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A NEW CHARACTERIZATION OF χ-SPACES

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All spaces are assumed to be regular and T_1 in this paper.

A family $\mathcal{F}$ of subsets of a space X is called a k-network if for any compact set K and and open set U which contains K, there exists a finite subfamily $\mathcal{F}'$ such that $K \subset \cup\mathcal{F}' \subseteq U$. A space X with a σ-locally finite k-network is called an $\aleph$-space ([7]). A family $\mathcal{F}$ of subsets of X is called a cs-network if for any sequence $\{x_n : n \in N\}$ which converges to a point $x \in U$, where U is open in X, there exists an $F \in \mathcal{F}$, such that , $F \subseteq U$ and $\{x_n : n \in N\}$ is eventually in F ([2]). In [1] it was proved that a space X is an $\aleph$-space if and only if X has a σ-locally finite cs-network, and it is well known that the condition of having a σ-hereditarily closure-preserving k-network is strictly weaker than that of being an $\aleph$-space. But, what about the condition of having a σ-hereditarily closure-preserving cs-network? Is it equivalent to the former condition or to the latter one? In this paper, we prove that having a σ-hereditarily closure-preserving cs-network is a characterization of $\aleph$-spaces. As an application of this result, we show that $\aleph$-spaces are preserved under open and closed mappings, which answers a question raised in [5] (See [5] Question 4.4).

Let S be the subspace $\{1/n : n \in N\} \cup \{0\}$ of the real line in the usual topology. For each $\alpha < \omega_1$, let $S^{(\alpha)}$ be a copy of S. We denote by S_{ω_1} the quotient space obtained from the topological union $\bigoplus_{\alpha<\omega_1} S^{(\alpha)}$ by mapping all the nonisolated points into one point. Then S_{ω_1} is a Lašnev space and by [9] S_{ω_1} is not an $\aleph$-space.

Theorem 1. *A space X is an $\aleph$-space if and only if X has a σ-hereditarily closure-preserving cs-network.*

The proof of Theorem 1 is based on Lemma 1 and the following Theorem 2 proved in [3].

Theorem 2. *Let X be a space which has a σ-hereditarily closure preserving k-network. Then X is an $\aleph$-space if and only if X contains no closed copy of S_{ω_1} ([3] Theorem 1).*

Lemma 1. *S_{ω_1} has no σ-hereditarily closure-preserving cs-network.*

Proof. By [4] Lemma 1, if a space has a σ-hereditarily closure-preserving cs-network, then the space also has a σ-hereditarily closure-preserving closed cs-network. Denote $S_{\omega_1} = \{(1/n, \alpha) : n \in N \text{ and } \alpha < \omega_1\} \cup \{(0,0)\}$, where $(0,0)$ is the limit point of S_{ω_1}, and denote $S^{(\alpha)} = \{(1/n, \alpha) : n \in N\}$. By [6] Theorem 2.1, if $\mathcal{F}$ is a σ-hereditarily closure-preserving family of closed subsets of S_{ω_1}, then $\{F \in \mathcal{F} : | F \cap S^{(0)} |= \omega\}$ is countable. Hence to prove Lemma 1 it is suffice to show that for every closed cs-network $\mathcal{F}$ of S_{ω_1} the family $\{F \in \mathcal{F} : | S^{(0)} \cap F |= \omega\}$ is uncountable.

Let $\mathcal{F}$ be a closed cs-network of S_{ω_1}. By transfinite induction we can choose $F_\alpha \in \mathcal{F}$ and $z_\alpha \in S^{(\alpha)} \cap F_\alpha$ for each $\alpha, 0 < \alpha < \omega_1$, such that $F_\alpha \neq F_\beta$ whenever $\alpha \neq \beta$, as follows:

For each α, such that $0 < \alpha < \omega_1$, we regard $S^{(0)} \cup S^{(\alpha)}$ as a sequence such that the subsequences consisting of the odd terms and the even terms of which are $S^{(0)}$ and $S^{(\alpha)}$ respectively. Then $S^{(0)} \cup S^{(\alpha)}$ is a sequence which converges to $(0,0)$ in S_{ω_1}. We choose $F_1 \in \mathcal{F}$ such that $S^{(0)} \cup S^{(1)}$ is eventually in F_1 and a $z_1 \in S^{(1)} \cap F_1$. It is obvious that $| S^{(0)} \cap F_1 |= \omega$. Let $\alpha < \omega_1$ and assume that for every $\beta < \alpha$ $(\beta > 0)$, we have chosen $F_\beta \in \mathcal{F}$ and $z_\beta \in S^{(\beta)} \cap F_\beta$ such that $| S^{(0)} \cap F_\beta |= \omega$. Note that the set $S_{\omega_1} - \{z_\beta : \beta < \alpha\}$ is open and $(0,0) \in S_{\omega_1} - \{z_\beta : \beta < \alpha\}$; hence there exists $F_\alpha \in \mathcal{F}$ such that $S^{(0)} \cup S^{(\alpha)}$ is eventually in F_α and $F_\alpha \subseteq S_{\omega_1} - \{z_\beta : \beta < \alpha\}$. It is obvious that $| S^{(0)} \cap F_\alpha |= \omega$ and we can choose $z_\alpha \in S^{(\alpha)} \cap F_\alpha$. Since $F_\alpha \subseteq S_{\omega_1} - \{z_\beta : \beta < \alpha\}$, we have that $F_\alpha \neq F_\beta$ for each $\beta < \alpha$. This completes the induction construction.

The Proof of Theorem 1. The "only if" part is obvious.

The "if" part: Assume that X has a σ-hereditarily closure-preserving cs-network $\mathcal{F}$. Since the condition of being a cs-network is stronger than that of being a cover which satisfies (C_2) of [8], by [8] Proposition 1.2, $\mathcal{F}$ is also a σ-hereditarily closure-preserving k-network of X. On the other hand, X cannot contain any closed copy of S_{ω_1}; otherwise, $\mathcal{F} \cap S_{\omega_1}$ would be a σ-hereditarily closure-preserving cs-network of S_{ω_1}, which contracts Lemma 1. Hence, by Theorem 2, X is an $\aleph$-space.

Theorem 3. *Let X be an $\aleph$-space and let $f : X \to Y$ be an open and closed onto mapping, then Y is an $\aleph$-space.*

Proof. Assume that $\mathcal{F}$ is a σ-hereditarily closure-preserving cs-network of X. Since f is a closed mapping, the family $f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\}$ is σ-hereditarily closure-preserving. We only need to show that $f(\mathcal{F})$ is a cs-network of Y. Let $\{y_n : n \in N\}$ be a sequence which converges to a point $y \in U$ where U is open in Y. Take an $x \in f^{-1}(y)$. Since X is an $\aleph$-space, $\{x\}$ is a G_δ subset of X. Let $\{W_n : n \in N\}$ be a sequence of neighborhoods of x such that $\cap_{n\in N} W_n = \{x\}$ and, for each $n \in N$, $\overline{W}_{n+1} \subseteq W_n$. Since f is an open mapping, we can assume, without loss of any generality, that for each $n \in N$, $W_n \cap f^{-1}(y_k) \neq \emptyset$ whenever $k \geq n$. Take $x_n \in W_n \cap f^{-1}(y_n)$ for each $n \in N$. Since f is closed, $\{x_n\}$ is a sequence which converges to x. Hence there exists an $F \in \mathcal{F}$ such that $\{x_n : n \in N\}$ is eventually in F and $F \subseteq f^{-1}(U)$. Consequently $\{y_n : n \in N\}$ is eventually in $f(F)$ and $f(F) \subseteq U$.

References

[1] L. Foged, *Characterizations of $\aleph$-spaces*, Pacific J. Math. **110** (1984) 59-64.

[2] J. Guthrie, *A characterization of $\aleph$-spaces*, General Topology and its Appl. **1** (1971) 105-110.

[3] H. Junnila and Z. Yun, *$\aleph$-spaces and spaces with a σ-hereditarily closure-preserving k-network*, Proceedings of the Symposium on General Topology and Applications (Oxford 1989), to appear.

[4] S. Lin, *On a problem of K. Tamano*, Q. and A. in General Topology, **6** (1988) 99-102.

[5] S. Lin, *A survey of the theory of $\aleph$-spaces*, Q. and A. in General Topology, **8** (1990) 405-419.

[6] A. Okuyama, *On a generalization of Σ-spaces*, Pacific J. Math. **42** (1972) 485-495.

[7] P. O'Meara, *On paracompactness in function spaces with the compact-open topology*, Proc. Amer. Math. Soc. **29** (1971) 183-189.

[8] Y. Tanaka, *Point-countable covers and k-networks*, Topology Proc. **12** (1987) 327-349.

[9] Y. Tanaka, *Closed mappings of metric spaces*, Top. Appl. **11** (1980) 80-92.

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