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SUBGROUPS, QUOTIENT GROUPS AND PRODUCTS OF $\mathbb{R}$-FACTORIZABLE GROUPS

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1. Introduction

Given a compact topological group G and a continuous function $g : G \to \mathbb{R}$, one can find a second-countable topological group H, a continuous homomorphism $\pi : G \to H$ and a continuous function $h : H \to \mathbb{R}$ such that $g = h \cdot \pi$ (see Example 37 of [21]), i.e., a compact topological group is $\mathbb{R}$-factorizable in the sense of Definition 1.12 of [29]. The conclusion remains valid for every pseudocompact topological group G, a result due to W. W. Comfort and K. A. Ross [11].

How far can one generalize the assertions above? The following results were obtained in this direction.

A. Every Lindelöf topological group is $\mathbb{R}$-factorizable (see Assertion 1.1 of [29] and Assertion 10 of [28]).

B. Every (not necessarily closed) subgroup of a Lindelöf $\sum$-group is $\mathbb{R}$-factorizable (Corollary 1.13 of [29]).

The definition of a Lindelöf $\sum$-group can be found in [31,29], where some useful properties of these groups are established. We recall only that the class of Lindelöf $\sum$-groups contains all σ-compact groups and is closed with respect to countable products, passing to closed subgroups and to continuous homomorphic images. Since the completion of a totally bounded topological group is compact, the statement B implies that every totally bounded group is $\mathbb{R}$-factorizable (see Theorem 3.8 of [27]).

One of our aims is to show that the assertions A and B can not be improved simultaneously, as Example 2.1 shows.

A topological group G is said to be $\aleph_0$-bounded provided that for each neighborhood V of the identity there exists a countable subset $K \subseteq G$ such that $KV = G$ (see [15,4]). In [15] I. I. Guran characterized $\aleph_0$-bounded groups as subgroups of Cartesian products of second-countable topological groups. Since $\aleph_0$-boundedness is inherited by arbitrary subgroups and every Lindelöf topological group is $\aleph_0$-bounded, Example 2.1 below shows that $\aleph_0$-bounded groups need not be $\mathbb{R}$-factorizable. Thus we answer a question of M. Hušek in the negative. Note that every $\mathbb{R}$-factorizable topological group is $\aleph_0$-bounded (see the comment after Definition 1.12 of [29]). Another connection between these notions is established in Assertion 2.2, which implies that the class of $\aleph_0$-bounded groups coincides with the class of closed subgroups of $\mathbb{R}$-factorizable groups.

The question on monotonicity of the dimension function *dim* in the class of topological groups is considered at the end of Section 2. It is well-known that the dimension *dim* can increase when passing to a (closed) subspace of a Tychonoff space [13]. Our Theorem 2.7 states that the inequality $dim H \leq dim G$ is valid for any $\mathbb{R}$-factorizable subgroup H of an *arbitrary* topological group G. This result generalizes Theorem 2.2 of [23], in which a subgroup H is assumed to be totally bounded. Additional information on the problem of monotonicity is contained in the papers of D. B. Shakhmatov [23,24]. Other dimensional properties of $\mathbb{R}$-factorizable groups were investigated in [30].

In Section 3 we discuss some delicate properties of $\mathbb{R}$-factorizable groups. It is proved that every locally connected, $\mathbb{R}$-factorizable group is pseudo-ω_1-compact (Theorem 3.8). However it is an open problem whether the assumption of local connectedness in the theorem can be omitted. Also we prove that the $\mathbb{R}$-factorization property is preserved by quotient groups (Theorem 3.10). The section contains a number of problems concerning $\mathbb{R}$-factorizable groups.

The behavior of the $\mathbb{R}$-factorization property under the product operation is considered in Section 4. The main problem is the following one: must a product of an $\mathbb{R}$-factorizable group with a compact group be $\mathbb{R}$-factorizable? We prove in Theorem 4.3 that the answer is 'yes', if the first factor either has countable cellularity, or is $\aleph_0$-stable, or is locally connected. More restrictive requirements on the first factor are necessary if the second factor is assumed to be pseudocompact (see Theorem 4.13). It is also shown that the product of an $\aleph_0$-stable Lindelöf group and an arbitrary subgroup or a Lindelöf $\sum$-group is $\mathbb{R}$-factorizable (Theorem 4.16). This implies that the product of a Lindelöf P-group with a totally bounded group is $\mathbb{R}$-factorizable as well (Corollary 4.18).

Theorem 4.8 seems to be somewhat surprising: it asserts that every k-group has countable o-tightness. This result is used in the proof of Theorem 4.13.

In the end of the paper we introduce the notion of a weakly $\aleph_0$-stable group and prove that a product of an $\mathbb{R}$-factorizable group with this property and a compact group is $\mathbb{R}$-factorizable (Theorem 4.20). This is the most complicated result of the paper which generalized Theorem 4.3 (modulo Theorem 3.8).

2. Subgroups of Lindelöf and $\mathbb{R}$-factorizable groups. Monotonicity of the dimension *dim*.

Example 2.1. There exists a Hausdorff Abelian topological group G such that the completion $\widehat{G}$ of G is a Lindelöf group, but G is not $\mathbb{R}$-factorizable.

Construction. Let $\mathbb{Z}_2 = \{0,1\}$ be a discrete group. Consider the Cartesian product $\Pi = \Pi_{\alpha<\omega_1} G_\alpha$, where $G_\alpha = \mathbb{Z}_2$ for each $\alpha < \omega_1$, and endow Π with the $\aleph_0$-box topology $\mathcal{T}$, the base of which consists of the sets of the form $p_\alpha^{-1}(x)$ with $\alpha < \omega_1$ and $x \in \Pi_\alpha = \Pi_{\beta<\alpha} G_\beta$; here $p_\alpha : \Pi \to \Pi_\alpha$ is the projection. One can easily see that $\Pi = (\Pi, \mathcal{T})$ is a Hausdorff Abelian topological group, each G_δ-subset of which is open. In other words, Π is a P-group.

For every $g \in \Pi$ denote $\mathrm{supp}(g) = \{\alpha < \omega_1 : \pi_\alpha(g) = 1\}$, where π_α is the projection of Π onto G_α. Let G^* be a weak sum of the groups $G_\alpha, \alpha < \omega_1$, i.e., $G^* = \{g \in \Pi : |\ \mathrm{supp}(g)| < \aleph_0\}$. Consider G^* as a subgroup of Π. Obviously G^* is a P-group, and hence G^* is zerodimensional. By the result of Comfort [9], the group G^* is Lindelöf. Let $\bar{0}$ and $\bar{0}_\alpha$ be neutral elements of the groups G^* and Π_α resp., $\alpha < \omega_1$. The family $\mathcal{T}(\bar{0}) = \{p_\alpha^{-1}(\bar{0}_\alpha) \cap G^* : \alpha < \omega_1\}$ constitutes a base of G^* at the point $\bar{0}$, consequently $\chi(G^*) = \aleph_1$. Since $|\ G^*\ | = \aleph_1$, we conclude that $w(G^*) = \aleph_1$. Hence there exists a base $\mathcal{B} = \{O_\alpha : \alpha < \omega_1\}$ for G^* consisting of clopen sets.

The group G will be defined as a dense subgroup of G^*. Fix an element $g^* \in G^* \setminus \{\hat{O}\}$ and define by recursion a sequence $\{H_\alpha : \alpha < \omega_1\}$ of countable subgroups of G^* and sequences $\{U_\alpha : \alpha < \omega_1\}, \{V_\alpha : \alpha < \omega_1\}$ of clopen subsets of G^* satisfying the following conditions for each $\alpha < \omega_1$:

(1) $H_\alpha \cap O_\beta \neq \emptyset$, if $\beta < \alpha$;
(2) $U_\alpha \cap V_\alpha = \emptyset$;
(3) $H_\beta \subseteq H_\alpha, U_\beta \subseteq U_\alpha, V_\beta \subseteq V_\alpha$, if $\beta < \alpha$;
(4) $H_\alpha \subseteq U_{\alpha+1} \cup V_{\alpha+1}$;
(5) $p_\alpha(H_\alpha \cap U_\alpha) \cap p_\alpha(H_\alpha \cap V_\alpha) \neq \emptyset$, if α is a non-limit ordinal;
(6) $g^* \notin H_\alpha \cup U_\alpha \cup V_\alpha$.

The construction is based on the following easy observation: (*) for every non-empty open subset O of G^* and every countable subgroup $H \subseteq G^* \setminus \{g^*\}$ there exists an element $g \in O$ such that $g^* \notin H + < g >$, where $< g > = \{\bar{0}, g\}$.

Note that $< g >$ is a subgroup of G^*, and $H + < g >$ is also. Assume that a countable subgroup $H_\alpha \subseteq G^*$ and the sets U_α, V_α satisfying (1)-(6) are defined. Since $|\ H_\alpha\ | \leq \aleph_0$, one can find disjoint clopen subsets $U_{\alpha+1}, V_{\alpha+1}$ of G^* so that $U_\alpha \subseteq U_{\alpha+1}, V_\alpha \subseteq V_{\alpha+1}$ and $H_\alpha \subseteq U_{\alpha+1} \cup V_{\alpha+1}$. Obviously, the sets $U_{\alpha+1}$ and $V_{\alpha+1}$ can also be chosen to satisfy the conditions $g^* \notin U_{\alpha+1} \cup V_{\alpha+1}$ and $T = p_{\alpha+1}(U_{\alpha+1}) \cap p_{\alpha+1}(V_{\alpha+1}) \neq \emptyset$. Pick a point $t \in T$. The set $U' = U_{\alpha+1} \cap p_{\alpha+1}^{-1}(t)$ is open and non-empty; hence (*) implies that there exists an element $a_\alpha \in U'$ such that $g^* \notin H_\alpha + < a_\alpha > = H'_\alpha$. Apply assertion (*) once more to find an element b_α of $V_{\alpha+1} \cap p_{\alpha+1}^{-1}(t)$ such that $g^* \notin H'_\alpha + < b_\alpha >= H''_\alpha$. Finally, there exists an element $c_\alpha \in O_\alpha$ such that $g^* \notin H''_\alpha + < c_\alpha >= H_{\alpha+1}$. It is easily seen that $H_{\alpha+1}, U_{\alpha+1}$ and $V_{\alpha+1}$ satisfy the conditions (1)-(6).

At a limit step $\alpha \leq \omega_1$ put $H_\alpha = \cup_{\beta<\alpha} H_\beta, U_\alpha = \cup_{\beta<\alpha} U_\beta$ and $V_\alpha = \cup_{\beta<\alpha} V_\beta$. Thus, we have defined a dense subgroup $G = H_{\omega_1}$ of G^* (see condition (1)) and disjoint open subsets $U = U_{\omega_1}, V = V_{\omega_1}$ of G^*. Clearly, $G \subseteq U \cup V$ (see condition (2)).

Let f be a function on G defined by the rule: $f(x) = 0$ for each $x \in U \cap G$, and $f(x) = 1$ for each $x \in V \cap G$. Obviously, f is continuous. Let $\pi : G \to H$ be a continuous homomorphism of G to a metrizable group H. Then the kernel of π is of countable pseudocharacter in G, consequently, one can find $\alpha < \omega_1$ such that $p_{\alpha+1}^{-1}(\bar{0}_{\alpha+1}) \cap G \subseteq \ker \pi$. The condition (5) implies that $p_{\alpha+1}(G \cap U) \cap p_{\alpha+1}(G \cap V) \neq \emptyset$, which in turn implies $\pi(G \cap U) \cap \pi(G \cap V) \neq \emptyset$. Hence there exist points $x \in U \cap G$ and $y \in V \cap G$ such that $\pi(x) = \pi(y)$, whereas $f(x) = 0$ and $f(y) = 1$. This means that the group G is not $\mathbb{R}$-factorizable.

Every G_δ-subset of the Lindelöf group G^* is open, so G^* is complete. Since G is dense in G^*, the completion of G coincides with the Lindelöf group G^*.

Example 2.1 shows that (dense) subgroups of $\mathbb{R}$-factorizable groups need not be $\mathbb{R}$-factorizable. The result does not change when passing to a closed subgroup.

Assertion 2.2. Every $\aleph_0$-bounded Abelian group G embeds in some $\mathbb{R}$-factorizable Abelian group H as a closed subgroup.

Proof. Being $\aleph_0$-bounded, G embeds in a product of second-countable groups, say K, as a subgroup [15]. Denote $\Pi = \Pi_{n=0}^{\infty} K_n$, where $K_n = K$ for each integer n. Let σ be the weak sum of the groups K_n's, i.e. the subgroup of Π consisting of all points which coincide with the neutral element of Π on almost all coordinates. Obviously, σ is dense in Π provided that Π is endowed with the Tikhonov topology. Consider the mapping i of G to Π defined by the rule: $\pi_n i(g) = g$ for all $g \in G$ and $n \in N$, where π_n is the projection of Π onto K_n. It is clear that i is a topological monomorphism of G to Π, and $H = i(G) + \sigma$ is a dense subgroup of Π. The group K is a product of second-countable groups, and so is Π. Being dense in Π, the group H is $\mathbb{R}$-factorizable by Corollary 1.10 of [29] (another way is to apply Theorem 2 of [5]). From the definition of H follows that for every point $p \in H \setminus i(G)$ there exist integers m and n such that $\pi_n(p) \neq \pi_m(p)$, and hence $i(G)$ is closed in H. It remains to identify G with $i(G)$.

A similar method was applied in Theorem 2.4 of [12] to show that the torsion subgroup of the circle group (in fact, every totally bounded group) embeds in some pseudocompact group as a closed subgroup. With the help of a little modification of the previous proof one can show that a non-Abelian version of Assertion 2.2 holds (define H as the minimal subgroup of Π containing $i(G)$ and σ).

We omit an easy proof of the following result which show that some subgroups inherit the $\mathbb{R}$-factorizable property.

Assertion 2.3. Every C-embedded subgroup of an $\mathbb{R}$-factorizable group is $\mathbb{R}$-factorizable.

It was mentioned in the introduction that the covering dimension *dim* is not monotone in the realm of Tikhonov spaces. What can one say about monotonicity of *dim* in the class of topological groups? The question seems to be surprisingly difficult: no example that raises the dimension of a subgroup is known (see Problem 2.1 of [23]). A positive answer to this question was given by D. B. Shakhmatov in the special case of a totally bounded subgroup (announcement [23, Th. 2.2], proof is in [24]). Here we strengthen this result by showing that the inequality $dim H \leq dim G$ remains valid for each $\mathbb{R}$-factorizable subgroup H of a topological group G. The proof of this theorem requires three lemmas, the first of which is folklore.

Lemma 2.4. *The following conditions are equivalent for a completely regular space X:*

(1) *$dimX \leq n$;*

(2) *for every continuous mapping f of X to a separable metrizable space Y there exist a separable metrizable space Z and continuous mappings $g : Z \to Y, h : X \to Z$ such that $f = gh$ and $dimZ \leq n$.*

The proof of Lemma 2.4 can be found in [30, Section 2]. In what follows the phrase "a uniformly continuous metric (function) on a group G" means that one should consider the *left* group uniformity $^*\mathcal{V}$ on G and a metric (function), which is uniformly continuous with respect to $^*\mathcal{V}$.

Lemma 2.5. *For every uniformly continuous function $f : G \to \mathbb{R}$ on $\mathbb{R}$-factorizable group G there exist a second-countable group H, a continuous homomorphism $\pi : G \to H$ and a uniformly continuous function $h : H \to \mathbb{R}$ such that $f = h \circ \pi$.*

Proof. Since f is uniformly continuous, for each $n \in \mathbb{N}^+$ one can find a neighborhood U_n of the identity e_G such that $| f(x) - f(y) | < 1/n$ whenever $x^{-1} \cdot y \in U_n$. Being $\mathbb{R}$-factorizable, the group G is $\aleph_0$-bounded. This fact and Corollary 1 of [15] imply that for every $n \in \mathbb{N}^+$ there exist a continuous homomorphism π_n of G onto a second-countable group H_n and a neighborhood V_n of the identity of H_n such that $\pi_n^{-1}(V_n) \subseteq U_n$. Denote by π the diagonal product of homomorphisms $\pi_n, n \in \mathbb{N}^+$, and put $H = \pi(G)$. Then H is a subgroup of the product $\Pi = \Pi_{n \in \mathbb{N}^+} H_n$; hence H is second-countable. Use the definition of π and H in order to find, for every $n \in \mathbb{N}^+$, an open neighborhood W_n of the identity e_H such that $\pi^{-1}(W_n) \subseteq U_n$. It is clear that there exists a function $h : H \to \mathbb{R}$ such that $f = h \circ \pi$. Let us verify that h is uniformly continuous.

Fix a positive integer n and elements $x, y \in H$ with $x^{-1} \cdot y \in W_n$. Pick elements $x_1, y_1 \in G$ such that $\pi(x_1) = x$ and $\pi(y_1) = y$. Then $\pi(x_1^{-1} \cdot y_1) = x^{-1} \cdot y \in W_n$, and the choice of W_n implies that $x_1^{-1} \cdot y_1 \in U_1$. Therefore, $|f(x_1) - f(y_1)| <$

$1/n$, *i.e.*, $|h(x) - h(y)| < 1/n$. Thus h is uniformly continuous on the group H.

Lemma 2.6. *Let H be an $\mathbb{R}$-factorizable subgroup of a topological group G. Then every uniformly continuous real-valued function on H extends to a continuous function on G.*

Proof. Suppose that a function $f : H \to \mathbb{R}$ is uniformly continuous. Apply Lemma 2.5 to find a continuous homomorphism π of H onto second-countable group K and a uniformly continuous function h on K such that $f = h \circ \pi$. Let $\mathcal{B} = \{V_n : n \in \mathbb{N}\}$ be a countable base at the identity of K. Obviously, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open symmetric neighborhoods of the identity e_G such that $U_n \cap H \subseteq \pi^{-1}(V_n)$ and $U_{n+1}^2 \subseteq U_n$ for each $n \in \mathbb{N}$. Set $P = \cap_{n=0}^{\infty} U_n$. Then P is a closed G_δ-subgroup of G, and $P \cap H \subseteq \ker \pi$. Denote by p the quotient mapping of G onto coset space G/P, $p(x) = xP$ for each $x \in G$. By Theorem 3 of [14], there exists a continuous left-invariant pseudometric d on G with the following property:

(i) the implications $x^{-1} \cdot y \in U_{n+1} \Rightarrow d(x,y) < 2^{-n} \Rightarrow x^{-1} \cdot y \in U_n$ hold for each $n \in \mathbb{N}$ and $x, y \in G$.

The property (i) implies that the equality $d(x,y) = 0$ is equivalent to $x^{-1} \cdot y \in P$; hence one can define a metric $\tilde{d}$ on G/P by the rule $\tilde{d}(p(x), p(y)) = d(x,y)$ for each $x, y \in G$. Obviously, this metric is well-defined. Endow G/P with the topology generated by the metric $\tilde{d}$. The mapping $p : G \to G/P$ remains continuous. Put $\tilde{H} = p(H)$. Note that, if $x, y \in H$ and $x^{-1} \cdot y \in P$, then $x^{-1} \cdot y \in P \cap H \subseteq \ker\pi$, and hence $\pi(x) = \pi(y)$. Consequently, there exists a mapping $j : \tilde{H} \to K$ such that $\pi = jp|_H$. We claim that j is uniformly continuous with respect to the metric $\tilde{d}$. Indeed, let $n \in \mathbb{N}$ and $y_1, y_2 \in \tilde{H}$ satisfying the condition $\tilde{d}(y_1, y_2) < 2^{-n}$ be chosen. It is sufficient to show that $j(y_1)^{-1} \cdot j(y_2) \in V_n$. To this end, pick elements $x_1, x_2 \in H$ so that $p(x_i) = y_i$, $i = 1, 2$. Then $d(x_1, x_2) = \tilde{d}(y_1, y_2) < 2^{-n}$, and hence (i) implies that $x_1^{-1} \cdot x_2 \in U_n$. In turn, this fact and the choice of U_n together imply that $\pi(x_1^{-1} \cdot x_2) \in V_n$, i.e., $\pi(x_1)^{-1} \cdot \pi(x_2) \in V_n$. Evidently, $\pi(x_i) = jp(x_i) = j(y_i)$, $i = 1, 2$, so $j(y_1)^{-1} j(y_2) \in V_n$.

Consider the uniformly continuous mapping $g = hj$ of $\tilde{H}$ to $\mathbb{R}$ and extend g to a uniformly continuous mapping $\tilde{g} : F \to \mathbb{R}$, where F is the closure of $\tilde{H}$ in G/P (apply Theorem 8.3.10 of [13]). Since the space G/P is metric, $\tilde{g}$ extends to a continuous function $g^* : G/P \to \mathbb{R}$. It is clear that the function $f^* = g^* p$ is the required extension of f. The diagram below clarifies the proof.

$$\begin{tikzcd}
& H \arrow[rr, hook] \arrow[dl, "f"'] \arrow[d, "\pi"] \arrow[dr, dotted, "p|_H"] & & G \arrow[d, "p"] \\
\mathbb{R} & K \arrow[l, "h"'] & \tilde{H} \arrow[l, "j"'] \arrow[ll, bend left, "g"] \arrow[r, hook] & G/P \arrow[lll, bend left=50, "g^*"]
\end{tikzcd}$$

Figure 1.

Theorem 2.7. *If H is an $\mathbb{R}$-factorizable subgroup of an arbitrary topological group G, then $dimH \le dimG$.*

Proof. Let f be a continuous mapping of H to a separable metrizable space Z. Since the group H is $\mathbb{R}$-factorizable, one can find a continuous homomorphism π of H onto a second-countable group H^* and a continuous mapping $\varphi : H^* \to Z$ such that $f = \varphi \circ \pi$. By Theorem 3 of [14], the topology of the group H^* is generated by some left-invariant metric d. Evidently, d is uniformly continuous on H^*. Let $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset of H^*. For every $n \in \mathbb{N}$ define a real-valued function $\psi_n : H^* \to \mathbb{R}$ by the rule $\psi_n(x) = d(x, x_n)$, $x \in H^*$. We claim that the functions ψ_n, $n \in \mathbb{N}$, are uniformly continuous. Indeed, let $n \in \mathbb{N}$ and $\epsilon > 0$. Put $V = \{z \in H^* : d(z, e) < \epsilon\}$, where e is the identity of H^*. Then $| \psi_n(x) - \psi_n(y) | = | d(x, x_n) - d(y, x_n) | \le d(x, y)$; and if $x^{-1} \cdot y \in V$, then $d(x, y) = d(e, x^{-1} y) < \epsilon$.

One easily verifies that the diagonal product $\psi = \Delta\{\psi_n : n \in \mathbb{N}\}$ is a *homeomorphic* embedding of H^* into $R^{\mathbb{N}}$. By Lemma 2.6, for each $n \in \mathbb{N}$ the uniformly continuous function $\lambda_n = \psi_n \circ \pi|_H$ extends to a continuous function $\chi_n : G \to \mathbb{R}$, and we put $\chi = \Delta\{\chi_n : n \in \mathbb{N}\}$, the diagonal product of functions χ_n. Obviously, $\chi|_H = \psi \circ \pi|_H$. Lemma 2.4 implies that there exist a separable metrizable space P and continuous mappings $\xi : G \to P$, $\eta : P \to \mathbb{R}^{\mathbb{N}}$ such that $\chi = \eta \circ \xi$ and $dimP \leq dimG$. Put $Y = \xi(H), g = \xi|_H$ and $h = \varphi \circ \psi^{-1}|_{\psi(H^*)} \circ \eta|_Y$. The definition of h is correct because ψ is a homeomorphism of H^* onto $\psi(H^*)$. It is clear that g and h are continuous, $f = hg$ and $dimY \leq dimP \leq dimG$ (use the fact that the function dim is monotone in the realm of separable metrizable spaces, see Theorems 7.1.1 and 7.3.3 of [13]). By equivalence of the conditions (1) and (2) of Lemma 2.4, we conclude that $dimH \leq dimG$. Figure 2 clarifies the proof.

$$
\begin{tikzcd}
& H \arrow[rr, hook] \arrow[ddl, "f"'] \arrow[dd, "\pi"'] \arrow[dr, "\lambda_n"] & & G \arrow[dl, "\chi_n"'] \arrow[dd, "\chi"] \arrow[ddr, "\xi"] & \\
& & \mathbb{R}_{(n)} & & \\
Z & H^* \arrow[l, "\phi"'] \arrow[rr, "\psi"] \arrow[ur, "\psi_n"] & & \mathbb{R}^{\mathbb{N}} \arrow[ul, "pr_{(n)}"'] & P \arrow[l, "\eta"'] \\
& & & & Y \arrow[u, hook] \arrow[ullll, bend left, "h"]
\end{tikzcd}
$$

Figure 2.

We say that a topological group G is σ-precompact, if G is a union of a countable family of totally bounded subsets. By Assertion 1.15 of [29], every σ-precompact group is $\mathbb{R}$-factorizable. Hence, Theorem 2.7 implies immediately the following.

Corollary 2.8. *The inequality $dimH \leq dimG$ holds for every σ-precompact subgroup of a topological group G.*

3. Some positive results and open problems

Now we can summarize a part of results of Section 2 in the following way: $\mathbb{R}$-factorizable groups constitute a proper class in the variety of $\aleph_0$-bounded groups and, in turn, Lindelöf groups and subgroups of Lindelöf $\sum$-groups are (proper) subclasses of the class of $\mathbb{R}$-factorizable groups.

It is easy to see that every topological group with the Souslin property is $\aleph_0$-bounded [15]. Moreover, if a topological group G has countable cellularity (abbrev. G has *c.c.c.*), then for every continuous real-valued function g on G there exists a closed normal subgroup N of type G_δ in G such that g is constant on each coset on N in G [22]. The following problem arose as an attempt to improve the previous result.

Problem 3.1. Is every *c.c.c.* topological group $\mathbb{R}$-factorizable? What if a group is separable?

This problem remains non-trivial in a very concrete situation.

Problem 3.2. Let S be the Sorgenfrey line and $A(S)$ the free Abelian topological group over S. Is $A(S)$ $\mathbb{R}$-factorizable?

It is known that $c(G) \leq 2^{\aleph_0}$ for every $\aleph_0$-bounded group G [26, Th.2.8]. Hence, one may expect positive solution to the following problem.

Problem 3.3. Let g be a continuous real-valued function defined on an $\aleph_0$-bounded group G. Are there a continuous homomorphism π of G onto a group H of weight $\leq 2^{\aleph_0}$ and a continuous function h on H such that $g = h \circ \pi$?

The problem below remains open for about ten years.

Problem 3.4. [A. V. Arhangel'skiĭ]. Does every subgroup of $\mathbb{Z}^\tau$ satisfy *c.c.c.* for each cardinal τ?

Problem 3.5. Is every subgroup G of $\mathbb{Z}^\tau$ necessarily $\mathbb{R}$-factorizable?

The answer to Problem 3.5 is "yes", if G is dense in $\mathbb{Z}^\tau$. To see this, apply Corollary 1 of [6], or use Corollary 1.10 of [29]. The following problem seems to be important. Its solution will help to answer some natural questions concerning products of $\mathbb{R}$-factorizable groups.

Problem 3.6. a) Must every locally finite family of open subsets of an $\mathbb{R}$-factorizable group be countable? b) Is every $\mathbb{R}$-factorizable group G weakly Lindelöf (i.e., every open cover of G contains a countable subfamily, a union of which is dense in G)?

Here we give a positive solution to Problem 3.6, a) in the case of locally connected, $\mathbb{R}$-factorizable groups. First we need to establish an auxiliary result.

Lemma 3.7. *Let $f : G \to X$ be a continuous mapping of a locally connected, $\mathbb{R}$-factorizable group G to a separable metrizable space X. Then there exist a continuous homomorphism $\pi : G \to H$ onto a group H with a countable base $\mathcal{B}$ and a continuous mapping $h : H \to X$ such that $f = h \circ \pi$ and $\pi^{-1}(V)$ is connected for every $V \in \mathcal{B}$. In particular, H is locally connected.*

Proof. Note that the group G has a base at the identity e consisting of open, connected cozero-sets. Indeed, let U be an open neighborhood of e, and a sequence $\{U_n : n \in \mathbb{N}\}$ of open neighborhoods of e can be chosen satisfying the conditions $U_0 = U$, $U_n^{-1} = U_n$ and $U_{n+1}^2 \subseteq U_n$ for each $n \in \mathbb{N}$. Then the set $U^* = \cup\{U_1U_2 \dots U_n : n \in \mathbb{N}\}$ is a connected, open neighborhood of e, which is a union of open connected sets. Apply Theorem 8.2 of [16] to conclude that U^* is a cozero-set and $U^* \subseteq U_0 = U$.

Now we claim that for every open cozero-set U of G there exist a group H_U of countable weight, an open subset $V_U \subseteq H_U$ and a continuous homomorphism λ_U of G onto H_U such that $U = \lambda_U^{-1}(V_U)$. Indeed, let g be a continuous real-valued function on G such that $U = g^{-1}(0, +\infty)$. Since G is $\mathbb{R}$-factorizable, one can find a group H_U of countable weight, a continuous homomorphism $\lambda_U : H \to H_U$ and a continuous function $h : H_U \to \mathbb{R}$ such that $g = h \circ \lambda_U$. It remains to put $V_U = h^{-1}(0, +\infty)$.

By induction on n, define a sequence $\{\pi_n : n \in \mathbb{N}\}$ of continuous homomorphisms of G onto second-countable groups. Embed X into $\mathbb{R}^{\aleph_0}$, and find a continuous homomorphism π_0 of G onto a second-countable group H_0 and a continuous mapping $h_0 : H_0 \to X$ such that $f = h_0 \circ \pi_0$. Let a continuous homomorphism $\pi_n : G \to H_n$ be defined, $w(H_n) \leq \aleph_0$. Choose a countable base $\mathcal{C}_n$ at the identity of H_n, and put $\mathcal{B}_n = \{U_V : V \in \mathcal{C}_n\}$, where U_V is an open, connected cozero-set, and $e \in U_V \subseteq \pi_n^{-1}(V)$ for each $V \in \mathcal{C}_n$. Then define a homomorphism π_{n+1} as the diagonal product of π_n and λ_U's, $U \in \mathcal{B}_n$. Obviously, $H_{n+1} = \pi_{n+1}(G)$ is a subgroup of the product $H_n \times \Pi_{U \in \mathcal{B}_n} H_U$; hence $w(H_{n+1}) \leq \aleph_0$. Note that the equality $U = \pi_{n+1}^{-1}\pi_{n+1}(U)$ holds for each $U \in \mathcal{B}_n$.

Let π be the diagonal product of homomorphisms $\pi_n, n \in \mathbb{N}$, and $H = \pi(G)$. Then the group H is second-countable and $U = \pi^{-1}\pi(U)$ for each $U \in \mathcal{B}^* = \cup\{\mathcal{B}_n : n \in \mathbb{N}\}$. Furthermore, $\mathcal{B}^*$ consists of open connected neighborhoods of the identity e, and $\pi(U)$ is open in H for each $U \in \mathcal{B}^*$. From the construction it follows that $\mathcal{B} = \{\pi(U) : U \in \mathcal{B}^*\}$ is a base of H. Let π_0^∞ be a homomorphism of H onto H_0 such that $\pi_0 = \pi_0^\infty \circ \pi$. Then π_0^∞ is continuous and we complete the proof by putting $h = h_0 \circ \pi_0^\infty$.

Theorem 3.8. *Every locally finite family of open subsets of a locally connected* $\mathbb{R}$*-factorizable group* G *is countable.*

Proof. Suppose that there exists an uncountable locally finite family of open subsets of G. Then there exists an uncountable discrete family $\{O_\alpha : \alpha < \omega_1\}$ of non-void open subsets of G (see Lemma 1 of [22]). For every $\alpha < \omega_1$ pick a point $x_\alpha \in U_\alpha$ and define a continuous function $f_\alpha : G \to [0,1]$ such that $f_\alpha(x_\alpha) = 1$ and $f_\alpha(G \setminus O_\alpha) = \{0\}$. Then $f = \sum_{\alpha<\omega_1} f_\alpha$ is a continuous function, $0 \leq f \leq 1$. By Lemma 3.7, there exist a continuous homomorphism $\pi : G \to H$, a countable base $\mathcal{B}$ for H and a continuous function $h : H \to [0,1]$ such that $f = h \circ \pi$ and $\pi^{-1}(V)$ is connected for each $V \in \mathcal{B}$. Put $\mathcal{B} = \{V_n : n \in \mathbb{N}\}$ and $U_n = \pi^{-1}(V_n)$, $n \in \mathbb{N}$. Obviously $h(\pi(x_\alpha)) = f(x_\alpha) = 1$, so for each $\alpha < \omega_1$ there exists $n \in \mathbb{N}$ such that $h(V_n y_\alpha) \subseteq (0,1]$, where $y_\alpha = \pi(x_\alpha)$. Hence $\pi(U_n x_\alpha) \subseteq (0,1]$, $\alpha < \omega_1$. The set $U_n x_\alpha$ is connected, $x_\alpha \in U_n x_\alpha \cap O_\alpha \neq \emptyset$, and $U_n x_\alpha$ is contained in the union of mutually disjoint sets O_β's. Consequently, $U_n x_\alpha \subseteq O_\alpha$. Since the set of integers is countable, one can find $k \in \mathbb{N}$ and an uncountable set $A \subseteq \omega_1$ such that $U_k x_\alpha \subseteq O_\alpha$ for each $\alpha \in A$. Then $U_k x_\alpha \cap U_k x_\beta = \emptyset$ whenever $\alpha, \beta \in A, \alpha \neq \beta$. Let W be an open symmetric neighborhood of the identity of G such that $W^2 \subseteq U_k$. Being $\mathbb{R}$-factorizable, the group G is $\aleph_0$-bounded [29, §1]. Therefore there exists a countable subset $K \subseteq G$ such that $G = WK$. Since A is uncountable, one can find $x \in K$ and distinct $\alpha, \beta \in A$ such that $\{x_\alpha, x_\beta\} \subseteq Wx$. Then $x_\alpha x_\beta^{-1} \in W^2 \subseteq U_k$, *i.e.*, $x_\alpha \in U_k x_\beta$. It contradicts the fact that $U_k x_\alpha \cap U_k x_\beta = \emptyset$.

Let $\mathcal{G}$ be either the class of all Lindelöf groups or the class of all subgroups of Lindelöf $\sum$-groups. If $G \in \mathcal{G}$ and $\pi : G \to H$ is a continuous homomorphism of G onto a group H, then $H \in \mathcal{G}$. It is not known if the class of $\mathbb{R}$-factorizable groups has the same property.

Problem 3.9. Does a continuous homomorphic image of an $\mathbb{R}$-factorizable group inherit the property of $\mathbb{R}$-factorization? Conversely: is every $\aleph_0$-bounded group a continuous homomorphic image of an $\mathbb{R}$-factorizable group?

A special case of open homomorphisms is considered below.

Theorem 3.10. *An open homomorphic image of an $\mathbb{R}$-factorizable group is $\mathbb{R}$-factorizable.*

Proof. Let π be an open continuous homomorphism of an $\mathbb{R}$-factorizable group G onto a group H and f be a continuous real-valued function on H. Since G is $\mathbb{R}$-factorizable, there exist a continuous homomorphism φ of G onto a second-countable group P and a continuous real-valued function g on P such that $f \circ \pi = g \circ \varphi$. Choose a countable base $\{U_n : n \in \mathbb{N}\}$ at the identity of P and put $V_n = \pi\varphi^{-1}(U_n)$, $n \in \mathbb{N}$. Then every set V_n is open in H, and an easy verification shows that for every point $x \in H$ and every real $\epsilon > 0$ one can find $n \in \mathbb{N}$ such that $f(xV_n) \subseteq (f(x) - \epsilon, f(x) + \epsilon)$. Being a quotient group of G, the group H is $\aleph_0$-bounded; hence there exist a second-countable group Q with a base $\{W_n : n \in \mathbb{N}\}$ at the identity e_Q and a continuous homomorphism λ of H onto Q such that $\lambda^{-1}(W_n) \subseteq V_n$ for each $n \in \mathbb{N}$ (apply Corollary 1 of [15] characterizing $\aleph_0$-bounded groups as subgroups of products with second-countable factors). It is easy to see that there exists a function $q : Q \to \mathbb{R}$ such that $f = q \circ \lambda$. The continuity of q follows from the choice of the families $\{W_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$.

4. The product operation and $\mathbb{R}$-factorizable groups

A final dose of results and problems concentrates on the behaviour of the $\mathbb{R}$-factorization property under the product operation. A general problem is the following.

Problem 4.1. Is a product $G \times H$ $\mathbb{R}$-factorizable for every $\mathbb{R}$-factorizable group G and H?

A concrete version of the previous problem seems to be more natural.

Problem 4.2 Is a product $G \times K$ $\mathbb{R}$-factorizable for every $\mathbb{R}$-factorizable group G and a compact group K?

Here Problem 4.2 is solved under some additional assumptions. Recall that a space X is said to be $\aleph_0$-stable [5], if $nw(Y) \leq \aleph_0$ for every continuous image Y of X, which admits a continuous one-to-one mapping onto a separable metrizable space.

Theorem 4.3. *Let K be a compact group and G be an $\mathbb{R}$-factorizable group satisfying one of the following conditions:*

(1) *G has a dense σ-precompact (or Lindelöf) subset;*
(2) *G has countable cellularity;*
(3) *G is locally connected;*
(4) *G is $\aleph_0$-stable.*

Then the group $G \times K$ is $\mathbb{R}$-factorizable.

Proof. I. We consider the cases 1)-3) simultaneously. Note that if G has a dense σ-precompact subset, then G is *c.c.c.* Indeed, let $B = \cup_{n=0}^{\infty} B_n$ be a dense subset of G, where each B_n is totally bounded in G. Then the subgroup G_n of G generated by B_n has *c.c.c.* by Assertion 9 of [28]. Hence $G^* = \cup_{n=0}^{\infty} G_n$ is a dense *c.c.c.* subset of G, and so is G.

Let $f : G \times K \to \mathbb{R}$ be a continuous function. Denote by $C(K)$ the space of all continuous real-valued functions on K with the sup-norm topology and consider a continuous mapping $\Psi : G \to C(K)$ defined by $\Psi(x) = f|_{\{x\}\times K}$, $x \in G$. In each of the cases 1)-3) the image $\Psi(G)$ has countable weight. Indeed, if G is locally connected, then every locally finite family of open subsets of G is at most countable (Theorem 3.8), i.e., G is pseudo- ω_1-compact. However, G has the same property in cases 1) and 2). Thus $\Psi(G)$ is a pseudo- ω_1-compact subset of the metric space $C(K)$, whence it follows that $w\Psi(G)) \leq \aleph_0$.

Since G is $\mathbb{R}$-factorizable, one can find a continuous homomorphism π of G onto a second-countable group H and a continuous mapping $\psi : H \to C(K)$ such that $\Psi = \psi \circ \pi$. Let id_K be the identity mapping of K onto itself. We claim that the equality $(\pi \times id_K)(x,y) = (\pi \times id_K)(x_1,y_1)$ implies $f(x,y) = f(x_1,y_1)$ for any $x, x_1 \in G$ and $y, y_1 \in K$. Indeed, assume that $\pi(x) = \pi(x_1)$ and $y = y_1$, but $f(x,y) \neq f(x_1,y_1)$. Then $\Psi(x_1)(y) \neq \Psi(x)(y)$, i.e., $\Psi(x_1) \neq \Psi(x)$. Hence from the equality $\Psi = \psi \circ \pi$ it follows $\pi(x_1) \neq \pi(x)$, which is a contradiction. Thus, there exists a mapping $h : H \times K \to \mathbb{R}$ such that $h \circ (\pi \times id_K) = f$. Now we need to verify that h is continuous.

Pick a point $(t^*, y^*) \in H \times K$ and a positive real number

ϵ. Since ψ is continuous, there exists an open neighborhood O of t^* in H such that $\| \psi(t) - \psi(t^*) \| < \epsilon/2$ for each $t \in O$, i.e., $| f(x,y) - f(x^*,y) | < \epsilon/2$ whenever $\pi(x) \in O, \pi(x^*) = t^*$ and $y \in K$. There exists a neighborhood V of y^* in K such that $| f(x^*,y) - f(x^*,y^*) | < \epsilon/2$ for each $y \in V$. Let $(t,y) \in O \times V$ and $x \in G$, $\pi(x) = t$. Then we have: $| h(t,y) - h(t^*,y^*) | = | f(x,y) - f(x^*,y^*) | \leq | f(x,y) - f(x^*,y) | + | f(x^*,y) - f(x^*,y^*) | < \epsilon/2 + \epsilon/2 = \epsilon$. Consequently, h is continuous.

The group $H \times K$ is Lindelöf, being a product of the second-countable group H with the compact group K. Hence the group $H \times K$ is $\mathbb{R}$-factorizable by Assertion 1.1 of [29] (see also [28]) and one can find a continuous homomorphism λ of $H \times K$ onto a second-countable group P and a continuous function $j : P \to \mathbb{R}$ such that $h = j \circ \lambda$. Then $\pi^* = \lambda \circ (\pi \times id_K)$, a continuous homomorphism of $G \times K$ onto P, satisfies the equality $f = j \circ \pi^*$. So the group $G \times K$ is $\mathbb{R}$-factorizable.

II. Now consider the case of an $\aleph_0$-stable group G. Again, let $f : G \times K \to \mathbb{R}$ be a continuous function. Define a continuous mapping $\Phi : K \to C_p(G)$ by the rule $\Phi(y) = f\ |_{G \times \{y\}}$, where $C_p(G)$ is the space of all continuous real-valued functions on G with the pointwise convergence topology. Since G is $\aleph_0$-stable, the space $C_p(G)$ is $\aleph_0$-monolithic, i.e., every separable subspace of $C_p(G)$ has countable network ([5, Theorem 11] or [7, Theorem II.6.8]). Every compact topological group is dyadic by the result of Ivanovskiĭ and Kuz'minov (see [17,18] and also [8], where a short modern proof of the dyadicity theorem is given). Hence, $\Phi(K)$ is a dyadic $\aleph_0$-monolithic compact space. Now from [1] it follows that $\Phi(K)$ has countable weight. Since every compact group is $\mathbb{R}$-factorizable, there exist a continuous homomorphism μ of K onto a group L of countable weight and a continuous mapping $\varphi : K \to C(G)$ such that $\Phi = \varphi \circ \mu$. An argument analogous to the one applied in the part I shows that there exists a function $h : G \times L \to \mathbb{R}$ such that $f = h \circ (id_G \times \mu)$. Note that the homomorphism μ is open because K is a compact group. Hence the homomorphism $id_G \times \mu$ is open, and

h is continuous. To complete the proof, consider the mapping $\Psi : G \to C(L)$ defined in the part I, note that $C(L)$ has a countable base, and deduce the existence of a continuous homomorphism π of G onto a second-countable group H and a continuous function $j : H \times L \to \mathbb{R}$ such that $h = j \circ (\pi \times id_L)$. Then $\pi \times \mu$ is a continuous homomorphism of $G \times K$ onto the group $H \times L$ of countable weight, and $f = j \circ (\pi \times \mu)$.

Remark 4.4. a) The first part of the previous proof makes clear that a product of an arbitrary $\mathbb{R}$-factorizable group with a compact metrizable group is $\mathbb{R}$-factorizable. This follows from the fact that the weight of $C(K)$, the space of continuous real-valued functions on a compact space K, is equal to the weight of K.
b) Denote by LH the hypothesis that $2^{\aleph_0} < 2^{\aleph_1}$. With the help of LH and easy cardinal estimates one can show that every $\mathbb{R}$-factorizable group of weight $< 2^{\aleph_1}$ is pseudo-ω_1-compact. Therefore, an argument of the previous proof can be applied to get the assertion that, under LH, a product of $\mathbb{R}$-factorizable group of weight $< 2^{\aleph_1}$ with a compact group is $\mathbb{R}$-factorizable.
c) It seems to be likely that Theorem 4.3 remains valid if the second factor is assumed to be pseudocompact. However, our argument does not work in this case (Theorem 4.13 below presents some positive results in this direction).

Let us consider a case of non-compact factors. The notion of o-tightness will be useful in what follows. A space X is said to be of countable *o-tightness*, briefly, $ot(X) \leq \aleph_0$, if for every family γ of open subsets of X and each point $x \in cl(\cup\gamma)$ there exists a countable subfamily $\mu \subseteq \gamma$ such that $x \in cl(\cup\mu)$. Some properties of o-tightness are established in [25] where this notion appeared. We begin with three lemmas.

Lemma 4.5. *A product $X \times Y$ of a space X with $ot(X) \leq \aleph_0$ and a first-countable space Y has countable o-tightness.*

Proof. Let γ be a family of open subsets of the product space and $(x, y) \in cl(\cup\gamma)$. One can assume that every element of γ

has rectangular form $U \times V$. Choose a countable base $\{V_n : n \in \mathbb{N}\}$ at y in Y and put $\gamma_n = \{U \times V \in \gamma : V \subseteq V_n\}$, $n \in \mathbb{N}$. Then $(x, y) \in cl(\cup\gamma_n)$, and the condition $ot(X) \leq \aleph_0$ implies the existence of a countable subfamily $\mu_n \subseteq \gamma_n$ such that $x \in cl\pi(\cup\mu_n)$, where π is the projection of $X \times Y$ onto X. It is easy to see that the point (x, y) belongs to the closure of the union of a countable family $\mu = \cup_{n=0}^{\infty}\mu_n \subseteq \gamma$.

Lemma 4.6. *Let G and H be topological groups, and $ot(G) \leq \aleph_0$. If H is totally bounded, then $ot(G \times H) \leq \aleph_0$.*

Proof. Denote by $\widehat{H}$ the group completion of H. Since $G \times H$ is dense in $G \times \widehat{H}$, it is sufficient to show that $ot(G \times \widehat{H}) \leq \aleph_0$. Let γ be a family of open subsets of $G \times \widehat{H}$ and $(x, y) \in cl(\cup\gamma)$. We may assume that every element O of γ has the form $U \times V$, where V is an open cozero-set in H. Since the compact group $\widehat{H}$ is $\mathbb{R}$-factorizable, a cozero-set V is representable in the form $V = \pi_O^{-1}\pi_O(V)$, where π_O is a continuous homomorphism of H onto some second-countable group.

An easy inductive construction is required now. Assume that a continuous homomorphism $\lambda_n : \widehat{H} \to H_n$ is defined for some $n \in \mathbb{N}, w(H_n) \leq \aleph_0$. Lemma 4.5 implies that $ot(G \times H_n) \leq \aleph_0$. hence there exists a countable subfamily $\gamma_n \subseteq \gamma$ such that $(x, \lambda_n(y)) \in cl(\cup\{U \times \lambda_n(V) : U \times V \in \gamma_n\})$. (Use the fact that λ_n is an open homomorphism). Denote by λ_{n+1} the diagonal product of homomorphisms $\pi_O, O \in \gamma_n$, and λ_n. Obviously, the group $H_{n+1} = \lambda_{n+1}(\widehat{H})$ is second-countable.

Let λ be the diagonal product of homomorphisms λ_n, $n \in \mathbb{N}$, $H_\infty = \lambda(\widehat{H})$, and $\gamma_\infty = \cup_{n=0}^{\infty}\gamma_n$. Then for every $n \in \mathbb{N}$ there exists a continuous homomorphism λ_n^∞ of H_∞ onto H_n such that $\lambda_n = \lambda_n^\infty \circ \lambda$. Since $\widehat{H}$ is a compact group, the topology of H_∞ is initial with respect to the family of mappings λ_n^∞, $n \in \mathbb{N}$. Therefore, $(x, \lambda(y))$ is a cluster point of the family $\{\varphi(O) : O \in \gamma_\infty\}$, where $\varphi = id_G \times \lambda$. Obviously, φ is open because it is the product of two open mappings. From the construction it follows that $O = \varphi^{-1}\varphi(O)$ for each $O \in \gamma_\infty$, so $(x, y) \in cl(\cup\gamma_\infty)$. Since $\gamma_\infty \subseteq \gamma$ and $| \gamma_\infty |\leq \aleph_0$, we are done.

It is easy to see that the conclusion of Lemma 4.6 remains valid for *spaces* G and H such that $ot(G) \leq \aleph_0$ and H has a σ-lattice of d-open mappings onto first-countable spaces (the necessary definitions are given in [29]). The following result admits an analogous generalization.

Lemma 4.7. *Let G be a topological group of countable o-tightness, and $\widehat{H}$ be a completion of a pseudocompact group H. Then $G \times H$ is C-embedded into $G \times \widehat{H}$.*

Proof. Apply an argument of [25, Theorem 1]. Let $f : G \times H \to \mathbb{R}$ be a continuous function. By Π and $\widehat{\Pi}$ denote $G \times H$ and $G \times \widehat{H}$, resp. It is sufficient to show that $cl_{\widehat{\Pi}} f^{-1}(F_1) \cap cl_{\widehat{\Pi}} f^{-1}(F_2) = \emptyset$ for every pair of disjoint closed subsets F_1, F_2 of the reals (note that H meets every non-empty G_δ-subset of $\widehat{H}$, and hence the same is true for Π and $\widehat{\Pi}$). Suppose that the closures of the sets $f^{-1}(F_1)$ and $f^{-1}(F_2)$ in $\widehat{\Pi}$ are not disjoint. Choose open sets $U_i \supseteq F_i$, $i = 1, 2$, such that $clU_1 \cap clU_2 = \emptyset$. Then $cl_\Pi f^{-1}(U_1) \cap cl_\Pi f^{-1}(U_2) = \emptyset$. There exist open subsets V_1, V_2 of $\widehat{\Pi}$ such that $V_i \cap \Pi = f^{-1}(U_i)$, $i = 1, 2$. Then $cl_{\widehat{\Pi}} V_1 \cap cl_{\widehat{\Pi}} V_2 \neq \emptyset$, and we can choose a point $p = (x, y) \in \widehat{\Pi}$ from this intersection. Let γ_1 and γ_2 be families of open rectangular sets in $\widehat{\Pi}$ (as in the proof of Lemma 4.6) such that $V_i = \cup\gamma_i$, $i = 1, 2$. Then every set $O \in \gamma_1 \cup \gamma_2$ is of the form $O = U \times W$, where $W = \pi_O^{-1}\pi_O(W)$ for some continuous homomorphism π_O of $\widehat{H}$ onto a second-countable group. Since $ot(\widehat{\Pi}) \leq \aleph_0$ (Lemma 4.6), there exist countable subfamilies $\mu_i \subseteq \gamma_i (i = 1, 2)$ such that $p \in cl_{\widehat{\Pi}}(\cup\mu_1) \cap cl_{\widehat{\Pi}}(\cup\mu_2)$. Denote by π the diagonal product of homomorphisms $\pi_O, O \in \mu_1 \cup \mu_2$, and put $K = \pi(\widehat{H})$. The definition of π implies that $O = \varphi^{-1}\varphi(O)$ for each $O \in \mu_1 \cup \mu_2$, where $\varphi = id_G \times \pi$. Since H meets every G_δ-set in $\widehat{H}$, there exists a point $z \in H$ such that $\pi(z) = \pi(y)$. It is clear that φ is open, and hence the fact that $\varphi(p) = (x, \pi(z)) \in cl_{\widehat{\Pi}}\varphi(\cup\mu_1) \cap cl_{\widehat{\Pi}}\varphi(\cup\mu_2)$ implies $(x, z) \in cl_{\widehat{\Pi}}(\cup\mu_1) \cap cl_{\widehat{\Pi}}(\cup\mu_2)$. However, $\cup\mu_i \subseteq V_i$ $(i = 1, 2)$, which implies $(x, z) \in cl_{\widehat{\Pi}} V_1 \cap cl_{\widehat{\Pi}} V_2$. Since $f^{-1}(U_i)$ is dense in V_i, we have : $(x,z) \in cl_\Pi f^{-1}(U_1) \cap cl_\Pi f^{-1}(U_2) \neq \emptyset$. This contradicts the choice of U_1 and U_2.

Suppose that the topology of a group G is generated by the family of all compact subsets of G, i.e., G is a k-group. Is there any estimate of o-tightness of G? We answer this question in the affirmative.

Theorem 4.8. *$ot(G) \leq \aleph_0$ for every k-group G. Moreover, for every family $\mathcal{F}$ of G_δ-sets in G and a cluster point x of $\mathcal{F}$ there exists a countable subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that $x \in cl(\cup\mathcal{F}')$.*

Proof. We need some notions. A subgroup H of G is said to be *admissible* provided that there exists a sequence $\{V_n : n \in \mathbb{N}\}$ of open neighborhoods of the identity e_G such that $V_n^{-1} = V_n$, $V_{n+1}^3 \subseteq V_n$ for each $n \in \mathbb{N}$, and $H = \cap_{n=0}^{\infty} V_n$ (see Definition 2.19 of [27]). A G_δ-set F in G is said to be *standard* if one can find an admissible subgroup H of G and a G_δ-set Φ in the coset space G/H such that $F = \pi^{-1}(\Phi)$, where $\pi : G \to G/H$ is the quotient mapping. It is easily seen that every G_δ-subset of G is a union of standard G_δ-sets in G. We claim that $cl(\cup\mu)$ is a $G_{\delta,\Sigma}$-set in G, i.e., a union of G_δ-sets in G, for every countable family μ consisting of standard G_δ-subsets of G. Indeed, for every $F \in \mu$ choose H_F, an admissible subgroup of G, and Φ_F, a G_δ-set in G/H_F, such that $F = \pi_F^{-1}(\Phi_F)$, where $\pi_F : G \to G/H_F$ is the coset mapping. Then $H = \cap\{H_F : F \in \mu\}$ is an admissible subgroup of G, and the equality $F = \pi^{-1}\pi(F)$ holds for each $F \in \mu$, where $\pi : G \to G/H$. Note that the coset space G/H is of countable pseudocharacter, and hence every subspace of G/H is of type $G_{\delta,\Sigma}$ in G/H. Put $F^* = \cup\mu$ and $\Phi^* = \pi(F^*)$. Then $F^* = \pi^{-1}(\Phi^*)$. Since π is open, we have $cl\, F^* = \pi^{-1}(cl\Phi^*)$, which implies that $cl\, F^*$ is a $G_{\delta,\Sigma}$-set.

Let $\mathcal{F}$ be a family of G_δ-sets in G. We need to prove that the set $X = \cup\{cl(\cup\mu) : \mu \subseteq \mathcal{F}, |\mu| \leq \aleph_0\}$ is closed in G. Without the loss of generality, one can assume that every element F of $\mathcal{F}$ is standard. Choose an arbitrary compact subset B of

G and denote by $< B >$ the subgroup of G generated by B. Obviously, the group $< B >$ is σ-compact and $X \cap < B >$ is a $G_{\delta,\Sigma}$-set in $< B >$ by the assertion above. From the definition of X it follows that there exists a family $\mathcal{P}$ of G_δ-sets in $< B >$ such that $\cup\mathcal{P} = X \cap < B >$, and each element P of $\mathcal{P}$ is contained in a closure of a union of some countable subfamily $\mu_P \subseteq \mathcal{F}$. By the result of Uspenskii [31, Theorem 2], the family $\mathcal{P}$ in σ-compact group has a countable subfamily λ whose union is dense in the union of $\mathcal{P}$. Put $\mu^* = \cup\{\mu_P : P \in \lambda\}$. Then $cl(\cup\mathcal{P}) = cl(\cup\lambda) \subseteq cl(\cup\mu*) \subseteq X$, because $\mu*$ is countable. On the other hand, $X \cap < B >= \cup\mathcal{P}$, whence it follows that $X \cap < B >= cl(\cup\mu*) \cap < B >$. Thus, $X \cap < B >$ is closed in $< B >$, and $X \cap B$ is closed in B. Since G is a k-group and B is an arbitrary compact subset of G, X is closed in G.

Corollary 4.9. *A closure of every $G_{\delta,\Sigma}$-set in a k-group is a $G_{\delta,\Sigma}$-set.*

Proof. Let $\mathcal{F}$ be a family consisting of G_δ-sets in a k-group G and $P = \cup\mathcal{F}$. Every G_δ-set in G is a union of standard G_δ-sets, hence one can assume that each element of $\mathcal{F}$ is standard. Since $cl(\cup\mu)$ is a $G_{\delta,\Sigma}$-subset of G for each countable subfamily μ of $\mathcal{F}$ (see the first part of the previous proof), it remains to apply Theorem 4.8.

Corollary 4.10. *Let νG be the Hewitt realcompactification of a k-group G. Then group operations admit a continuous extension from G to νG, i.e., νG has the natural structure of a topological group.*

Proof. Apply Theorem 4.8 above and Proposition 7 of [32].

Problem 4.11. Is it possible to improve Corollary 4.9 by showing that a closure of a $G_{\delta,\Sigma}$-set in a k-group H is a G_δ-set? What if a group H is sequential or Fréchet?

Problem 4.12. Suppose G is an $\mathbb{R}$-factorizable group of countable o-tightness and K is a compact group. Is the product $G \times K$ $\mathbb{R}$-factorizable ? What if G is a k-group?

The following result is closely related with Theorem 4.3.

Theorem 4.13. *Let G be an $\mathbb{R}$-factorizable group and H a pseudocompact group. Then the group $G \times H$ is $\mathbb{R}$-factorizable in each of the following cases :*

(1) *G is a k-group with a dense Lindelöf subset;*
(2) *G has countable cellularity;*
(3) *G is an $\aleph_0$-stable group of countable o-tightness;*
(4) *G is Lindelöf.*

Proof. In each of the cases 1)-3) the group G has countable o-tightness (use Theorem 4.8), and Lemma 4.7 implies that $G \times H$ is C-embedded into $G \times \widehat{H}$, where $\widehat{H}$ is the completion of H. The conclusion follows now from Theorem 4.3.

The case 4) is more complicated. Let f be a continuous real-valued function on $G \times H$, where G is Lindelöf. Since f is continuous, for every point $y \in H$ there exists a closed G_δ-set $\Phi(y)$ in H such that $y \in \Phi(y)$ and $f(x,y) = f(x,y')$ whenever $x \in G$ and $y' \in \Phi(y)$. One can assume that $\Phi(y) = F(y) \cap H$ for every $y \in H$, where $F(y)$ is closed G_δ-set in $\widehat{H}$ (in fact, $F(y) = cl_{\widehat{H}}\Phi(y))$. Every family of G_δ-sets in a Lindelöf Σ-group (in particular, in the compact group $\widehat{H}$) contains a countable dense subfamily by Theorem 2 of [31]. Hence there exists a countable set $M \subseteq H$ such that $\cup\{F(y) : y \in M\}$ is dense in $\cup\{F(y) : y \in H\}$. The family $\{F(y) : y \in M\}$ consists of zero-sets in the compact group $\widehat{H}$, and since $\widehat{H}$ is $\mathbb{R}$-factorizable, one can find a continuous homomorphism λ_0 of $\widehat{H}$ onto a second-countable group H_0 such that $F(y) = \lambda_0^{-1}\lambda_0(F(y))$ for each $y \in M$. Put $\lambda = \lambda_0|_H$. We claim that the continuous homomorphism $id_G \times \lambda$ of $G \times H$ onto the group $G \times H_0$ factorizes the function f.

First, we need to verify that for any points (x, y_1) and (x, y_2) of the group $G \times H$ the equality $\lambda(y_1) = \lambda(y_2)$ implies $f(x, y_1) = f(x, y_2)$. Assume the contrary. Then there exist open sets $U \ni x$ and $V_i \ni y_i$ ($i = 1, 2$) such that $f(U \times V_1) \cap f(U \times V_2) = \emptyset$. The set $W = \lambda(V_1) \cap \lambda(V_2)$ is an open neighborhood of $\lambda(y_1)$ in

H_0, because λ_0 and $\lambda_0|_H$ are open homomorphism (note that $\widehat{H}$ is compact, H intersects every non-empty G_δ-subset of $\widehat{H}$, and $H_0 = \lambda_0(H)$ is first-countable). Since $\bigcup\{\Phi(y) : y \in M\}$ is dense in H, the set $W \cap \lambda(\Phi(y))$ is not empty for some $y \in M$. Note that $\Phi(y) = \lambda^{-1}\lambda(\Phi(y))$, and pick two points $z_1 \in V_1 \cap \Phi(y)$ and $z_2 \in V_2 \cap \Phi(y)$. Then $(x, z_i) \in U \times V_i$ $(i = 1, 2)$, whence it follows that $f(x, z_1) \neq f(x, z_2)$. This contradicts the choice of the set $\Phi(y)$. Thus, we proved the existence of a function $g : G \times H_0 \to \mathbb{R}$ such that $f = g \circ (id_G \times \lambda)$. Second, the continuity of g should be verified. However, this follows from the fact that $id_G \times \lambda$ is an open homomorphism.

Since H_0 is a compact second-countable group, an application of Remark 4.4, a) completes the proof.

Recall that a continuous mapping $\varphi : X \to Y$ is said to be z-closed [20,10] if $\varphi(F)$ is closed in Y for each zero-set F in X. The following result gives additional information on products of $\mathbb{R}$-factorizable groups in case one of the factors is pseudocompact.

Lemma 4.14. *Let G be a group, H be a pseudocompact group and $\widehat{H}$ the completion of H. Then the implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) hold, where*

(1) *$G \times H$ is an $\mathbb{R}$-factorizable group;*
(2) *$G \times H$ is C-embedded into $G \times \widehat{H}$;*
(3) *the projection $p : G \times H \to G$ is z-closed.*

Proof. (1) $\Rightarrow$ (2). Let $f : G \times H \to \mathbb{R}$ be continuous. By (1), one can find a second-countable group P, a continuous homomorphism φ of $G \times H$ onto P and a continuous function $g : P \to \mathbb{R}$ such that $f = g \circ \varphi$. Extend φ to a continuous homomorphism $\hat{\varphi}$ of $\widehat{G \times H}$ into $\widehat{P}$, where the symbol ^ denotes the group completion. It is clear that $G \times \widehat{H}$ is a subgroup of $\widehat{G \times H}$, and $G \times H$ meets every non-empty G_δ-set in $G \times \widehat{H}$. Since P and $\widehat{P}$ are first-countable, it follows that $\hat{\varphi}(G \times \widehat{H}) = \hat{\varphi}(G \times H) = P$. Then $\hat{f} = g \circ \hat{\varphi}|_{G \times \widehat{H}}$ is the required continuous extension of f.

(2) $\Leftrightarrow$ (3). Apply the arguments of [20].

Theorem 4.13 and Lemma 4.14 together imply the following.

Corollary 4.15. *Let G and H be a Lindelöf and a pseudocompact topological groups, resp. Then the projection $p : G \times H \to G$ is z-closed.*

A slight modification of our methods can be done to obtain another version of Theorem 4.13.

Theorem 4.16. *Let G be an $\aleph_0$-stable Lindelöf group and H a subgroup of a Lindelöf Σ-group. Then the group $G \times H$ is $\mathbb{R}$-factorizable.*

Proof. Consider a Lindelöf Σ-group N containing H as subgroup. The closed subgroup $\tilde{H} = cl_N H$ of N is a Lindelöf Σ-group. Apply an argument of the proof of Theorem 4.13 to a continuous real-valued function f on $G \times H$ in order to define an open homomorphism λ_0 of $\tilde{H}$ onto a group K_0 of countable pseudocharacter and a continuous function g on $G \times \lambda_0(H)$ such that $f = g \circ (id_G \times \lambda)$, where $\lambda = \lambda_0|_H$. Since the group K_0 is Lindelöf and $\psi(K_0) \leq \aleph_0$, Corollary 1.10 of [3] implies that there exists a continuous one-to-one mapping of K_0 onto a separable metrizable space. The group $\tilde{H}$ is $\aleph_0$-stable [6, Coro. 16], and hence $nw(K_0) \leq \aleph_0$. This implies that the group $K = \lambda_0(H) \subseteq K_0$ has countable netweight and is separable. Let S be a countable dense subset of K and i_y be the embedding of G into $G \times K$ defined by $i_y(x) = (x, y), x \in G$. Since G is $\mathbb{R}$-factorizable and $|S| \leq \aleph_0$, one can find a continuous homomorphism π of G onto a second-countable group G_0 and a family $\{\varphi_y : y \in S\}$ of continuous real-valued functions on G_0 such that $g \circ i_y = \varphi_y \circ \pi$ for each $y \in S$. We claim that for each $x_1, x_2 \in G$ and $y \in K$ the equality $\pi(x_1) = \pi(x_2)$ implies $g(x_1, y) = g(x_2, y)$.

Indeed, if $g(x_1, y) \neq g(x_2, y)$, then there exist open sets $U_i \ni x_i$ $(i = 1, 2)$ and $V \ni y$ such that $g(U_1 \times V) \cap f(U_2 \times V) = \emptyset$. Choose a point $z \in V \cap S$. Obviously, $g(x_1, z) \neq g(x_2, z)$, that contradicts the equalities $g(x_1, z) = \varphi_z(\pi(x_1)) = \varphi_z(\pi(x_2)) = g(x_2, z)$.

Thus, there exists a function $h : G_0 \times K \to \mathbb{R}$ such that $g = h \circ (\pi \times id_K)$. However, h is not necessarily continuous. Consider the quotient group $G_0^* = G/\ker \pi$ and the quotient homomorphism $\pi^* : G \to G_0^*$. Also let j be a continuous one-to-one homomorphism of G_0^* onto G_0 such that $\pi = j \circ \pi^*$. Note that the function $h^* = h \circ (j \times id_K)$ satisfies the equality $g = h^* \circ (\pi^* \times id_K)$. since π^* and $\pi^* \times id_K$ are open homomorphism, h^* is continuous. By the assumption, G is $\aleph_0$-stable, whence it follows that $nw(G_0^*) \leq \aleph_0$. Therefore, the group $G_0^* \times K$, the domain of h^*, has countable netweight. Being Lindelöf, the group $G_0^* \times K$ is $\mathbb{R}$-factorizable; hence one can find a continuous homomorphism θ of $G_0^* \times K$ onto separable metrizable group P and a continuous function $p : P \to \mathbb{R}$ such that $h^* = p \circ \theta$. Obviously, the homomorphism $\Theta = \theta \circ (\pi * \ \times \lambda)$ satisfies the equality $f = p \circ \Theta$. This completes the proof, which is illustrated by figure 3.

$$\begin{tikzcd}
G \times H \arrow[r, "\mathrm{id}_G \times \lambda"] \arrow[d, "\pi^* \times \lambda"] \arrow[ddd, bend right=60, "f"'] & G \times K \arrow[d, "\pi \times \mathrm{id}_K"'] \arrow[dddl, bend left=60, "g"] \\
G_0^* \times K \arrow[r, "j \times \mathrm{id}_K"] \arrow[d, "\theta"'] & G_0 \times K \arrow[ddl, "h"] \\
P \arrow[d, "p"'] & \\
\mathbb{R} &
\end{tikzcd}$$

Figure 3.

Corollary 4.17. *The product of a Lindelöf $\aleph_0$-stable group with a subgroup of a σ-compact group is $\mathbb{R}$-factorizable.*

Recall that X is said to be a P-space if every G_δ-set in X is open.

Corollary 4.18. *The product of a Lindelöf P-group with a totally bounded group is* $\mathbb{R}$*-factorizable.*

Proof. By Theorem 17 of [6], every Lindelöf P-group is $\aleph_0$-stable. Apply Theorem 4.16 to complete the proof.

Problem 4.19 Must the product of a Lindelöf group with a totally bounded group be $\mathbb{R}$-factorizable?

An examination of the proof of Theorem 4.16 shows that a totally bounded factor in the previous problem can be assumed second-countable.

The final comment to Theorem 4.3 seems to be useful. Call a topological group G *weakly* $\aleph_0$*-stable* if the following condition is fulfilled: given continuous homomorphism $G \xrightarrow{\pi} H \xrightarrow{\lambda} K$ with $\pi(G) = H$, $\ker \lambda = \{e_H\}$ and $w(K) \le \aleph_0$, every discrete family of open sets in H is countable, i.e., H is pseudo-ω_1-compact. The theorem below generalizes Theorem 4.3. We omit some parts of its proof duplicating earlier proofs.

Theorem 4.20. *If* G *is a weakly* $\aleph_0$*-stable,* $\mathbb{R}$*-factorizable group and* H *is a compact group, then the group* $G \times H$ *is* $\mathbb{R}$*-factorizable.*

Proof. For a given continuous function $f : G \times H \to \mathbb{R}$ consider a continuous mapping $\Psi : G \to C(H)$ defined by $\Psi(g) = f|_{\{g\}\times H}, g \in G$, where $C(H)$, the space of real-valued continuous functions on H, is endowed with the sup-norm topology. Being $\mathbb{R}$-factorizable, the group G is $\aleph_0$-bounded, and hence $c(G) \le 2^{\aleph_0}$ by Theorem 2.8 of [26]. Since $C(H)$ is metrizable, the inequalities $w(\Psi(G)) \le c(\Psi(G)) \le 2^{\aleph_0}$ hold. Now use the $\mathbb{R}$-factorizable property of G to define a continuous homomorphism ψ of G onto a group G_1 with $w(G_1) \le 2^{\aleph_0}$, and a continuous function $f_1 : G_1 \times H \to \mathbb{R}$ such that $f = f_1 \circ (\psi \times id_H)$. Let $C_P(G_1)$ be the space of continuous real-valued functions on G_1 endowed with the topology of pointwise convergence. Then $nw(C_P(G_1)) \le w(G_1) \le 2^{\aleph_0}$, by the theorem of E. Michael [19]. Define the mapping $\Phi : H \to G_P(G_1)$ by the rule $\Phi(h) = f_1|_{G_1\times\{h\}}$, $h \in H$. Obviously, Φ is continuous. Since the compact group H is $\mathbb{R}$-factorizable, one can find a continuous homomorphism φ of H onto a group H_1 with $w(H_1) \leq 2^{\aleph_0}$ and a continuous mapping $\tilde{f} : H_1 \to C_P(G_1)$ such that $\Phi = \tilde{f} \circ \varphi$ (use the equality of the weight and the netweight of a compact subset $\Phi(H)$ of $C_P(G_1)$). The mapping $\tilde{f}$ can easily be transformed to the function $f_2 : G_1 \times H_1 \to \mathbb{R}$ such that $f_1 = f_2 \circ (id_{G_1} \times \varphi)$. The homomorphisms φ and $id_{G_1} \times \varphi$ are open, and hence f_2 is continuous. Compact groups H and H_1 are dyadic by the theorem of Kuz'minov. Since $w(H_1) \leq 2^{\aleph_0}$, the group H_1 is separable (see [2]). Clearly, the continuous function $g = f_2 \circ (\psi \times id_{H_1})$ on $G \times H_1$ satisfies the equality $f = g \circ (id_G \times \varphi)$. Use the method of the proof of Theorem 4.16 along with the separability of H_1 to define an open continuous homomorphism ψ_1 of G onto a group P of countable pseudocharacter and a continuous function $p : P \times H_1 \to \mathbb{R}$ such that $g = p \circ (\psi_1 \times id_{H_1}.)$ Since P is an $\aleph_0$-bounded group of countable pseudocharacter, there exists a one-to-one continuous homomorphism of P onto a separable metrizable group. This fact and the weak $\aleph_0$-stability of G together imply that P is pseudo-ω_1-compact. To complete the proof, consider the continuous mapping $\Psi_1 : P \to C(H_1)$ defined by $\Psi_1(x) = p|_{\{x\}\times H_1}, x \in P$, and note that $w(\Psi_1(P)) \leq \aleph_0$, because $\Psi_1(P)$ is a pseudo-ω_1-compact subspace of the metrizable space $C(H_1)$. Use the $\mathbb{R}$-factorizable property of P (Theorem 3.10) to define a continuous homomorphism λ of P onto a second-countable group Q and a continuous function $q : Q \times H_1 \to \mathbb{R}$ such that $p = q \circ (\lambda \times id_{H_1})$. The product $Q \times H_1$ of the separable metrizable group Q with the compact group H_1 is Lindelöf; hence this product is $\mathbb{R}$-factorizable. Therefore, one can find a continuous homomorphism μ of $Q \times H_1$ onto a second-countable group T and a continuous function $t : T \to \mathbb{R}$ such that $q = t \circ \mu$. Obviously, the homomorphism $\pi = \mu \circ (\lambda \circ \psi_1 \times \varphi)$ of $G \times H$ onto T and the function t satisfy the equality $f = t \circ \pi$. Figure 4 clarifies the proof.

Added in Proof. Recently V. V. Uspenskiĭ solved Problem

$$\begin{tikzcd}
G \times H \arrow[r, "\psi \times \mathrm{id}_H"] \arrow[d, "\mathrm{id}_{G_1} \times \varphi"'] \arrow[dr, "f"'] & G_1 \times H \arrow[r, "\mathrm{id}_{G_1} \times \varphi"] \arrow[d, "f_1"] & G_1 \times H_1 \arrow[dl, "f_2"] \\
G \times H_1 \arrow[r, "g"] \arrow[d, "\psi_1 \times \mathrm{id}_{H_1}"'] & \mathbb{R} & \\
P \times H_1 \arrow[r, "\lambda \times \mathrm{id}_{H_1}"] \arrow[ur, "p"] & Q \times H_1 \arrow[r, "\mu"] \arrow[u, "q"'] & T \arrow[ul, "t"']
\end{tikzcd}$$

Figure 4.

3.4 by constructing a subgroup H of $\mathbb{Z}^{\omega_1}$ with $c(H) = \aleph_1$.

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