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THE SPACE OBTAINED BY SPINNING THE MENGER CURVE ABOUT INFINITELY MANY OF ITS HOLES IS NOT HOMOGENEOUS

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A continuum is a compact, connected metric space. A continuum X is homogeneous if for each x and y in X, there exists a homeomorphism $h\colon (X,x) \to (X,y)$. A continuum X is aposyndetic if for each pair of distinct points x and y in X, there is a subcontinuum S of X such that $x \in \text{int}\,(S)$ and $y \in X \setminus S$.

Let M be the Menger curve, and let B_1 be a circle embedded in one of the faces of M. Let $r\colon M \to B_1$ be a map which is the projection of M onto the face of M containing B_1, followed by a retraction of the face onto B_1. Let S be a solenoid, and let $f\colon S \to B_1$ be projection onto the first coordinate. Then the space

$$\tilde{M} = \{(x,y) \in M \times S : r(x) = f(y)\}$$

is a Case continuum [1]. This was the first known example of a homogeneous continuum which is aposyndetic but not locally connected [2], [3].

For another construction, let B_1 be a bouquet of n circles embedded in a face of M. Let T_n be the n-dimensional torus, and let 1 represent the identity element of S^1. Note that there is a natural embedding of B_1 into T_n, which identifies B_1 with the set $\{(x_1,\dots,x_n)\colon x_i = 1$ except for at most one coordinate $\}$. Let $r\colon M \to T_n$ be the projection of M onto the face containing B_1, followed by a retraction onto B_1, followed by the natural embedding of B_1 into T_n. Let $S = \prod_{i=1}^{n} S_i$ be a product of n solenoids. Let $p_i\colon S \to S_i$ be projection, and let $f_{1,i}\colon S_i \to S^1$

be projection onto the first coordinate. Let $f\colon S \to T_n$ be the map $\prod_{i=1}^{n} f_{1,i} \circ p_i$. If $\tilde{M} = \{(x,y) \in M \times S : r(x) = f(y)\}$, then $\tilde{M}$ is a homogeneous continuum which is aposyndetic, but not locally connected [4].

A question implied in [4] was whether $\tilde{M}$ would still be a homogeneous continuum which is aposyndetic, but not locally connected, if we replace S with an infinite product of solenoids, and B_1 with a Hawaiian earring—a union of a sequence of circles with decreasing diameters, such that there is a point which is equal to the intersection of every pair of the circles. We will show that, in this case, $\tilde{M}$ is not homogeneous.

Theorem . *Let B_1 be a Hawaiian earring embedded in a face of the Menger curve M. Let $r\colon M \to T$ be the projection of M onto the face of M containing B_1, followed by a retraction onto B_1, followed by the natural embedding into the infinite dimensional torus T. Let $S = \prod_{n=1}^{\infty} S_n$ be a product of solenoids. Let $p_n\colon S \to S_n$ be projection, and $f_{1,n}\colon S_n \to S^1$ be projection onto the first coordinate. Let $f\colon S \to T$, where $f = \prod_{n=1}^{\infty} f_{1,n} \circ p_n$. Then the space*

$$\tilde{M} = \{(x,y) \in M \times S : r(x) = f(y)\}$$

is not homogeneous.

Proof. We need some more notation. Let $v_n \in S_n$ be $(1,1,1,\dots)$ and let $v \in S$ be $(v_1, v_2, v_3, \dots)$. Let $\hat{v} \in T$ be $f(v)$. Note that r maps the point of intersection of the circles of B_1 onto $\hat{v}$. Let $G_n = f_{1,n}^{-1}(1)$, and let $G = f^{-1}(\hat{v}) = \prod_{n=1}^{\infty} G_n$. Let $\pi_1\colon \tilde{M} \to M$ and $\pi_2\colon \tilde{M} \to S$ be projections. Also, let $q_i\colon T \to S^1$ be projection onto the ith coordinate. We will first prove a lemma.

Lemma. *If $(m,s) \in \tilde{M}$ such that $r(m) \neq \hat{v}$, then there exists an open neighborhood U of m in M and a homeomorphism*

$$h\colon \pi_1^{-1}(U) \to U \times G,$$

such that $\pi_1 \circ h^{-1}\colon U \times G \to U$ is projection onto the first coordinate. Furthermore, we can choose U so that it is connected.

Proof. Since $f(s) = r(m) \neq \hat{v}$, $f(s)$ lies on exactly one of the circles of B_1, so, for some n,

$$f(s) \in \prod_{k=1}^{n-1} \{1\} \times (S^1 - \{1\}) \times \prod_{k=n+1}^{\infty} \{1\}.$$

Since S_n is a fiber bundle over S^1 with projection $f_{1,n}$ and fiber G_n, there exists an open neighborhood U' of $f_{1,n}(p_n(s))$ in S^1 and a homeomorphism

$$h' \colon f_{1,n}^{-1}(U') \to U' \times G_n,$$

such that $f_{1,n} \circ (h')^{-1} \colon U' \times G_n \to U'$ is projection onto the first coordinate. Since $f_{1,n}(p_n(s)) \neq 1$, we can choose U' so that $1 \notin U'$. Let $U = r^{-1}(q_n^{-1}(U'))$. Then U is an open neighborhood of m in M. We will show that $\pi_1^{-1}(U)$ is homeomorphic to $U \times G$.

Note that

$$\pi_1^{-1}(U) = \bigcup_{x \in U} [\{x\} \times \prod_{k=1}^{n-1} G_k \times f_{1,n}^{-1}(q_n(r(x))) \times \prod_{k=n+1}^{\infty} G_k].$$

Define $h \colon \pi_1^{-1}(U) \to U \times G$ as follows: If $(x, (s_1, s_2, \dots)) \in \pi_1^{-1}(U)$, let

$$h((x, (s_1, s_2, \dots))) = (x, (s_1, s_2, \dots, s_{n-1}, \rho_2(h'(s_n)), s_{n+1}, \dots))$$

where $\rho_2 \colon U' \times G_n \to G_n$ is the projection. Since $f((s_1, s_2, \dots)) = r(x)$, then $f_{1,n}(s_n) = q_n(r(x))$. Since $x \in U$, $q_n(r(x)) \in U'$, so s_n belongs to the domain of h'. Then $h \colon \pi_1^{-1}(U) \to U \times G$ is a continuous map.

If $h((x, (s_1, s_2, \dots))) = h((x', (s_1', s_2', \dots)))$, then $x = x'$ and $s_k = s_k'$ for $k \neq n$. Also, $\rho_2(h'(s_n)) = \rho_2(h'(s_n'))$. Let $\rho_1 \colon U' \times G_n \to U'$ be projection. Then

$$\rho_1(h'(s_n)) = (f_{1,n} \circ (h')^{-1})(h'(s_n)) = f_{1,n}(s_n)$$

and, likewise, $\rho_1(h'(s_n')) = f_{1,n}(s_n')$. But we must have $f_{1,n}(s_n) = q_n(r(x))$ and $f_{1,n}(s_n') = q_n(r(x')) = q_n(r(x))$, so $f_{1,n}(s_n) = f_{1,n}(s_n')$. Then $h'(s_n) = h'(s_n')$. Since h' is a homeomorphism, $s_n = s_n'$. Then h is an injection.

Let $(x,(g_1,g_2,\dots)) \in U \times G$. Then $x \in r^{-1}(q_n^{-1}(U'))$, so $(q_n(r(x)), g_n) \in U' \times G_n$. Then there exists an $s_n \in f_{1,n}^{-1}(U')$ such that $h'(s_n) = (q_n(r(x)), g_n)$. Then

$$\begin{aligned} q_n(r(x)) &= \rho_1((q_n(r(x)), g_n)) \\ &= (f_{1,n} \circ (h')^{-1})((q_n(r(x)), g_n)) = f_{1,n}(s_n). \end{aligned}$$

Also, since $q_n(r(x)) \in U'$ and $1 \notin U'$, $r(x)$ lies on exactly one of the circles of B_1 and $q_i(r(x)) = 1$ for $i \neq 1$. Then

$$\begin{aligned} r(x) &= (1,1,\dots, f_{1,n}(s_n), 1, 1, \dots) \\ &= f(g_1, g_2, \dots, g_{n-1}, s_n, g_{n+1}, \dots), \text{ and} \\ &(x,(g_1,g_2,\dots,g_{n-1},s_n,g_{n+1}\dots)) \in \pi_1^{-1}(U). \end{aligned}$$

We have

$$h((x,(g_1,g_2,\dots,g_{n-1},s_n,g_{n+1},\dots))) =$$

$$(x,(g_1,g_2,\dots,g_{n-1},g_n,g_{n+1},\dots)).$$

Then h is a surjection. If $(x,(g_1,g_2,\dots)) \in U \times G$, then

$$h^{-1}((x,(g_1,g_2,\dots))) =$$

$$(x,(g_1,g_2,\dots,g_{n-1},(h')^{-1}((q_n(r(x)),g_n)),g_{n+1},\dots)),$$

which is continuous, so h is a homeomorphism. Also, note that $\pi_1 \circ h^{-1}$ is projection onto the first coordinate, and the first part of the lemma is proved.

Since M is locally connected, we can replace U with the component of U containing m, proving the last assertion in the lemma.

Now we are ready to prove the theorem. Let $(m,s) \in \tilde{M}$ such that $r(m) \neq \hat{v}$, and let U be as in the lemma. Then the components of $\pi_1^{-1}(U)$ are each homeomorphic to $U \times \{g\}$ for some $g \in G$, so they are locally connected.

Suppose $\tilde{M}$ is homogeneous. We will not distinguish between the points of B_1 embedded in M and the points of B_1 embedded in T, so we will denote the point of intersection of the circles of B_1 in M by $\hat{v}$. There must be some neighborhood W of $(\hat{v}, v)$

in $\tilde{M}$ homeomorphic to $\pi_1^{-1}(U)$. Let K be the component of W containing $(\hat{v}, v)$. Then K is locally connected.

W contains some basic open set $(W_0 \times \prod_{k=1}^{n} W_k \times \prod_{k=n+1}^{\infty} S_k) \cap \tilde{M}$, where $\hat{v} \in W_0$, open in M, and $v_k \in W_k$, open in S_k for $k = 1, \dots, n$. W_0 must contain all but finitely many of the circles of B_1, so there exists an $m > n$ such that the mth circle of B_1 is contained in W_0. Then

$$B = \prod_{k=1}^{m-1} \{v_k\} \times S_m \times \prod_{k=m+1}^{\infty} \{v_k\} \subset \pi_2(W).$$

B is connected since it is homeomorphic to S_m. Note that we can think of $f \mid \pi_2(\tilde{M})\colon \pi_2(\tilde{M}) \to B_1$ as a map into M since B_1 is embedded in M. Let $B' = \{(f(b), b)\colon b \in B\}$. Since r restricted to B_1 is the identity, $r(f(b)) = f(b)$, and $B' \subset \tilde{M}$. In fact, $(\hat{v}, v) \in B' \subset W$. B' is connected, since it is a continuous image of B, so $B' \subset K$. Then $p_m(\pi_2(K)) = S_m$.

Let A be an open arc of S^1 containing 1. Then

$$E = \pi_2^{-1}(p_m^{-1}(f_{1,m}^{-1}(A))) \cap K$$

is an open subset of K containing $(\hat{v}, v)$. Let C be the component of E containing $(\hat{v}, v)$. C must be open in K, since K is locally connected. Then C contains some basic open set

$$L = (L_0 \times \prod_{k=1}^{\infty} L_k) \cap K,$$

where L_0 is open in M, the L_k are open in S_k, and $(\hat{v}, v) \in L$. Note that since $p_m(\pi_2(K)) = S_m$, $p_m(\pi_2(L)) = L_m$. Since $p_m(\pi_2(C))$ is a connected set contained in $f_{1,m}^{-1}(A)$, $p_m(\pi_2(C))$ must be contained in a component of $f_{1,m}^{-1}(A)$. Then L_m must be contained in this component. But all components of $f_{1,m}^{-1}(A)$ in S_m must have empty interior. Therefore, $\tilde{M}$ cannot be homogeneous.

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