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A NOTE ON THE CLUSTER POINTS OF SEQUENCES OF SUCCESIVE APPROXIMATIONS

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Abstract. Suppose Y is a compact metric space and $T : Y \to Y$ is continuous. Let $\mathcal{L}$ denote the set of cluster points of $\{T^n x_0\}$, $x_0 \in Y$, and let $\mathcal{L}'$ denote the derived set of $\mathcal{L}$. We consider questions, raised in an earlier note, about the cardinality of $\mathcal{L} \setminus \mathcal{L}'$ and $T\mathcal{L}' \setminus \mathcal{L}'$, and discuss briefly the (non)existence of fixed points of T.

Let Y be a compact metric space, let $T : Y \to Y$ be continuous and let $x_0 \in Y$. For $n = 0, 1, 2, \ldots$, set $x_n = T^n(x_0)$, let $s = \{x_n\}_{n=0}^{\infty}$ and let $X = \mathrm{cl}(s)$. For any $A \subset Y$, we will denote the derived set of A by A'. Following [1], which followed [2], we write $\mathcal{L}$ for X'. Assume s is infinite. Then $\mathcal{L} \neq \emptyset$, $T\mathcal{L} = \mathcal{L}$ and $\mathcal{L}' \subset T\mathcal{L}'$. It was shown in [1] that $\mathcal{L} \setminus \mathcal{L}'$ is countable (more generally, of course, if Z is a second countable, Hausdorff space and $A \subset Z$ then $A \setminus A'$ is countable); necessary and sufficient conditions for $T\mathcal{L}' = \mathcal{L}'$ were given; it was shown that one does not always have $T\mathcal{L}' = \mathcal{L}'$; and it was also shown that $\mathcal{L}'$ need not consist entirely of fixed points, even that $\mathcal{L}'$ need not have a single fixed point. This answered two questions raised in [2]. The authors of [1] then asked the following questions.

(1) Suppose $\mathcal{L} \setminus \mathcal{L}' \neq \emptyset$. Is $\mathcal{L} \setminus \mathcal{L}'$ infinite? Does $\mathcal{L}'$ contain a fixed point of T?

(2) What are necessary and sufficient conditions that $\mathcal{L}'$ consist entirely of fixed points of T?

(3) Can $T\mathcal{L}' \setminus \mathcal{L}'$ be infinite?

The answer to both questions in (1) is "not necessarily." We will give a sufficient condition for $\mathcal{L}\backslash\mathcal{L}'$ to be infinite. Question (2) seems quite difficult, and we will not say too much about it. The answer to (3) is "yes."

Our approach will be to begin with a set and a map having the desired properties of T, $\mathcal{L}$ and $\mathcal{L}'$ and work backwards to obtain $\{x_n\}$. Naturally, most of our answers involve the construction of examples, even example schemas. Before proceeding, we give some terminology and technical results which will make the construction of our examples more efficient. From here on, A denotes a compact metric space, $Z = A \times \boldsymbol{I}$ ($\boldsymbol{I} = [0,1]$) and $g : A \to A$ is a continuous surjection. Suppose $\boldsymbol{s} = \{x_n\}_{n=0}^{\infty} \subset Z$ such that (i) $A = \boldsymbol{s}'$, (ii) the function T_0 defined by $T_0 x_n = x_{n+1}$ is continuous on $\boldsymbol{s}$ and (iii) T_0 has a continuous extension T to $\mathrm{cl}(\boldsymbol{s})$ such that $T|A = g$. Then $\boldsymbol{s}$ will be called a *generating sequence for* g. Thus, if a generating sequence exists, we have $\mathcal{L} = A$ and $g = T|\mathcal{L}$, whence many questions about T, $\mathcal{L}$ and $\mathcal{L}'$ become questions about g, A and A'. Given a sequence $\boldsymbol{s} = \{x_n\}_{n=0}^{\infty}$, set $\boldsymbol{s}_0 = \{x_{2n}\}_{n=0}^{\infty}$ and $\boldsymbol{s}_1 = \{x_{2n+1}\}_{n=0}^{\infty}$. If each of $\boldsymbol{s}_0$ and $\boldsymbol{s}_1$ is a generating sequence for g, we will say $\boldsymbol{s}$ can be *split.*

Suppose A is connected and D is a countable dense subset of A. It is easy to obtain a sequence $\{p_n\} \subset A$ such that the distance between succeeding terms goes to zero and such that each element of D occurs infinitely often in the sequence. Then $\{(p_n, 1/n)\}$ is a generating sequence for the identity on A. The reader may easily convince himerself that any such generating sequence can be split. On the other hand, suppose A is the disjoint union of two open sets U and V and $g|U$ is the identity on U. Suppose, further, that $\boldsymbol{s}$ is a generating sequence for g. Then every subsequence of $\boldsymbol{s}$ in $U \times \boldsymbol{I}$ must have all of its cluster points in U, which means that $\boldsymbol{s} \cap (V \times \boldsymbol{I})$ is finite. Therefore, no such sequence exists. It follows that if a generating sequence for g exists and p is a fixed point of g then $p \in A'$. (We will say more about this later.) However, Example 3 will show that g may have any finite number of fixed points.

Now suppose there is a generating sequence for $g : A \to A$, suppose $q : A \to B$ is continuous and suppose there is a continuous map $f : B \to B$ such that $q \circ g = f \circ q$. Let $\bar{q}$ be the extension of q to Z where $\bar{q}(a, b) = (q(a), b)$. Then if $\{x_i\}$ is a generating sequence for g, $\{\bar{q}(x_i)\}$ is a generating sequence for f.

Define a transitive relation $\leq$ on A by $a \leq b \equiv$ for some n, $g^n(b) = a$. Let $\mathcal{C}$ be the set of all maximal $\leq$-chains. Note that if $c \in C \in \mathcal{C}$ then C is the union of $\{g^n(c)\}_{n=0}^{\infty}$ and a maximal comparable subset of $\{g^{-n}(c)\}_{n=1}^{\infty}$. We will use C to denote both the chain C and its underlying set in A. Thus, if C (as the underlying set) has a single cluster point p, we may write $C \to p$. Consider the following properties:

(1) For all $a \in A$, there is an n such that $g^n(a)$ is a fixed point of g.

(2) Every $\leq$-chain is well ordered.

(3) There are no nontrivial cycles. (A *cycle* is a finite subset $\{a_0, \ldots, a_{n-1}\}$ of A such that $g(a_i) = a_{i+1(\text{mod n})}$ for all i. A cycle is nontrivial if necessarily $n \geq 2$.)

(4) $\leq$ is a partial order.

Obviously, (1) $\iff$ (2) $\implies$ (3) $\iff$ (4), and (4) $\not\Longrightarrow$ (2). Also observe that if $C \in \mathcal{C}$ then $g|C$ is one-to-one.

Lemma. *Suppose (2) holds, and let $C \in \mathcal{C}$. Then C converges to a fixed point of g.*

Proof. Observe that for all $C \in \mathcal{C}$, either C is infinite or $C = \{a\}$ where a is a fixed point. Suppose C is infinite. Let x be a cluster point of C. By hypothesis, there is an n such that $p = g^n(x)$ is a fixed point of g. Let $\{x_i\}$ be a sequence of distinct points of C converging to x. Since $g|C$ is one-to-one, $\{g^n(x_i)\}$ is a sequence of distinct points of C converging to p. Suppose that necessarily, $n \geq 1$. Let U be an open neighborhood of p missing $(g^{-1}(p) \cap C) \setminus \{p\}$. There is a sequence $\{p_j\}$, cofinal in $\{g^n(x_i)\}$, such that for all j, $g(p_j) \notin U$. But then $g(p) \notin U$

which contradicts the fact that p is a fixed point. Therefore $n = 0$ and p is the only cluster point of C.

In general, the point to which C converges need not be in C. Let A be the union of the harmonic sequence and $\{0\}$ and define g by $g(0) = 0$, $g(1) = 1$ and for $n = 2, 3, \dots$, $g(1/n) = 1/(n-1)$. If C is the maximal chain whose least element is 1, then $C \to 0$ but $0 \notin C$. Note that there is no generating sequence for this function, however. Naturally, we may redefine g by setting $g(1) = 0$.

Corollary. *Suppose, in addition to the hypotheses of the lemma, that g has exactly one fixed point and C is countable. Then there is a generating sequence for g.*

Proof. Well order C as $\{C_i\}$ and for all i, let $C_i = \{a_k(i)\}_{k=0}^{\infty}$ such that for $k \geq 1$, $g(a_k(i)) = a_{k-1}(i)$. For all $n, i \geq 1$, let $s(i,n)$ denote the string $a_n(i), a_{n-1}(i), \dots, a_0(i)$. Let $\{x_k\}$ denote the sequence which arises from concatenating strings as follows:

$$s(1,1) \cdot s(2,2) \cdot s(1,3) \cdot s(2,4) \cdot s(3,5) \cdot s(1,6) \cdot s(2,7) \cdot \cdots .$$

Then $\{(x_k, 1/k)\}$ is a generating sequence for g.

Theorem 3 will show that, under the hypotheses of the lemma, the fixed point must be unique.

We now turn to question (1).

Example 1. Let $\boldsymbol{s} = \{x_i\}$ be a generating sequence for the identity on $\boldsymbol{I}$ and let $A = \boldsymbol{s} \cup \boldsymbol{I}$. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of copies of A such that for all i and j, A_i' and A_j' are coplanar and the A_n's converge to a single point p_0. Let $\{p_i\}_{i=1}^{\infty}$ be a sequence of distinct points in the plane containing the arcs where $p_i \to p_0$. For $n = 1, 2, \dots$, let $h_n : A_{n+1} \to A_n$ be a homeomorphism which preserves order in each coordinate, *i.e.*, h_n merely scales up A_{n+1}. Let $\mathcal{L}_0 = \{\cup A'_n\} \cup \{p_0\}$, and for $k \geq 1$, set $\mathcal{L}_k = \mathcal{L}_{k-1} \cup \{p_k\}$. Define $g_k : \mathcal{L}_k \to \mathcal{L}_k$ by

$$g_k(x) = \begin{cases} h_n(x) & \text{if } x \in A'_{n+1},\ n \geq 1 \\ p_1 & \text{if } x \in A'_1 \\ p_{n+1} & \text{if } x = p_n,\ 1 \leq n < k \\ p_0 & \text{if } x = p_0 \text{ or } p_k. \end{cases}$$

This is shown schematically below.

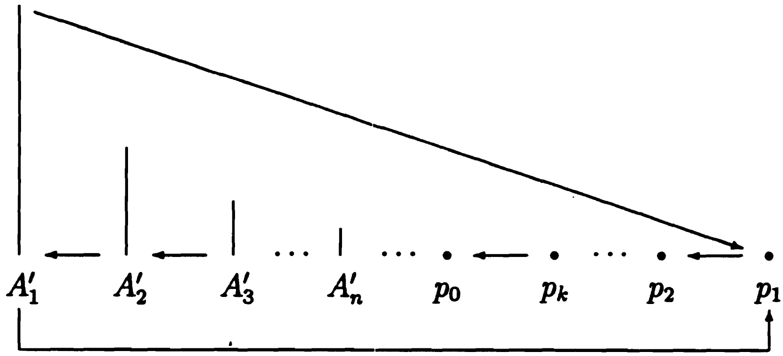

Figure 1

Denote the generating sequence for the identity on A'_n by $\{x_i(n)\}$ and let $y_i(n) = (p_n, 1/(n+i))$. The sequence $x_1(1), y_2(1), \ldots,$ $y_{k+1}(k), x_{k+2}(2), x_{k+3}(1), y_{k+4}(1), \ldots, y_{2k+4}(k), x_{2k+5}(3), \ldots,$ is a generating sequence for g_k. It is not too hard to see that if we set $\mathcal{L}_\infty = \cup \mathcal{L}_k$ with $g = \lim g_k$ then we can obtain a generating sequence for $\mathcal{L}_\infty$. Note that $\mathcal{L}_k \setminus \mathcal{L}'_k = \{p_1, \ldots, p_k\}$. This schema gives proof to our first theorem.

Theorem 1. *$\mathcal{L} \setminus \mathcal{L}'$ may have any cardinality Λ where $0 \leq \Lambda \leq \aleph_0$.*

Theorem 2. *Suppose $\mathcal{L}' \neq \emptyset$ and T is at most $\aleph_0$-to-1 . Then $\mathcal{L} \setminus \mathcal{L}'$ is either empty or infinite.*

Proof. Suppose $\mathcal{L} \setminus \mathcal{L}' = \{p_1, \ldots, p_m\}$. Then there exist pairwise disjoint open sets $U, V_1, \ldots, V_m$ with $\mathcal{L}' \subset U$ and $p_j \in V_j$ for $j = 1, \ldots, m$ and there exists a subsequence $\{x^j_n\} \subset V_j$ which converges to p_j for $j = 1, \ldots, m$. For some

j, there must be a subsequence $\{x^j_{n_k}\}$, each point of which is mapped to U. It follows that $Tp_j \in U$. Since $T\mathcal{L} = \mathcal{L}$, there must be an $x \in \mathcal{L} \setminus \mathcal{L}'$ such that $Tx = p_i$ for some i. Then there is an open neighborhood N of x, with $N \subset U$, such that $TN \subset V_i$. By Theorem 3 of [1], $\mathcal{L}'$ is perfect which means that $N \cap \mathcal{L}'$ is uncountable. Since T is at most $\aleph_0$-to-1 and $T(N \cap \mathcal{L}') \subset V_i \cap \mathcal{L}$, $V_i \cap \mathcal{L}$ is uncountable. But $V_i \cap \mathcal{L}$ is countable.

Suppose g has a generating sequence $\{x_i\}$ that can be split. Let $\boldsymbol{Y}$ be the disjoint union $Y \cup \tilde{Y}$ where $\tilde{Y}$ is a copy of Y. Given any point or set γ of Y, $\tilde{\gamma}$ will denote the corresponding point or set in $\tilde{Y}$. Let $S : \boldsymbol{Y} \to \boldsymbol{Y}$ be the map which switches x and $\tilde{x}$. The set of cluster points of $\{(ST)^n x_0\}$ is $\mathcal{L} \cup \tilde{\mathcal{L}}$ and $(\mathcal{L} \cup \tilde{\mathcal{L}})' = \mathcal{L}' \cup \tilde{\mathcal{L}}'$. However, there is no fixed point of ST. This leads one to ask, If $\mathcal{L} \setminus \mathcal{L}' \neq \emptyset$ *and* $\mathcal{L}'$ is connected, does $\mathcal{L}'$ contain a fixed point of T? Again, the answer is "not necessarily" as the next example shows.

Example 2. For convenience, we employ complex notation. For all n, let $a_n = 1 + 1/n$, let $S^1 = \{z : |z| = 1\}$ and set $A = S^1 \cup \{a_n\}_{n=1}^{\infty} \cup \{-a_n\}_{n=1}^{\infty}$. Define g by

$$g(z) = \begin{cases} -z & \text{if } z \in S^1 \\ a_n & \text{if } z = a_n, n \geq 2 \\ -a_n & \text{if } z = -a_n, n \geq 2 \\ 1 & \text{if } z = -a_1 \\ -1 & \text{if } z = a_1 \\ -z & \text{otherwise.} \end{cases}$$

For even n let s_n denote the string $\{z_0, \ldots, z_n\} \subset S^1$ where $z_0 = z_n = -1$ if $4|n$, and $z_0 = z_n = 1$ if $4 \nmid n$; and let t_n be the string $a_n, -a_{n-1}, \ldots, a_2, -a_1$ if $4|n$ and $-a_n, \ldots, a_1$ otherwise. Let $\{z_i\}$ be the sequence formed by concatenating strings as follows: $s_2 \cdot t_2 \cdot s_4 \cdot t_4 \cdot s_6 \cdot t_6 \cdot \cdots$. For all $n \geq 1$, set $x_n = (z_n, 1/n) \in Z$. Then $\{x_n\}$ is a generating sequence for g.

Hence, if $\mathcal{L}'$ is a continuum, it may consist entirely of fixed points, have no fixed points or have some, but not all, of its points be fixed points. The same trichotomy occurs if $\mathcal{L}'$ is not connected. This leads to question (2). We note that if $\mathcal{L}' \neq \emptyset$ and $x \in X$ is a fixed point of T then $x \in \mathcal{L}'$. The obvious sufficeint condition for $\mathcal{L}'$ to consist entirely of fixed points is that $\mathcal{L}'$ be a singleton. Question (Q2) at the end of this note suggests another condition.

Example 3. Let $A = \{(x, y)|y = 0, 1; 0 \leq x \leq 1\} \cup \{(x, y)|x = 0, 1; y = 1/n, 1 - 1/n \text{ for } n = 2, 3, \ldots\}$ Define g by

$$g(x,y) = \begin{cases} (0, 1/(n-1)) & \text{if } y = 1/n, n \geq 3 \\ (0, 1 - 1/(n+1)) & \text{if } y = 1 - 1/n, n \geq 2 \\ (1, 1 - 1/(n-1)) & \text{if } y = 1 - 1/n, n \geq 3 \\ (1, 1/(n+1)) & \text{if } y = 1/n, n \geq 2 \\ (x, y) & \text{otherwise.} \end{cases}$$

For all n, we define the following finite sequences:

$$\begin{aligned} F_{n,1} &= \langle (k/2^n, 1, 1/n) \rangle_{k=0}^{2^n} \\ F_{n,2} &= \langle (1, 1 - 1/k, 1/n) \rangle k = n^2 \\ F_{n,3} &= \langle (1, 1/k, 1/n) \rangle_{k=3}^{n} \\ F_{n,4} &= \langle (1 - k/2^n, 0, 1/n) \rangle_{k=0}^{n} \\ F_{n,5} &= \langle (0, 1/k, 1/n) \rangle_{k=n}^{2} \\ F_{n,6} &= \langle (0, 1 - 1/k, 1/n) \rangle_{k=3}^{n} \end{aligned}$$

Note that in $F_{n,2}$ and $F_{n,5}$ the sequence runs "backwards". Define S_n by concatenating the $F_{n,i}$'s in order, *i.e.*, $S_n = F_{n,1} \cdot F_{n,2} \cdot F_{n,3} \cdot F_{n,4} \cdot F_{n,5} \cdot F_{n,6}$. Let $s = S_1 \cdot S_2 \cdot S_3 \cdot \cdots$.

We may easily modify the example in such a way that A' consists of any finite number of arcs and $g|A'$ is the identity on A'. Furthermore, we may replace any of the arcs with points.

Example 3 shows that if $\mathcal{L}'$ has a finite number of components then it may consist entirely of fixed points. Is this condition necessary for $\mathcal{L}'$ to consist entirely of fixed points? I do not know, and again, refer the reader to (Q2). Obviously, if card($\mathcal{L}'$) ≥ 2 then one necessary condition is that $\cap\mathcal{C} = \emptyset$. Although we have nothing more to say about question (2), we do have more to say about $\cap\mathcal{C}$.

Theorem 3. *Suppose $g : A \to A$ contains no nontrivial cycles. Then if $\cap\mathcal{C} \neq \emptyset$, g has exactly one fixed point.*

Proof. Note that A can not be finite. Note, too, that there is at most one fixed point. Now, if $\cap\mathcal{C}$ has no upper bound then there can be only one maximal chain. If there were only one maximal chain and there were no least element, then g would be one-to-one and every point of A would be a cluster point of A, which is impossible since $\mathcal{C}$ is countable. Thus $\cap\mathcal{C}$ has a least element which must be the one and only fixed point. Suppose $\cap\mathcal{C}$ has a least upper bound, say a. Then $\cap\mathcal{C} = \{g^n(a)\}_{n=0}^{\infty}$. Suppose $\cap\mathcal{C}$ is infinite. Let p be a cluster point of $\cap\mathcal{C}$. For some m, $g^m(p) \in \cap\mathcal{C}$, and for all $n \geq 0$, $g^{m+n}(p)$ is a cluster point of $\cap\mathcal{C}$ contained in $\cap\mathcal{C}$. But again, this implies that A is uncountable. It follows that $\cap\mathcal{C}$ has a least element and hence, g has a fixed point.

Finally we show that $T\mathcal{L}' \setminus \mathcal{L}'$ can be infinite by modifying the first example given in [1].

Example 4. Let (a_n) be a strictly decreasing sequence with limit 0 and $a_1 = 1$. For each n, let $(b_{n,k})_k$ be a strictly increasing sequence with limit a_n such that $b_{n,1} > a_{n+1}$. For $n \geq 2$ and for all k, let $(c_{n,k,j})_j$ be a strictly increasing sequence with limit $b_{n,k}$, where $c_{n,1,1} > a_{n+1}$ and for $k \geq 2$, $c_{n,k,1} > b_{n,k+1}$. Let $A = \{0,\ a_n,\ b_{n,k},\ c_{n,k,j} | n, k, j \geq 1\}$. Define $g : A \to A$ by $g^{-1}(0) = \{0, 1, b_{1,k} | k \geq 1\}$; $g^{-1}(b_{1,k}) = \{b_{2,k}, c_{2,k,j} | j \geq 1\}$ for all k; for $n \geq 2$ and all k, $g(a_n) = a_{n-1}$ and $g(b_{n,k}) = b_{n-1,k}$; and for $n \geq 3$, for all k, for all j, $g(c_{n,k,j}) = c_{n-1,k,j}$. Now, $g(A') \setminus A' = \{b_{1,k} | k \geq 1\}$ is infinite. Easily, $\mathcal{C}$ is countable and $\cap\mathcal{C} = \{0\}$. Therefore, there is a generating sequence for g. Hence, $T\mathcal{L}' \setminus \mathcal{L}'$ can be infinite. Notice that g is at most $\aleph_0$-to-1.

We pose two questions.

(Q1) When do generating sequences exist?

(Q2) Assume A is countable. Suppose a generating sequence exists for g and suppose g has a fixed point p. Let A^{α} denote the α^{th} derived set of A and let α_0 be the least ordinal such that A^{α_0} is finite. Is $p \in A^{\alpha_0}$? (If so, then if card$(\mathcal{L}') = \omega$, $\mathcal{L}'$ can not consist entirely of fixed points.) More generally, is $A^{\alpha} \subset g(A^{\alpha})$ for all α?

I wish to thank Jim Solomon for calling these problems to my attention, and I wish to thank David Bellamy for an illuminating discussion about fixed points, derived sets and countable ordinals. Finally, we point out that today $\mathcal{L}$ would most commonly be referred to as the ω-limit set of x (under T).

REFERENCES

1. L. F. Guseman, Jr. and J. L. Solomon, *Subsequential limit points of successive approximations*, Proc. Amer. Math. Soc., **34** no. 2 (1972) 573–577.
2. F. T. Metcalf and T. D. Rogers, *The cluster set of sequences of successive approximations*, J. Math. Anal. Appl. **31** (1970) 206–212.

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