

and let p_1 be the path defined by $p_1(k) = y + k \cdot \delta_{j'}$ for $k \leq M'$ where M' is the least integer such that $M' \neq 0$ and $D_t(\{y + M' \cdot \delta_{j'}\}, \overline{f_1}) \geq n$. Notice that, because $D_t(\{y\}, \overline{f_0}) \geq 3n$ and $M' \leq 2n$ it follows that $D_t(\overline{p_1}, \overline{f_0}) \geq n$. Moreover, $D_t(\overline{p_0}, \overline{f_1}) \geq 7n - 6n = n$ because of the minimality of M. Hence, letting p be the concatenation of p_0 and p_1 yields the desired finite monotone path. The construction shows that p has length $K = M + M'$ and that $K \leq 8n$.

Lemma 3.6. *If f_0 and f_1 are monotone paths in $\mathbb{N}^3$ and $x \in \mathbb{N}^3$ is such that $7n > D_t(\{x\}, \overline{f_i}) > n$ for each $i \in 2$ then there is a finite monotone path p such that*

- *$dom(p) = K \neq 0$*
- *$D_t(\{p(K)\}, \overline{f_i}) \geq n$ for $i \in 2$*
- *$D_t(\{z\}, \overline{f_k}) \leq 16n$ for each $k \in 2$ and $z \in \overline{p}$*
- *$D_t(\overline{p}, \overline{f_i}) \geq n/4$ for $i \in 2$.*

Proof. From Lemma 3.3 it follows that there is at most one $j(k) \in 3$ such that $D_t(\{x + i \cdot \delta_{j(k)}\}, \overline{f_k}) < n/4$ for some integer i. Let $j \in 3 \setminus \{j(0), j(1)\}$.

Let M be the least integer such that there is some $K \in 2$ such that

$$D_t(\{x + M \cdot \delta_j\}, \overline{f_K}) > 7n$$

noting that such an integer exists because $\overline{f_k} \notin \mathfrak{B}_3$ for $k \in 2$. Let p_0 be the path defined by $p_0(m) = x + m \cdot \delta_j$ for $m \leq M$. It follows that $D_t(\overline{p_0}, \overline{f_k}) \geq n/4$ for each $k \in 2$ and also that

$D_t(\overline{p_0}, \overline{f_k}) \leq 8n$ for each $k \in 2$. Then use Lemma 3.5 to find a finite monotone path p_1 starting at $p_0(M)$ such that

- $\text{dom}(p_1) = K$
- $D_t(\{p_1(K)\}, \overline{f_i}) \geq n$ for $i \in 2$
- $K \leq 8n$
- $D_t(\overline{p_1}, \overline{f_i}) \geq n/4$ for $i \in 2$.

The concatenation of p_0 and p_1 is the desired path.

Lemma 3.7. *If f_0 and f_1 are monotone paths in $\mathbb{N}^3$ and $x \in \mathbb{N}^3$ is such that $D_t(\{x\}, \overline{f_i}) > n$ for each $i \in 2$ then there is a finite monotone path p such that*

- *$dom(p) = K \neq 0$*
- *$D_t(\{p(K)\}, \overline{f_i}) \geq n$ for $i \in 2$*
- *one of the following two conditions holds*
  - *$D_t(\{z\}, \overline{f_k}) \leq 16n$ for each $k \in 2$ and $z \in \overline{p}$*
  - *$K \leq 8n$*
- *$D_t(\overline{p}, \overline{f_i}) \geq n/4$ for $i \in 2$.*

Proof. Apply either Lemma 3.5 or Lemma 3.6 depending on whether or not there is some $i \in 2$ such that $D_t(\{x\}, \overline{f_i}) < 7n$.

The preceding lemmas will be employed to show that Martin's Axiom can be applied to the following partial order. In what follows the ball of radius n with respect to the taxicab metric will be denoted by $B_t(n)$. Note that if h is a finite monotone path of length $n+1$ and starting at $(0,0,0)$ then $h(n)$ lies on the boundary of $B_t(n)$. The taxicab distance between two[7] sets A and B — $D_t(A, B)$ — will refer to

$$D_t(A, B) = \inf\{\| a - b \|: a \in A \text{and} b \in B\}$$

Definition 3.2. *The partial order $\mathbb{P}$ consists of all quadruples $(h, \mathcal{A}, m, n)$ such that*

[7]Since the distance between a point and a set has not been defined, this will be denoted by $D_t(\{x\}, A)$. The reader will be forced to endure the author's pedantry on this point.

- *$h : m+1 \to \mathbb{N}^3$ is a finite monotone path in $\mathbb{N}^3$ starting at $(0,0,0)$*
- *$m \in \omega$ and $n \in \omega$*
- *$\mathcal{A} \in [\mathcal{F}]^{<\aleph_0}$*
- *for each $x \in \mathbb{N}^3 \setminus B_t(m)$ there do not exist f_0, f_1 and f_2, distinct members of $\mathcal{A}$, such that the $D_t(\overline{f_i} \setminus B_t(m), \{x\}) \leq 17n$ for each $i \in 3$*
- *if $f \in \mathcal{A}$ then $D_t(\overline{f} \setminus B_t(m), \{h(m)\}) \geq n$*

and the ordering on these quadruples is defined by $(h, \mathcal{A}, m, n) \leq (h', \mathcal{A}', m', n')$ if and only if $h' \supseteq h$, $\mathcal{A}' \supseteq \mathcal{A}$, $m' \geq m$, $n' \geq n$ and $D_t(\overline{h'} \setminus \overline{h}, \overline{f}) \geq n/4$ for all $f \in \mathcal{A}$.

Lemma 3.8. *If $\mathcal{D}(m) = \{(h, \mathcal{A}, i, j) \in \mathbb{P} : i \geq m\}$ then $\mathcal{D}(m)$ is dense in $\mathbb{P}$ for each $m \in \omega$. Moreover, if $(h, \mathcal{A}, i, n) \in \mathbb{P}$ then there is h' and i' such that $(h', \mathcal{A}, i', n) \in \mathcal{D}(m)$ and $(h', \mathcal{A}, i', n) \geq (h, \mathcal{A}, i, n)$.*

Proof. Let $(h, \mathcal{A}, i, n) \in \mathbb{P}$. It suffices to find h' and $i' \geq i+1$ such that $(h', \mathcal{A}, i', n) \geq (h, \mathcal{A}, i, n)$ because then a sequence of length m of extensions of this type can be found. Let f_0 and f_1 be two members of $\mathcal{A}$ such that $D_t(\{h(i)\}, \overline{f_0})$ is minimal and $D_t(\{h(i)\}, \overline{f_1})$ is minimal among $\{D_t(\{h(i)\}, \overline{f}) : f \in \mathcal{A} \setminus \{f_0\}\}$. Note that $D_t(\{h(i)\}, \overline{f_k}) > n$ for each $k \in 2$ by the definition of $\mathbb{P}$. It is therefore possible to use Lemma 3.7 to find a finite monotone path p starting at $h(i)$ such that

- $\mathrm{dom}(p) = K \neq 0$
- $D_t(\{p(K)\}, \overline{f_k}) \geq n$ for $k \in 2$
- one of the following two conditions holds
  - $D_t(\{z\}, \overline{f_k}) \leq 16n$ for all $z \in \overline{p}$ and $k \in 2$
  - $K \leq 8n$
- $D_t(\overline{p}, \overline{f_k}) \geq n/4$ for $k \in 2$.

It suffices to show that if $g \in \mathcal{A} \setminus \{f_0, f_1\}$ then $D_t(\overline{p}, \overline{g}) \geq n$. There are two cases to consider. First suppose that $z \in \overline{p}$ and $D_t(\{z\}, \overline{f_k}) \leq 16n$ for $k \in 2$ and $z \in \overline{p}$. Then, by the definition of $\mathbb{P}$, $D_t(\{z\}, \overline{g}) > 17n$ for each $z \in \overline{p}$ and so $D_t(\overline{p}, \overline{g}) > 17n \geq n$. The other possibility is that the length of p is not greater than $8n$. In this case the minimality of f_0 and f_1 guarantees that $D_t(\{h(i)\},\overline{g}) \geq 17n$. The length of p then ensures that $D_t(\{z\},\overline{g}) \geq 8n$ for each $z \in \overline{p}$.

Lemma 3.9 *If $g \in \mathcal{F}$ and $\mathcal{E}(g) = \{(h,\mathcal{A},i,n) \in \mathbb{P} : g \in \mathcal{A}\}$ then $\mathcal{E}(g)$ is dense in $\mathbb{P}$.*

Proof. Let $(h,\mathcal{A},i,n) \in \mathbb{P}$ and $g \in \mathcal{A}$. Find m such that

$$D_t(\overline{f} \setminus B_t(m), \overline{f'} \setminus B_t(m)) > 17n$$

for $\{f,f'\} \in [\mathcal{A} \cup \{g\}]^2$. Then use the fact that $\mathcal{D}(m)$ is dense to find a finite monotone path $h' : m \to \mathbb{N}^3$ and i' such that $(h',\mathcal{A},i',n) \geq (h,\mathcal{A},i,n)$. Then define $\mathcal{A}' = \mathcal{A} \cup \{g\}$ and note that $(h',\mathcal{A}',i',n) \geq (h,\mathcal{A},i,n)$.

Lemma 3.10. *Suppose that*

- $W \subseteq \mathbb{N}^3$
- $W \notin \mathfrak{B}_3$
- $W \not\equiv \overline{f}$ *for each* $f \in \mathcal{F}$
- $\mathcal{D}'(W) = \{(h,\mathcal{A},i,j) \in \mathbb{P} : W \cap \overline{h} \neq \emptyset\}$

then $\mathcal{D}'(W)$ is dense in $\mathbb{P}$.

Proof. Let $(h,\mathcal{A},i,n) \in \mathbb{P}$. Let $|\mathcal{A}| = L$ and let k be such that

$$D_t(\overline{f} \setminus B_t(k), \overline{f'} \setminus B_t(k)) > (L+16)n$$

From Lemma 3.8 it is possible to find h' and i' such that $(h',\mathcal{A},i',j) \geq (h,\mathcal{A},i,j)$ and $i' \geq k$.

Let $h'(i') = (x,y,z)$ and let P be the plane $\{(a,b,z) \in \mathbb{N}^3 : (a,b) \in \mathbb{N}^2\}$. There is then a monotone path p such that $\overline{p} \subseteq P$ and $D_t(\overline{p}, \overline{f} \setminus B_t(k)) \geq n/4$ for each $f \in \mathcal{A}$. To construct this path proceed by induction. Given $p \restriction N+1$ such that $2n \geq D_t(\{p(N)\},\overline{f}) > n$ there exists, by Lemma 3.3, $j \in 3$ such that $D_t(\{p(N) + i \cdot \delta_k\},\overline{f}) \geq n/4$ for all $i \geq 0$ and $k \in 3 \setminus \{j\}$. Consequently it is possible to find a straight, finite monotone path p' such that $D_t(\overline{p'},\overline{f}) \geq n/4$ and $\overline{p'} \subseteq P$. If $D_t(\overline{p'},\overline{f'}) \geq n/4$ for all $f' \in \mathcal{A}$ then let p be the concatenation of p and p'. Otherwise there is some $f' \in \mathcal{A}$ which is first approached with distance $n/4$ by the path p'. In this case p' can be followed until a point, N, is reached where $2n \geq D_t(\{p(N)\}, \overline{f'}) > n$ and, moreover, if $f'' \in \mathcal{A}$ then the choice of k guarantees that $D_t(\{p(N)\}, \overline{f''}) > (L+16)n \geq 16n$. Since $\overline{f} \notin \mathfrak{B}_3$ for each $f \in \mathcal{A}$ it follows that there is some M such that $D_t(\{x\}, \overline{f}) > n$ for each $f \in \mathcal{A}$ and $x \in P$ such that $x \geq p(M)$.

Now, because $W \not\equiv \overline{f}$ for each $f \in \mathcal{A}$ it is possible to choose $w = (w_0, w_1, w_2) \in W$ such that

- $p(M) \leq w' = (w_0 - Ln, w_1 - Ln, w_2 - Ln)$
- $D_t(\overline{f}, \{w\}) > (L + 1/4)n$ for $f \in \mathcal{A}$

Let $C = \{y \in \mathbb{N}^3 : w' \leq y \leq w\}$. Obviously, it suffices to find a path q from $p(M)$ to some point in C because any point in C can be connected to w by a path entirely contained in C and therefore a distance at least $n/4$ from each $\overline{f}$ such that $f \in \mathcal{A}$. This is done by induction.

Let q_0 be a path perpendicular to the plane P which continues until it encounters some $f_0 \in \mathcal{A}$. In particular, assume that $\mathrm{dom}(q_0) = Q_0 + 1$ and that

- $D_t(\overline{f_0}, \{q_0(Q_0)\}) > n$
- $D_t(\overline{f_0}, \{q_0(Q_0) + (0,0,1)\}) \leq n$
- there is some integer v such that $D_t(\overline{f_0}, \{q_0(Q_0) + (0,0,v)\}) < n/4$

By Lemma 3.3 it follows that $D_t(\overline{f_0}, \{q_0(Q_0)+(v,0,0)\}) > n/4$ and $D_t(\overline{f_0}, \{q_0(Q_0) + (0,v,0)\}) > n/4$ for every $v \geq 0$. From Lemma 3.2 it may be concluded that there do not exist v and v' such that $D_t(\overline{f_0}, q_0(Q_0) + (0,n,v)) < n/4$ and $D_t(\overline{f_0}, q_0(Q_0) + (n,0,v')) < n/4$. Hence, by making a deviation of no more than n it is possible to continue perpendicular to the plane P and come no closer than $n/4$ to $\overline{f}_0$. In all it will be necessary to make no more than L deviations, each of size no more than n, in order to miss all the functions in $\mathcal{A}$ by at least $n/4$. This path will end up in C because C has been chosen sufficiently large.

Actually, the fact that no more than n deviations are required is not immediate because, *a priori* it may be that some function from $\mathcal{A}$ is encountered more than once. This is not possible however, because if $f \in \mathcal{A}$, $g \in \mathcal{A}$, $f(u)$ and $g(u')$ have the same third coordinate and $p(M) \leq f(u) \leq w$ then either $p(M) \not\leq g(u)$ or $g(u) \not\leq w$. The reason for this is that $D_t(\overline{f} \setminus B_t(k), \overline{g} \setminus B_t(k)) > (L+16)n$. Also, if f is a monotone path and $p(M) \leq f(u) \geq w$ and $p(M) \leq f(u') \geq w$ and $U \leq u'' \leq u'$ then $p(M) \leq f(u'') \geq w$. Therefore each function can be encountered at most once.

Theorem 3.1. *If Martin's Axiom holds then there is a strongly maximal family of monotone paths in* $\mathbb{N}^3$. *In fact, only Martin's Axiom for* σ*-linked*[8] *partial orders will be used.*

Proof. Given a family, $\mathcal{F}$, of monotone paths in $\mathbb{N}^3$ such that $\overline{f} \not\equiv \overline{g}$ for $\{f,g\} \in [\mathcal{F}]^2$ and an infinite set $W \subset \mathbb{N}^3$ such that $W \not\equiv \overline{f}$ for each $f \in \mathcal{F}$, it must be shown that there is a partial order which is σ-linked and which adds a monotone path g such that $g \equiv A$ and $\overline{g} \not\equiv \overline{f}$ for each $f \in \mathcal{F}$. The partial order $\mathbb{P}$ has been defined to do this.

It must first be shown that $\mathbb{P}$ is σ-linked. To this end let $\mathbb{P}(i,j,k,H,\mathcal{B})$ be the set of all $(h, \mathcal{A}, m, n)$ such that

- $H = h$, $i = m$ and $j = n$
- $\mathcal{B}$ is a family of monotone paths in $B_t(k)$
- $|\mathcal{B}| = |\mathcal{A}|$
- $\{\overline{f} \cap B_t(k) : f \in \mathcal{A}\} = \mathcal{B}$
- if f and g are distinct members of $\mathcal{A}$ then $D_t(f \setminus B_t(k), g \setminus B_t(k)) > 16n$

It is easy to check that given any *two* conditions $(H, \mathcal{A}_0, i, j)$ and $(H, \mathcal{A}_1, i, j)$ in $\mathbb{P}(i,j,k,H,\mathcal{B})$ that $(H, \mathcal{A}_0 \cup \mathcal{A}_1, i, j) \geq (H, \mathcal{A}_k, i, j)$ for $k \in 2$. Hence any two members of $\mathbb{P}(i,j,k,H,\mathcal{B})$ are compatible. It is easy to see that every member of $\mathbb{P}$ belongs to some $\mathbb{P}(i,j,k,H,\mathcal{B})$.

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[8] A partial order is σ-linked if it is the union of countably many subsets which have the property that any *two* elements have a common lower bound.

It is easy to see that if $(h, \mathcal{A}, i, j) \in \mathbb{P}$ the so is $(h, \mathcal{A}, i, j+1)$. Lemmas 3.10 and 3.9 provide the density of the other required sets.

4. Remarks and Questions

The main result of the preceding section obviously begs the following question.

Question 4.1 *Does there exist a strongly maximal family of monotone paths in* $\mathbb{N}^3$*?*

For $\mathbb{N}^k$ where $k \geq 4$ the status of the existence of a Cook set is just as unclear. However it is easy to see that if C_i is a Cook set $\mathbb{N}^{k_i}$ for $i \in 2J$ then there is a Cook set in $\mathbb{N}^{k_0+k_1}$. The reason for this is that for any two monotone paths $f_i : \omega \to \mathbb{N}^{k_i}$ it is possible to define the product path $f_0 * f_1 : \omega \to \mathbb{N}^{k_0+k_1}$ by $f_0 * f_1(n) = (f_0(n), f_1(n))$. It is easily checked that

$$\{f_0 * f_1 : (\forall i \in 2)(f_i \in C_i)\}$$

is a Cook set. By reparameterising it is also possible to show that if there are Cook sets in $\mathbb{N}^{k_i}$ for $i \in 2$ then there is one in $\mathbb{N}^{k_0+k_1-1}$. Hence if there is a Cook set in $\mathbb{N}^2$ then there is one in $\mathbb{N}^k$ for each $k \geq 2$.

Question 4.2. *For each $N \in \omega$, does there exist a model of set theory where there is a strongly maximal family of monotone paths in $\mathbb{N}^{N+1}$ but not in $\mathbb{N}^N$?*

Question 4.3. *Does the existence of a Cook set in $\mathbb{N}^3$ imply the existence of a Cook set in $\mathbb{N}^4$?*

Observe that the proof of Theorem 3.1 also show that it is possible to get small maximal families of monotone paths even in $\mathbb{N}^2$. To do this construct the family by an iterated forcing which adds the members of the family one at a time. To get a family of size $\aleph_1$ an iteration of length ω_1 is required so at each step a countable partial order is used to get the next monotone path — in other words adding $\aleph_1$ Cohen real adds a maximal family of monotone paths in $\mathbb{N}^k$ for each $k \in \omega$.

Recall that $\mathfrak{a}$ represents the least cardinality of a maximal almost disjoint family in $\mathcal{P}(\omega)$. Define $\mathfrak{a}_k$ to be the least cardinality of a maximal family of monotone paths in $\mathbb{N}^k$ — if such a family exists. A family of monotone paths in $\mathbb{N}^k$ will be called *weakly maximal* if any two paths are separated and the family can not be extended to a larger family with this property. Define $\mathfrak{a}_k^-$ to be the least cardinality of a weakly maximal family of monotone paths in $\mathbb{N}^k$.

Question 4.4. *What are the relationships between the cardinal $\mathfrak{a}$, the cardinals $\mathfrak{a}_k$ and the cardinals $\mathfrak{a}_k^-$?*

Question 4.5. *Does $\mathfrak{a}_k^- = \mathfrak{a}_k$?*

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