Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Vol 17, 1992

LOWER SEMIFINITE TOPOLOGY IN HYPERSPACES

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ABSTRACT. In this paper we use the lower semifinite topology in hyperspaces to obtain examples in topology such as pseudocompact spaces not countably compact, separable spaces not Lindelöf and in a natural way many spaces appear which are T_0 but not T_1. We also give, in a unified form, many examples of contractible locally contractible spaces, absolute extensor for the class of all topological spaces, absolute retract and many examples of spaces having the fixed point property. Finally we obtain the following characterization of compactness:

"A paracompact Hausdorff space is compact if and only if the hyperspace, with the lower semifinite topology, 2^X, has the fixed point property".

INTRODUCTION

This paper was born while the authors were rereading two excellent books: *Rings of Continuous Functions* by L. Gillman and M. Jerison, [4], and *Hyperspaces of Sets* by S. Nadler, [6]. But the subject of our paper is not the same as that in [4] or [6]. On the other hand there is some relation to the material in [3].

We found, in a unified form, many examples of different kinds in Topology, such as covering, homotopy, separation,

The authors have been supported by the "Acciones Concertadas" of the Universidad Politécnica de Madrid.

countability, extension, retraction and fixed point properties.

We must say that the separation properties we give here are not as good as that of other examples about the same topics that have appeared in the literature but we have obtained these examples from a common branch.

Among the original results, the most significant of them are those which relate the fixed point property in 2^X with the compactness in X. For example, as a consequence of the results in this paper we obtain the following characterization of compactness:

A paracompact Hausdorff space is compact if and only if 2^X *has the fixed point property*

In order to obtain this result we use Zorn's Lemma.

Other interesting properties from our point of view, of the spaces 2^X (for X Tychonov) are that they are contractible and locally contractible and they are absolute extensors for all topological spaces, and since 2^X is homeomorphic to 2^Y if and only if X is homeomorphic to Y (see 2.5), we have many different topological types of spaces with such a property. From 3.1 to 3.3 we point out some properties $\mathcal{P}$ such that $X \in \mathcal{P} \Leftrightarrow 2^X \in \mathcal{P}$

In this paper, all unconstructed spaces are assumed to be Tychonov and all undefined topological notions can be found in [1], [2] and [5]. We will denote by $C(X)$ the ring of all real valued continuous functions from X. A map means a continuous function.

We would like to dedicate this paper to Toñi and Ana who encouraged us.

We also want to thank the referee for his comments that improved the previous version of the paper.

1.- The Space 2^X.

Let X be a Tychonov space. We denote by 2^X the set of all non empty closed subsets of X. Suppose that $f : X \longrightarrow \mathbb{R}$, we denote by U_f the set $U_f = \{C \in 2^X$ such that $f|_C$ is not the constant zero map$\}$. Note that if $f(x) \neq 0$ for every $x \in X$ then $U_f = 2^X$ and that if $g : X \longrightarrow \mathbb{R}$ is the zero map then $U_g = \emptyset$.

If X is nontrivial (it has at least two points) then the family $\{U_f\}_{f \in C(X)}$ is not a base for a topology in 2^X. However $\{U_f\}_{f \in C(X)}$ is a subbase for a topology in 2^X. In the sequel we consider 2^X with such a topology and we will denote the corresponding topological spaces by 2^X again.

Since the topology in a Tychonov space X is completely determined by $C(X)$, we have the following

Proposition 1.1. *For every Tychonov space X the above topology in 2^X is precisely the lower semifinite topology defined in [6]* (page 179) (see [5] too).

First, it is easy to describe the closure of a point in 2^X.

Proposition 1.2. *For every $C \in 2^X$ the closure of C in 2^X, denoted by $\overline{C}$, is the family of all nonempty closed subsets of C.*

We have the following consequences

Corollary 1.3. a) *If X is a nontrivial Tychonov space then 2^X is a T_0 space that is not T_1.*

b) *2^X is always a Baire, pseudocompact, connected and separable space.*

Separability follows from the fact that $\overline{X} = 2^X$. As a consequence, every map $f : 2^X \longrightarrow Y$ into a T_1-space is constant.

Since every point of a Tychonov space X is closed, we have a map $\Phi : X \longrightarrow 2^X$ defined by $\Phi(x) = \{x\}$. We will call this map the *canonical map from* X to 2^X.

Now we can prove the following:

Proposition 1.4. *The canonical map $\Phi : X \longrightarrow 2^X$ is an homeomorphism onto $\Phi(X)$, $\Phi(X)$ is closed and if X is nontrivial $\Phi(X)$ is nowhere dense in 2^X.*

Proof: Obviously Φ is a one to one map onto $\Phi(X)$. Let $x \in X$ and suppose that $U_{f_1} \cap U_{f_2} \cap \ldots \cap U_{f_n}$ is a basic neighborhood of $\{x\}$ in 2^X. Then $f_i(x) \neq 0$ for every $i \in \{1, 2, \ldots, n\}$. Consequently $\{x\} \in U_{\min\{f_1^2, \ldots, f_n^2\}} \cap \Phi(X) \subset U_{f_1} \cap U_{f_2} \cap \ldots U_{f_n} \cap \Phi(X)$.

Let $V = (\min\{f_1^2, \ldots, f_n^2\})^{-1}(\mathbb{R} \backslash \{0\})$, then V is open in X and $x \in V$. It is clear that $\Phi(V) = \Phi(X) \cap U_{\min\{f_1^2, \ldots, f_n^2\}}$. It follows that Φ is continuous and since the cozero sets are a base for X then Φ is an open map, so $\Phi : X \longrightarrow \Phi(X)$ is a homeomorphism.

$\Phi(X)$ is nowhere dense if X is nontrivial because $\Phi(X)$ does not contain X as a point in 2^X.

If X is trivial then $2^X = \Phi(X)$ and $\Phi(X)$ is closed. Let us suppose that X is nontrivial, then there exist at least two different points x and x' in X. Since X is Hausdorff, there exist V and V' disjoint open neighborhoods of x and x' respectively. Then $X = (X \backslash V) \cup (X \backslash V')$ and $x \in X \backslash V', x' \in X \backslash V$. Since X is T_{3a} it follows that there exist $g, h : X \longrightarrow \mathbb{R}$ maps such that $X \backslash V \subset g^{-1}(0), X \backslash V' \subset h^{-1}(0)$ and $g(x) \neq 0$ $h(x') \neq 0$. Obviously $\{x, x'\} \in U_g \cap U_h$.

Let $C \in U_g \cap U_h$, since $X = g^{-1}(0) \cup h^{-1}(0)$ we obtain that C has at least two points. Then $U_g \cap U_h \subset 2^X \backslash \Phi(X)$. This procedure allows us to say that $2^X \backslash \Phi(X)$ is open in 2^X and then $\Phi(X)$ is closed. This completes the proof. □

Notice that using analogous arguments as in the final part of the last proposition one can prove:

Proposition 1.5. *Let $n \geq 2$ be a natural number. The set $F_n(X) = \{C \in 2^X : \text{Cardinal}(C) < n\}$ is a closed subset of 2^X.*

Remark 1.6. One can ask if the last proposition is true for infinite cardinal numbers. The answer is negative because if X has infinite cardinal then $F_{\aleph_0} = \{C \in 2^X : \text{cardinal}(C) < \aleph_0\}$ is a proper dense subset of 2^X and then is not closed.

2.-Maps between Hyperspaces induced by maps between spaces.

Proposition 2.1. *Let X, Y be Tychonov spaces. Suppose that $f : X \longrightarrow Y$ is a map. Then the function $2^f : 2^X \longrightarrow 2^Y$ defined by $2^f(C) = \overline{f(C)}$ is continuous.($C \in 2^X$ and $\overline{\quad}$ is the closure in Y).*

Proof: Let $U_{g_1} \cap U_{g_2} \cap \ldots \cap U_{g_n}$ be a basic open neighborhood of $\overline{f(C)}$ in 2^Y. It follows that there exist $x_1, x_2, \ldots x_n \in C$ such that $\{f(x_1), \ldots, f(x_n)\} \in U_{g_1} \cap U_{g_2} \cap \ldots \cap U_{g_n}$. Let us take the basic open set $U_{g_1 o f} \cap U_{g_2 o f} \cap \ldots \cap U_{g_n o f}$ in 2^X. It is clear that $C \in U_{g_1 o f} \cap U_{g_2 o f} \cap \ldots \cap U_{g_n o f}$ and $2^f(U_{g_1 o f} \cap U_{g_2 o f} \cap \ldots \cap U_{g_n o f}) \subset U_{g_1} \cap U_{g_2} \cap \ldots \cap U_{g_n}$. □

The next proposition is straightforward

Proposition 2.2. a) $2^{1_X} = 1_2X$ *(where 1_Z is the identity map on Z).*

b) *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous then $2^{g o f} = 2^g \circ 2^f$.*

Proposition 2.3. *Let $f : X \longrightarrow Y$ be a map, then f is closed if 2^f is closed.*

Let us prove the following

Proposition 2.4. *If $h : 2^X \longrightarrow 2^Y$ is a closed map then there exists a continuous and closed map $f : X \longrightarrow Y$ such that $h = 2^f$.*

Proof: Let $x \in X$. Since h is closed then the closed set $h(\{x\})$ of 2^Y has only one element. Then $h(\{x\})$ can be identified with a point of Y. Let us denote it by $y_{h(\{x\})}$. We define a closed map $f : X \longrightarrow Y$ such that $f(x) = y_{h(\{x\})}$.

Now let $C \in 2^X$, then $h(C)$ is a point of 2^Y and then a closed subset of Y. Since h is continuous and closed we have that $h(\overline{C}) = \overline{h(C)}$ (*), in particular for every $x \in C$ we obtain that $h(\{x\}) = f(x) \in h(C)$ then $f(C) \subset h(C)$. On the other hand, consider $y \in h(C)$. It is easy to see that there exist (from (*)) a point $x \in C$ such that $h(\{x\}) = \{y\}$ and then $f(x) = y$. Consequently $h(C) \subset f(C)$. Hence $2^f(C) = f(C) = h(C)$ for every $C \in 2^X$. □

From Proposition 2.2 and 2.4 we have

Corollary 2.5. *Let X, Y be Tychonov spaces. Then X is homeomorphic to Y if and only if 2^X is homeomorphic to 2^Y.*

Corollary 2.6. *In the class of topological spaces which are T_0 but not T_1 there are at least so many different topological types as in the class of Tychonov spaces.*

3.-RELATIONS BETWEEN SOME TOPOLOGICAL PROPERTIES OF X AND 2^X

In this section we are going to relate some covering and countability properties of X with the same properties in 2^X. In this way we have

Proposition 3.1. *A Tychonov space X is compact if and only if 2^X is compact.*

Proof: If 2^X is compact, then since X is closed in 2^X (see Proposition 1.4) we have that X is compact.

Conversely, if X is compact one obtains that the canonical image of X in 2^X is compact. Using the Alexander subbasis Theorem, it is enough to see that every covering of 2^X by elements of the family $\{U_f\}_{f \in C(X)}$ has a finite subcovering. Let us suppose that $2^X = \cup U_f$ where f is in a subfamily of $C(X)$. Then, up to an identification, $X = \cup U_f$ and since X is compact it follows the existence of $f_1, f_2, \ldots, f_n \in C(X)$ such that $X = \cup_{i=1}^n U_{f_i}$. Now, let C be a nonempty closed subset of X. There exist $x \in C$ and $i_0 \in \{1, 2, \ldots, n\}$ with $\{x\} \in U_{f_{i_0}}$. Then $C \in U_{f_{i_0}}$ and $2^X = \cup_{i=1}^n U_{f_i}$. Consequently 2^X is compact. □

Using similar arguments we have

Proposition 3.2 *Let X be a Tychonov space, then*

a) *X is countably compact if and only if 2^X is countably compact.*

b) *X is Lindelöf if and only if 2^X is Lindelöf.*

Let us prove now the following

Proposition 3.3. *A Tychonov space X is second countable if and only if 2^X is second countable.*

Proof: Let us suppose that X is second countably. Then there exists a countable family of maps $F = \{f_n : X \longrightarrow \mathbb{R} : n \in \mathbb{N}\}$ such that the family $\{f_n^{-1}(\mathbb{R}\setminus\{0\}) : n \in \mathbb{N}\}$ is a base for the topology in X. Let now $U_{g_1} \cap U_{g_2} \cap \ldots U_{g_m}$ a basic open subset of 2^X containing a closed subset C of X. Then there exists $\{x_1, x_2, \ldots, x_m\} \subset C$ such that $g_i(x_i) \neq 0 \;\; i = 1, 2, \ldots, m$. We can choose $n_1, n_2, \ldots, n_m \in \mathbb{N}$ such that $x_i \in f_{n_i}^{-1}(\mathbb{R}\setminus\{0\}) \subset g_i^{-1}(\mathbb{R}\setminus\{0\})$ where $f_{n_i} \in F$. It is clear that $C \in U_{f_{n_1}} \cap U_{f_{n_2}} \cap \ldots \cap U_{f_{n_m}} \subset U_{g_1} \cap U_{g_2} \cap \ldots \cap U_{g_m}$. Consequently the family $\mathcal{B} = \{U_{f_{k_1}} \cap U_{f_{k_2}} \cap \ldots \cap U_{f_{k_n}} : n \in \mathbb{N} \;\; k_i \in \mathbb{N} \;\; i = 1, 2, \ldots, n \;\; f_{k_i} \in F\}$ is a countable base for 2^X.

The part "if" is obvious. □

As we know, countably compactness and pseudocompactness are equivalent properties in the class of normal Hausdorff spaces. Using Proposition 3.2 and Corollaries 1.3, 2.5 we have the following examples

Corollary 3.4. a) *For every noncountably compact Tychonov space X, 2^X is an example of a pseudocompact non countably compact space.*

b) *In the class of pseudocompact spaces which are not countably compact, there are at least as many different topological types as in the class of non countably compact Tychonov spaces (in particular as in the class of non compact metrizable spaces).*

Corollary 3.5. a) *For every non Lindelöf Tychonov space X, 2^X is an example of a separable non Lindelöf space.*

b) *In the class of separable non Lindelöf spaces, there are at least as many different topological types as in the class of* *non Lindelöf Tychonov spaces (in particular as in the class of non separable metrizable spaces).*

4.-HOMOTOPY, EXTENSION OF MAPS AND FIXED POINT PROPERTIES IN 2^X.

We begin this section with the following result

Proposition 4.1. *For every Tychonov space X, 2^X has trivial homotopy type.*

Proof: Let $H : 2^X \times I \longrightarrow 2^X$ defined by

$$H(C,t) = \begin{cases} X & \text{if } t \in [0,1) \\ C & \text{if } t = 1 \end{cases}$$

In order to prove the continuity of H consider V to be a neighborhood of $H(C,t)$ in 2^X. Since every nonempty open subset of 2^Xcontains the point X we have, if $t = 1$, that $(C,1) \subset V \times I$ and $H(V \times I) \subset V$ and, if $t \in [0,1)$, that $(C,t) \in 2^X \times [0,1)$ and $H(2^X \times [0,1)) \subset V$, then H is continuous. □

Using the same arguments we have

Corollary 4.2. *For every Tychonov space X we have that 2^X is contractible and locally contractible.*

We are going now to point out a curious property of 2^X

Proposition 4.3. *For every Tychonov space X, the space 2^X is an absolute extensor for the class of all topological spaces in the sense that every continuous function g from a closed subspace of a space Z to 2^X, can be continuously extended to Z.*

Proof: Let Z be a topological space. Suppose that A is a closed subset of Z. If $g : A \longrightarrow 2^X$ is a continuous function, let us define $\tilde{g} : Z \longrightarrow 2^X$ by $\tilde{g}|_A = g$ and $g(z) = X$ if $z \notin A$. Consider $z \in Z \backslash A$ and suppose that V is a neighborhood of X in 2^Xthen $\tilde{g}(Z \backslash A) \subset V$. On the other hand, if $a \in A$ and V is a neighborhood of $g(a)$ in 2^X from the continuity of g, there exists a neighborhood U of a in A such that $g(U) \subset V$.

Let $\tilde{U}$ be a neighborhood of a in Z such that $U = \tilde{U} \cap A$, then $\tilde{g}(\tilde{U}) \subset V$. □

Corollary 4.4. *If 2^X can be embedded as a closed subset of a space Z then 2^X is a retract of Z.*

Proposition 4.5. *If A is a closed subset of X then 2^A is homeomorphic to $\overline{A}$ (where — is the closure in 2^X).*

Corollary 4.6. *For every Tychonov space X and every closed subset A of X we have that, up to a homeomorphism, 2^A is a retract of 2^X.*

From the results in this section we can assure

Corollary 4.7. a) *In the realm of spaces which are absolute extensor for the class of all topological spaces there are at least as many different topological types as in the class of Tychonov spaces.*

b) *In the class of spaces which are contractible and locally contractible there are at least as many different topological types as in the class of Tychonov spaces.*

In order to end we are going to prove some of the most significant, from our point of view, results in this paper. These results give us relations between compactness and the fixed point property. We can prove the following

Proposition 4.8. *Let X be a compact (Tychonov) space then 2^X has the fixed point property.*

Proof: Let $f : 2^X \longrightarrow 2^X$ be a continuous function. Denote by $\mathcal{K} = \{K \subset X : K \text{ is closed and } f(K) \subset K\}$ (in the above notation and in the sequel we identify the point $f(K)$ in 2^X with the closed subset $f(K)$ of X).

As we know $\mathcal{K}$ is preordered by the inclusion ($A < B$ if $B \subset A$). Obviously $\mathcal{K} \neq \emptyset$ because $X \in \mathcal{K}$. Another property of $\mathcal{K}$ is the following: "if $K \in \mathcal{K}$ then $f(K) \in \mathcal{K}$". In fact, if $K \in \mathcal{K}$ then $f(K) \subset K$ and $f(K) \in \overline{K}$. Since f is continuous it follows that $f(\overline{K}) \subset \overline{f(K)}$ we deduce that $f(f(K)) \in \overline{f(K)}$ and $f(f(K)) \subset f(K)$.

Now let $\{K_\alpha : \alpha \in \Lambda\}$ be a chain in $\mathcal{K}$, then $\cap_\alpha K_\alpha \neq \emptyset$. On the other hand, $\cap_\alpha K_\alpha \subset K_\alpha$ for every $\alpha \in \Lambda$. Then, from the continuity of f, $f(\cap_\alpha K_\alpha) \subset f(K_\alpha)$ and consequently $f(\cap_\alpha K_\alpha) \subset \cap_\alpha f(K_\alpha) \subset \cap_\alpha K_\alpha$. It follows that $\cap_\alpha K_\alpha \in \mathcal{K}$. We have proved then that every chain in $\mathcal{K}$ has an upper bound, now using Zorn's Lemma we obtain that there exists a maximal element K of $\mathcal{K}$, then $f(K) \subset K$ and since $f(K) \in \mathcal{K}$ we conclude that $f(K) = K$ and the proof is finished. □

Corollary 4.9. *If X is a compact Hausdorff space and $f : X \longrightarrow X$ is a continuous map, there exists a compact subspace $K \subset X$ such that $f(K) = K$.*

Proof: The result follows from Proposition 4.8 applies to 2^f.

Remark 4.10. Let us note that Proposition 4.8 can be generalized as follows. "Let X be a Tychonov space and suppose that $f : 2^X \longrightarrow 2^X$ is a map such that $f(X)$ is a compact subset of X, then f has a fixed point".

Conversely we have

Proposition 4.11. *If X is a Tychonov space such that 2^X has the fixed point property then X is countably compact.*

Proof: Let us suppose that X is not countably compact, then there exists a closed subset C of X such that C is homeomorphic to the set $\mathbb{N}$ of the natural numbers. Let us order the elements of C as a sequence, i.e., $C = \{a_n : n \in \mathbb{N}\}(a_n \neq a_m$ if $n \neq m)$. Define $f : C \longrightarrow C$ as $f(a_n) = a_{n+1}$ and let us consider $2^f : 2^C \longrightarrow 2^C$. It is clear that 2^f does not have a fixed point. Finally let $g : 2^X \longrightarrow 2^X$ be defined by

$$g(K) = \begin{cases} 2^f(K) & \text{if } K \text{ is contained in } C \\ C & \text{if } K \text{ is not contained in } C \end{cases}.$$

Obviously g is continuous and does not have fixed points.

□

As we know compactness and countable compactness are equivalent in the realm of paracompact spaces. Then we have

Corollary 4.12. *A paracompact Hausdorff space is compact if and only if* 2^X *has the fixed point property.*

Corollary 4.13. *In the class of spaces having the fixed point property at least there are so many different topological types as in the class of compact Hausdorff spaces.*

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