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DISCRETE CHAIN CONDITIONS AND THE $\mathcal{B}$-PROPERTY

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ABSTRACT. In this paper, we introduce and study λ-Lindelöf, λ-compact and DλCC spaces. Some characterizations of these spaces are given. It is shown that in T_1 spaces with the $\mathcal{B}$-property, ω_α-Lindelöfness, $\omega_{\alpha+1}$-compactness and the Dω_αCC coincide. Some known results in the literature are generalized.

1. PRELIMINARIES

Throughout the paper, by a space we mean a topological space without any separation axioms assumed unless especially stated. Regular spaces need not be T_1. Cardinals are initial ordinals. $\omega = \omega_0$ denotes the first infinite ordinal and λ always denotes an infinite cardinal. A space X is said to have the $\mathcal{D}$-property [MN] (resp. $\mathcal{B}$-property[Z]) if any increasing open cover $\{U_\alpha : \alpha \in A\}$ of X has an open refinement (resp. increasing open refinement) $\{V_\alpha : \alpha \in A\}$ such that $\bar{V}_\alpha \subset U_\alpha$ for any $\alpha \in A$. It is known that

(*) paracompactness $\longrightarrow$ $\mathcal{B}$-property $\longrightarrow$ $\mathcal{D}$-property $\longrightarrow$ countable paracompactness

The above implications are not reversible (see [MN]). A family $\mathcal{U}$ of subsets of X is called a point-λ (resp. point-finite) family if for every $x \in X, |\{U \in \mathcal{U} : x \in X\}| \leq \lambda$ (resp. $< \omega$). A family $\mathcal{A} = \{A_\alpha : \alpha \in A\}$ of subsets of a space X is hereditarily closure-preserving (shortly, HCP) if any family $\{B_\alpha : \alpha \in A\}$ with $B_\alpha \subset A_\alpha$ for each $\alpha \in A$ is closure-preserving. Clearly, local finiteness implies HCP and HCP implies closure-preservingness. The following symbols are used: $\mathcal{A}_F(X) = \{\mathcal{U} : \mathcal{U}$ is an open cover of X having no finite subcover $\}$. $\mathcal{A}_\lambda(X) = \{\mathcal{U} : \mathcal{U}$ is an open cover of X having no subcover with cardinality $\leq \lambda\}$. If $\mathcal{A}_F(X)$ and $\mathcal{A}_\lambda(X)$ are not empty, we put $\ell_F(X) = \min\{|\mathcal{U}| : \mathcal{U} \in \mathcal{A}_F(X)\}$ and $\ell_\lambda(X) = \min\{|\mathcal{U}| : \mathcal{U} \in \mathcal{A}_\lambda(X)\}$

2. Definitions

Definition 2.1. *A space X is said to be λ-Lindelöf if any open cover of X has a subcover with cardinality $\leq \lambda$.*

Definition 2.2. *A space X is said to be λ-compact if any subset with cardinality λ has an accumulation point.*

Definition 2.3. *A space X is called a* DλCC *(discrete λ chain condition) space if any discrete family of non-empty open sets has cardinality $\leq \lambda$.*

Clearly, ω-Lindelöf spaces coincide with Lindelöf spaces and a T_1 space is ω-compact iff it is countably compact. ω_1-compact spaces are well known. DωCC is also denoted by DCCC which was introduced in [W]. DFCC denotes that any discrete family of non-empty open sets is finite. In the following Figure 1, the implications are obvious and none of them is reversible.

Example 2.1. (1) It is well known that $[0, \omega_1)$ is ω-compact, but not compact.

(2) $[0, \omega_{\alpha+2})$ is $\omega_{\alpha+1}$-compact, but not $\omega_{\alpha+1}$-Lindelöf (hence not ω_α-Lindelöf). In fact, let $A \subset [0, \omega_{\alpha+2})$ and $|A| = \omega_{\alpha+1}$, then $\beta_0 = \sup A < \omega_{\alpha+2}$ since $\omega_{\alpha+2}$ is regular. Since $[0, \beta_0]$ is compact the infinite subset A of $[0, \beta_0]$ has an accumulation point $\xi \in [0, \beta_0]$ which is also an accumulation point of A in $[0, \omega_{\alpha+2})$. So $[0, \omega_{\alpha+2})$ is $\omega_{\alpha+1}$-compact. Take the open cover $\mathcal{U} = \{[0, \beta) : \beta \in (0, \omega_{\alpha+2})\}$ of $[0, \omega_{\alpha+2})$. For any $\tilde{\mathcal{U}} \subset \mathcal{U}$ with $|\tilde{\mathcal{U}}| \leq \omega_{\alpha+1}$, let $C = \{\beta : [0, \beta) \in \tilde{\mathcal{U}}\}$, then $|C| \leq \omega_{\alpha+1}$ and $\lambda_0 = \sup C < \omega_{\alpha+2}$ since $\omega_{\alpha+2}$ is regular. Take a λ such that $\lambda_0 < \lambda < \omega_{\alpha+2}$, then for any $[0, \beta) \in \tilde{\mathcal{U}}$, we have $\lambda \bar{\in} [0, \beta)$. This shows that $[0, \omega_{\alpha+2})$ is not $\omega_{\alpha+1}$-Lindelöf.

$$
\begin{array}{ccccc}
\text{compact} & \longrightarrow & \omega\text{-compact} & \xrightarrow{T_1} & \text{DFCC} \\
\downarrow & & \downarrow & & \downarrow \\
\omega\text{-Lindelöf} & \longrightarrow & \omega_1\text{-compact} & \xrightarrow{T_1} & \text{D}\omega\text{CC} \\
\downarrow & & \downarrow & & \downarrow \\
\omega_1\text{-Lindelöf} & \longrightarrow & \omega_2\text{-compact} & \xrightarrow{T_1} & \text{D}\omega_1\text{CC} \\
\downarrow & & \downarrow & & \downarrow \\
\cdots & & \cdots & & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
\omega_\alpha\text{-Lindelöf} & \longrightarrow & \omega_{\alpha+1}\text{-compact} & \xrightarrow{T_1} & \text{D}\omega_\alpha\text{CC} \\
\downarrow & & \downarrow & & \downarrow \\
\omega_{\alpha+1}\text{-Lindelöf} & \longrightarrow & \omega_{\alpha+2}\text{-compact} & \xrightarrow{T_1} & D\omega_{\alpha+1}CC \\
\downarrow & & \downarrow & & \downarrow \\
\cdots & & \cdots & & \cdots
\end{array}
$$

Figure 1

Example 2.2. (1) Let X be a set with $|X| = \omega_{\alpha+2}$ and $\mathcal{T} = \{\emptyset\} \cup \{U \subset X : |X - U| \leq \omega_{\alpha+1}\}$, then the T_1 space $(X, \mathcal{T})$ is DFCC (hence Dω_αCC), but is not $\omega_{\alpha+1}$-compact (hence not ω-compact).

(2) Let A be a set of cardinality λ and $\{N_S : s \in S\}$, where $S \cap A = \emptyset$ and each $N_S \subset A$ has cardinality ω, be an infinite family satisfying that the intersection $N_s \cap N_{\tilde{s}}$ is finite for every pair $s, \tilde{s}$ of distinct elements of S and $\{N_S : s \in S\}$ is maximal with respect to the last property. Generate a topology on the set $X = A \cup S$ by the neighborhood system $\{\mathcal{B}(x) : x \in X\}$, where $\mathcal{B}(x) = \{\{x\}\}$ if $x \in A$ and $\mathcal{B}(x) = \{\{s\} \cup (N_s - F) : F$ is a finite subset of $N_s\}$ if $x = s \in S$. The space X is Tychonoff and DFCC (hence DλCC), but not λ-compact (cf. 3.6.I(a) of [E]).

Example 2.3. Suppose X is a discrete space.

(1) Let $|X| = \omega$, then X is ω-Lindelöf (hence ω_1-compact, DωCC), but not DFCC (hence not ω-compact, not compact).

(2) Let $|X| = \omega_{\alpha+1}$, then X is $\omega_{\alpha+1}$-Lindelöf (hence $\omega_{\alpha+2}$-compact, D$\omega_{\alpha+1}$CC), but not Dω_αCC (hence not $\omega_{\alpha+1}$-compact,

not ω_α-Lindelöf).

3. Characterizations

Definition 3.1. *A space X is said to have the property (λ) if any increasing open cover $\mathcal{U}$ of X with $|\mathcal{U}| \le \lambda$ has a closed refinement $\mathcal{F}$ with $|\mathcal{F}| \le \lambda$.*

Clearly, for any λ, the $\mathcal{D}$-property (hence the $\mathcal{B}$-property) implies the property (λ). A space is countably metacompact[MN] if every countable open cover of it has a point-finite open refinement. Clearly, countable paracompactness implies countable metacompactness. Form the following proposition, we can see that in a normal space countable paracompactness and countable metacompactness are equivalent.

Proposition 3.1. *A space X is countable metacompact iff every increasing countable open cover $\{U_i : i < \omega\}$ of X has an increasing countable closed refinement $\{F_i : i < \omega\}$ such that $F_i \subset U_i$ for any $i < \omega$. Hence countable metacompactness implies the property (ω).*

Proof: Necessity: Let $\mathcal{V}$ be a point-finite open refinement of the increasing open cover $\mathcal{U} = \{U_i : i < \omega\}$. Put $F_i = \{x \in X : \mathrm{st}(x, \mathcal{V}) \subset U_i\}$, then $\{F_i : i < \omega\}$ is the desired refinement of $\mathcal{U}$.

Sufficiency: Let $\mathcal{U} = \{U_i : i < \omega\}$ be a countable open cover of X. Put $V_i = \cup_{j \le i} U_j$, then the increasing open cover $\{V_i : i \le \omega\}$ has an increasing closed refinement $\{F_i : i < \omega\}$ such that $F_i \subset V_i$ for any $i < \omega$. Put $W_0 = V_0$, $W_i = V_i - F_{i-1}$, $i \ge 1$, then $\{W_i \cap U_j : j \le i,\ i < \omega\}$ is a point-finite open refinement of $\mathcal{U}$.

Lemma 3.1. *(1) $\ell_F(X)$ is regular. (2) If the space X has the property (λ), $\ell_\lambda(X)$ is regular.*

Proof: (1) Suppose $\ell_F(X) = \kappa > \mathrm{cf}\kappa$ and $\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \in \mathcal{A}_F(X)$ such that $|\mathcal{U}| = \ell_F(X)$. Let $f : \mathrm{cf}\kappa \to \kappa$ be an increasing cofinal mapping and for each $\alpha < \mathrm{cf}\kappa$, $W_\alpha = \cup_{\beta < f(\alpha)} U_\beta$. Then $\mathcal{W} = \{W_\alpha : \alpha < \mathrm{cf}\kappa\}$ is an increasing open cover of X with $|\mathcal{W}| \leq \mathrm{cf}\kappa$. Thus $\mathcal{W}$ has a finite (increasing) subcover $\widetilde{\mathcal{W}}$. So there is a $W_{\alpha_0} \in \widetilde{\mathcal{W}}$ such that $X = W_{\alpha_0} = \cup_{\beta<f(\alpha_0)} U_\beta$. Since $f(\alpha_0) < \kappa$, $\{U_\beta : \beta < f(\alpha_0)\}$ has a finite subcover $\widetilde{\mathcal{U}} \subset \mathcal{U}$. This contradicts the choice of $\mathcal{U}$.

(2) Suppose $\ell_\lambda(X) = \kappa > \mathrm{cf}\kappa$ and $\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \in \mathcal{A}_\lambda(X)$ such that $|\mathcal{U}| = \ell_\lambda(X)$. Then the increasing open cover $\mathcal{W}$, as defined in case (1), has a (increasing) subcover $\widetilde{\mathcal{W}}$ with $|\widetilde{\mathcal{W}}| \leq \lambda$. By the property (λ), $\widetilde{\mathcal{W}}$ has a closed refinement $\mathcal{F}$ with $|\mathcal{F}| \leq \lambda$. For each $F \in \mathcal{F}$, there is a $W_{\alpha_F} \in \widetilde{\mathcal{W}}$ such that $F \subset W_{\alpha_F} = \cup_{\beta<f(\alpha_F)} U_\beta$. Therefore $\{X - F\} \cup \{U_\beta : \beta < f(\alpha_F)\}$ is an open cover of X with cardinality $< \kappa$. It follows that for each F there is some $\mathcal{U}_F \subset \mathcal{U}$ that covers F and $|\mathcal{U}_F| \leq \lambda$. Hence $\cup\{\mathcal{U}_F : F \in \mathcal{F}\}$ is a cover of X with cardinality $\leq \lambda$, contradicting the choice of $\mathcal{U}$.

It is interesting that a space X is compact iff any increasing open cover of X has a finite subcover [A]. For λ-Lindelöf spaces, we have

Theorem 3.1. *Let the space X have the property (λ). Then X is λ-Lindelöf iff any increasing open cover of X has a subcover with cardinality $\leq \lambda$.*

Proof: Necessity is obvious. Sufficiency: Suppose that X is not λ-Lindelöf. Take $\mathcal{U} \in \mathcal{A}_\lambda(X)$ such that $|\mathcal{U}| = \ell_\lambda(X) = \kappa$. Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ and $W_\alpha = \cup_{\beta<\alpha} U_\beta$. Then the increasing open cover $\mathcal{W} = \{W_\alpha : \alpha < \kappa\}$ of X has a subcover $\widetilde{\mathcal{W}} = \{W_{\alpha_\beta} : \beta < \kappa_1\}$, where $k_1 \leq \lambda$. Put $\widetilde{\mathcal{U}} = \cup_{\beta<\kappa_1}\{U_\xi \in \mathcal{U} : \xi < \alpha_\beta\}$. By Lemma 3.1, κ is regular. Since for every $\beta < \kappa_1, \alpha_\beta < \kappa$ and $\kappa_1 \leq \lambda < \kappa$, we have $|\widetilde{\mathcal{U}}| < k$. According to the definition of κ, $\widetilde{\mathcal{U}}$ has a subcover $\mathcal{V}$ with $|\mathcal{V}| \leq \lambda$. This contradicts the choice of $\mathcal{U}$. Noticing the case $\lambda = \omega$ and Proposition 3.1, we have

Corollary 3.1. *A countably metacompact space (hence a countably paracompact space) is Lindelöf iff every increasing open cover of it has a countable subcover.*

Remark 3.1. We do not know whether the condition that X has the property (λ) in Lemma 3.1 (hence in Theorem 3.1) can be removed. For the case $\lambda = \omega$, M. E. Rudin has a conjecture that there is a normal, non-Lindelöf space, every increasing open cover of which has a countable subcover (see [MR], Chapter 10). By Corollary 3.1, if such a space exists, it must be a Dowker space (i.e. a normal but not countably paracompact space).

Theorem 3.2. *For a T_1 space X, the following are equivalent:*

(1) *X is λ-compact.*
(2) *Any discrete closed subspace Y of X has cardinality $< \lambda$.*
(3) *Any discrete family $\mathcal{U}$ of non-empty subsets of X has cardinality $< \lambda$.*
(4) *Any discrete family $\mathcal{U}$ of non-empty closed subsets of X has cardinality $< \lambda$.*
(5) *Any irreducible open cover $\mathcal{U}$ of X has cardinality $< \lambda$.*
(6) *Any point-λ, irreducible open cover $\mathcal{U}$ of X has cardinality $< \lambda$.*

Proof: (1)$\rightarrow$(2), (2) $\rightarrow$ (3) and (3) $\rightarrow$ (4) are obvious. (4) $\rightarrow$ (5): Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an irreducible open cover. Since $\mathcal{U}$ is irreducible, for every $\alpha \in A$, we can take $x_\alpha \in U_\alpha - \cup_{\gamma \neq \alpha} U_\gamma$. Put $M = \{x_\alpha : \alpha \in A\}$. For every $x \in X$, there is an $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$ and $U_\alpha \cap M = \{x_{\alpha_0}\}$. So $\mathcal{K} = \{\{x_\alpha\} : \alpha \in A\}$ is a discrete family of closed sets. By (4), $|\mathcal{U}| = |\mathcal{K}| < \lambda$. (5) $\rightarrow$ (6) is obvious. (6) $\rightarrow$ (1): Suppose X is not λ-compact, then there is an $M \subset X$ with $|M| = \lambda$ such that M has no accumulation point. So M is closed. For every $x \in M$, there is an open set U_x such that $x \in U_x$ and $U_x \cap M = \{x\}$. Then $\mathcal{U} = \{U_x : x \in M\} \cup \{X - M\}$ ($\mathcal{U} = \{U_x : x \in M\}$ if $\cup\{U_x : x \in M\} = X$) is an irreducible open cover satisfying for any $x \in X$, $|\{U \in \mathcal{U} : x \in U\}| \leq \lambda$ and $|\mathcal{U}| = \lambda$. This contradicts (6).

Remark 3.2. In this remark, spaces are T_1. So a space is ω-compact iff it is countably compact. Since any point-finite cover of a space has an irreducible subcover[E], by the equivalence of (1) and (5) of Theorem 3.2, we obtain

(1) A metacompact space[MN] is $\omega_{\alpha+1}$-compact (resp. countably compact) iff it is ω_α-Lindelöf (resp. compact).

By the equivalence of (1) and (4) of Theorem 3.2, we obtain

(2) A subparacompact space[MN] is $\omega_{\alpha+1}$-compact iff it is ω_α-Lindelöf.

Noticing that if for any discrete family $\{\{x_\beta\} : \beta < \omega_{\alpha+1}\}$, there is a discrete family $\{G_\beta : \beta < \omega_{\alpha+1}\}$ of open sets such that $G_\beta \supset \{x_\beta\}$, then the D$\omega_\alpha$CC implies $\omega_{\alpha+1}$-compactness, we obtain

(3) A collectionwise normal space is $\omega_{\alpha+1}$-compact iff it satisfies the Dω_αCC.

We can prove the following (4) similar to Lemma 1.2 in [T]. Actually, Lemma 1.2 in [T] is the case $\alpha = 0$ in (4).

(4) Let X be a space and $\mathcal{K}$ a cover of X by ω_α-compact sets. If X has a (mod $\mathcal{K}$)-net $\mathcal{F}$ such that $|\mathcal{F}| \leq \omega_\alpha$, then X is $\omega_{\alpha+1}$-compact.

It is easy to show that in a pseudonormal space [H] for any discrete countable family $\{\{x_i\} : i < \omega\}$, there is a discrete family $\{G_i : i < \omega\}$ of open sets such that $x_i \in G_i$ for any $i < \omega$. So we have

(5) A pseudonormal space is countably compact iff it satisfies the DFCC.

There is a Tychonoff DFCC space which is not countably compact (see Example 2.2 (2)).

Theorem 3.3. *For a regular space X, the following are equivalent:*

1. *X satisfies the* DλCC *(resp.* DFCC*).*
2. *Any locally finite family $\mathcal{U}$ of open sets of X has cardinality $\leq \lambda$ (resp. $< \omega$).*
3. *Any locally finite open cover $\mathcal{U}$ of X has cardinality $\leq \lambda$ (resp. $< \omega$).*

(4) *Any locally finite irreducible open cover $\mathcal{U}$ of X has cardinality $\leq \lambda$ (resp. $< \omega$).*
(5) *Any point-λ (resp. point-finite),* HCP *family $\mathcal{U}$ of open sets of X has cardinality $\leq \lambda$ (resp. $< \omega$).*
(6) *Any point$-\lambda$ (resp. point-finite),* HCP *open cover $\mathcal{U}$ of X has cardinality $\leq \lambda$ (resp. $< \omega$).*
(7) *Any point-λ (resp. point-finite),* HCP *irreducible open cover $\mathcal{U}$ of X has cardinality $\leq \lambda$ (resp. $< \omega$).*

Proof: (1) $\rightarrow$ (5): Let X be well-ordered and $x \in X$. Put $\mathcal{U}(x) = \{U \in \mathcal{U} : x$ is the first element in $U\}$. Since $\mathcal{U}$ is a point-λ (resp. point-finite) family, $\mathcal{U}(x)$ is empty or $|\mathcal{U}(x)| \leq \lambda$ (resp. $|\mathcal{U}(x)| < \omega$). Each $U \in \mathcal{U}$ is in one and only one $\mathcal{U}(x)$ and

(#) $y < x$ implies $y \bar{\in} U$ for $U \in \mathcal{U}(x)$

Put $A = \{x \in X : \mathcal{U}(x) \neq \emptyset\}$. We only need to show $|\{\mathcal{U}(x) : x \in A\}| \leq \lambda$ (resp. $< \omega$). To see this, for each $x \in A$, take a $U(x) \in \mathcal{U}(x)$, then there is an open set $V(x)$ such that $x \in V(x) \subset \bar{V}(x) \subset U(x)$ since X is regular. Put $W(x) = V(x) - \cup\{\bar{V}(y) : y > x\}$, then $\{W(x) : x \in A\}$ is a disjoint family of open sets since $\mathcal{U}$ is HCP. By (#) and the regularity of X there is an open set $X(x)$ such that $x \in X(x) \subset \bar{X}(x) \subset W(x)$. The family $\{X(x) : x \in A\}$ is discrete. In fact, for any $x \in X$, if $x \in W(y)$ for some $y \in A, W(y)$ only meets $X(y)$. If $x \bar{\in} W(y)$ for any $y \in A$, then $G = X - \cup_{y \in A} \bar{X}(y)$ is an open neighborhood of x since $\mathcal{U}$ is HCP and $X(y) \subset W(y)$ for each $y \in A$. For any $y \in A$, $G \cap X(y) = \emptyset$. Since X satisfies the DλCC (resp. DFCC) the discrete family $\{X(x) : x \in A\}$ of open sets has cardinality $\leq \lambda$ (resp. $< \omega$). (5) $\rightarrow$(2), (2) $\rightarrow$(3), (3) $\rightarrow$(4), (5) $\rightarrow$(6), (6) $\rightarrow$(7) and (7) $\rightarrow$(4) are obvious. (4) $\Rightarrow$ (1): Let $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ be a discrete family of non-empty open sets of X. If $\mathcal{G}$ covers X, then it is a locally finite irreducible open cover of X. Thus $|\mathcal{G}| \leq \lambda$ (resp. $< \omega$). Otherwise, take an $x_\alpha \in G_\alpha$ and an open set V_α such that $x_\alpha \in V_\alpha \subset \bar{V}_\alpha \subset G_\alpha$. Put $F = \cup_{\alpha \in A} \bar{V}_\alpha$. then F is a closed set and $\mathcal{U} = \{G_\alpha : \alpha \in A\} \cup \{X - F\}$ is a locally finite irreducible open cover. By (4), $|\mathcal{G}| \leq \lambda$ (resp. $< \omega$). This show that X satisfies the DλCC (resp. DFCC).

Remark 3.3. In [W], it is proved that a regular space satisfies the DCCC iff every locally finite family of open sets is countable. Our Theorem 3.3 generalizes the above result. Noticing the case DFCC in Theorem 3.3, we obtain

(1) A Tychonoff space satisfies the DFCC iff it is pseudocompact (cf. Theorem 3.10.22 of [E].)

(2) A regular DFCC T_1 space is countably compact iff it is countably paracompact.

4. Relationships

Lemma 4.1. *The space X has the $\mathcal{B}$-property (resp. countable paracompactness) iff every increasing open cover (resp. increasing countable open cover) of X has an open star refinement.*

Proof: Sufficiency: Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be an increasing open cover of X and $\mathcal{G}$ an open star refinement of $\mathcal{U}$. Put $V_\alpha = \cup\{G \in \mathcal{G} : \text{st}(G, \mathcal{G}) \subset U_\alpha\}$, then $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ is an increasing open refinement of $\mathcal{U}$ satisfying $\bar{V}_\alpha \subset U_\alpha$ for any $\alpha < \kappa$. Thus X has the $\mathcal{B}$-property.

Necessity: Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be an increasing open cover of X. By Theorem 1 in [Y] (its proof does not use the fact X is regular and T_1), $\mathcal{U}$ has an open refinement $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ such that for every $\alpha < \kappa$, $V_\alpha \subset U_\alpha$ and for every $x \in X$, there is an open neighborhood O_x of x and an $\alpha_x < \kappa$ such that $O_x \cap (\cup\{V_\alpha : \alpha \geq \alpha_x\}) = \emptyset$. Hence $\text{st}(0_x, \mathcal{V}) \subset \cup\{V_\alpha : \alpha < \alpha_x\} \subset U_{\alpha_x}$. Put $W_\alpha = \cup\{O : O \subset X$ is open and $\text{st}(O, \mathcal{V}) \subset U_\alpha\}$, then $\mathcal{W} = \{W_\alpha : \alpha < \kappa\}$ is an increasing open cover of X and $\text{st}(W_\alpha, \mathcal{V}) \subset U_\alpha$ for any $\alpha < \kappa$. Put $\mathcal{G} = \{W_\alpha \cap V_\beta : \alpha, \beta < \kappa\}$, then the open cover $\mathcal{G}$ of X satisfies for any $G \in \mathcal{G}$, there is a $U_\alpha \in \mathcal{U}$ such that $\text{st}(G, \mathcal{G}) \subset U_\alpha$.

For the countably paracompact case, the proof is similar noticing that in a countably paracompact space every increasing countable open cover $\{U_i : i < \omega, \}$ has a locally finite open refinement $\{V_i : i < \omega\}$ such that $V_i \subset U_i$ for any $i < \omega$.

Theorem 4.1. *If a* DλCC *(resp.* DFCC*) space X has the $\mathcal{B}$-property, then X is λ-Lindelöf (resp. compact).*

Proof: Suppose not. Take a $\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \in \mathcal{A}_\lambda(X)$ (resp. $\in \mathcal{A}_F(X)$) such that $|\mathcal{U}| = \ell_\lambda(X) = \kappa$ (resp. $|\mathcal{U}| = \ell_F(X) = \kappa$). By Lemma 3.1, $\mathrm{cf}\kappa = \kappa$. Put $V_\alpha = \cup_{\beta<\alpha} U_\beta$. Then the increasing open cover $\{V_\alpha : \alpha < \kappa\}$ has an increasing open refinement $\mathcal{W} = \{W_\alpha : \alpha < \kappa\}$ such that $\bar{W}_\alpha \subset V_\alpha$ for any $\alpha < k$ since X has the $\mathcal{B}$-property. By Lemma 4.1 $\mathcal{W}$ has an open star refinement $\mathcal{G}$. Take a non-empty $G_0 \in \mathcal{G}$ and an $\alpha_0 < \kappa$ such that $\mathrm{st}(G_0, \mathcal{G}) \subset W_{\alpha_0}$. Since $X - \bar{W}_{\alpha_0} \neq \emptyset$, there is a $\tilde{G}_1 \in \mathcal{G}$ and an $\alpha_1 > \alpha_0$ such that $G_1 = \tilde{G}_1 \cap (X - \bar{W}_{\alpha_0}) \neq \emptyset$ and $\mathrm{st}(\tilde{G}_1, \mathcal{G}) \subset W_{\alpha_1}$. Thus $\mathrm{st}(G_1, \mathcal{G}) \subset W_{\alpha_1}$. Suppose for the ordinal ν, when $\eta < \nu$, G_η and α_η have been defined. If $\tilde{\alpha}_\nu = \sup\{\alpha_\eta : \eta < \nu\} < \kappa$, then take a $\tilde{G}_\nu \in \mathcal{G}$ and an $\alpha_\nu > \tilde{\alpha}_\nu$ such that $G_\nu = \tilde{G}_\nu \cap (X - \bar{W}_{\tilde{\alpha}_\nu}) \neq \emptyset$ and $\mathrm{st}(\tilde{G}_\nu, \mathcal{G}) \subset W_{\alpha_\nu}$. Thus $\mathrm{st}(G_\mu, \mathcal{G}) \subset W_{\alpha_\nu}$. If $\tilde{\alpha}_\nu = \kappa$, the definition is finished. There is certainly some τ at which the definition is finished. Then the family $\{G_\eta : \eta < \tau\}$ of non-empty open sets is discrete. In fact, for every $x \in X$, if $x \in X - \mathrm{st}(\cup_{\eta<\tau} G_\eta, \mathcal{G})$, there is a $G_x \in \mathcal{G}$ such that $x \in G_x$ and $G_x \cap (\cup\{G_\eta : \eta < \tau\}) = \emptyset$. If $x \in \mathrm{st}(\cup_{\eta<\tau} G_\eta, \mathcal{G})$, there is $G_x \in \mathcal{G}$ and $\eta_x < \tau$ such that $G_x \cap G_{\eta_x} \neq \emptyset$. If $\eta_x \neq \eta < \tau$, $G_x \cap G_\eta = \emptyset$. To see this, observe that if $\nu < \eta$, $\mathrm{st}(G_\nu, \mathcal{G}) \subset \mathrm{st}(\tilde{G}_\nu, \mathcal{G}) \subset W_{\alpha_\nu}$ and $G_\eta \cap \bar{W}_{\alpha_\nu} = \emptyset$. So $\mathrm{st}(G_\nu, \mathcal{G}) \cap G_\eta = \emptyset$ (and hence $\mathrm{st}(G_\eta, \mathcal{G}) \cap G_\nu = \emptyset$). Therefore, $\mathrm{st}(G_\nu, \mathcal{G}) \cap G_\eta = \emptyset$ for any distinct $\nu, \eta < \tau$. So if $G_x \cap G_{\eta_x} \neq \emptyset$, then $G_x \cap G_\eta = \emptyset$ for $\eta \neq \eta_x$. This shows that $\{G_\eta : \eta < \tau\}$ is discrete. Since $\sup\{\alpha_\eta : \eta < \tau\} = \kappa$, $\{\alpha_\eta : \eta < \tau\}$ is cofinal in κ. So $\tau \geq \mathrm{cf}\kappa = \kappa > \lambda$ (resp. $\kappa \geq \omega$). This contradicts the assumption that X satisfies the DλCC (resp. DFCC).

Noticing Figure 1 and $(*)$, we have

Corollary 4.1. *A T_1 (resp. regular T_1) space is compact (resp. Lindelöf) iff it has the $\mathcal{B}$-property and satisfies the* DFCC *(resp.* DCCC*).*

Corollary 4.2. *(1) If X is a regular* DCCC *space, then $\mathcal{B}$-property $\leftrightarrow$ Lindelöfness $\leftrightarrow$ paracompactness.*

(2) If X is a regular DFCC *space, then*

$\mathcal{B}$-property $\leftrightarrow$ compactness $\leftrightarrow$ Lindelöfness $\leftrightarrow$ paracompactness.

Corollary 4.3. *Let X be a T_1 space having the $\mathcal{B}$-property, then the following are equivalent:*

(1) *X is ω_α-Lindelöf (resp. compact).*
(2) *X is $\omega_{\alpha+1}$-compact (resp. countably compact).*
(3) *X satisfies the* Dω_αCC *(resp.* DFCC*).*

From [D], we know that in a regular space the DFCC (resp. DCCC) and ω-starcompactness (resp. ω-star Lindelöfness) are equivalent, so we have

Corollary 4.4. *A regular ω-starcompact (resp. ω-star-Lindelöf) space with the $\mathcal{B}$-property is compact (resp. Lindelöf).*

Remark 4.1. (1) In Theorem 4.1, hence in its corollaries, the $\mathcal{B}$-property can not be replaced by the $\mathcal{D}$-property (resp. monotone normality [C], shrinkable property[MN], countable paracompactness). Actually, for any $\alpha \geq 0$, $[0, \omega_\alpha)$ has $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfies chain (F), where $\mathcal{W}(x) = \{[x, \beta] : x \leq \beta < \omega_\alpha\}$. So by Theorem 3 in [C], $[0, \omega_\alpha)$ is a monotonically normal space. It is known that in a monotonically normal space, every open cover is shrinkable (see Corollary 2.2 in [B]). Since the shrinkable property implies the $\mathcal{D}$-property, the space $[0, \omega_\alpha)$ has the $\mathcal{D}$-property (hence countable paracompactness). In Example 2.1, we have shown that $[0, \omega_{\alpha+2})$ (resp. $[0, \omega_1)$) is $\omega_{\alpha+1}$-compact (resp. ω-compact), and hence Dω_αCC (resp. DFCC), but it is not ω_α-Lindelöf (resp. compact).

(2) From Example 2.1 and Theorem 4.1, we can see that for any $\alpha \geq 0$, $[0, \omega_{\alpha+1})$ does not have the $\mathcal{B}$-property.

(3) Noticing paracompactness implies the $\mathcal{B}$-property and the implication is not reversible (see $(*)$), Corollary 4.1 improves Theorem 2.3 of [W] (i.e. a regular T_1-space is Lindelöf iff it is paracompact and satisfies the DCCC). Since in a T_1-space, ω_1-compact (resp. countably compact) implies the D-CCC (resp. DFCC) and the implication is not reversible (see Example 2.2), Corollary 4.1 also improves Theorem 2.1 and Corollary 2.2 of [Z] (i.e. a regular T_1 space is Lindelöf (resp. compact) iff it is ω_1-compact (resp. countably compact) and has the $\mathcal{B}$-property).

(4) In [D], the authors gave the following two figures:

compact	Lindelöf
$\downarrow$	$\downarrow$
countably compact	ω_1 – compact
$\downarrow$	$\downarrow$
strongly-1-starcompact	strongly-1-star-Lindelöf
$\downarrow$	$\downarrow$
1-starcompact	1 – star-Lindelöf
$\downarrow$	$\downarrow$
strongly-2-starcompact	strongly-2-star-Lindelöf
$\downarrow$	$\downarrow$
2-starcompact	2 – star-Lindelöf
$\downarrow$	$\downarrow$
$\cdots$	$\cdots$
$\downarrow$	$\downarrow$
ω – starcompact	ω – star-Lindelöf
Figure 2	Figure 3

The $\mathcal{B}$-property implies countable paracompactness (see $(*)$), which, in turn, implies pseudonormality [H]. Pseudonormal (hence countably paracompact) ω-starcompact spaces are known to be countably compact[D], but need not be compact. From Corollary 4.4, we see that in a regular T_1 space with the $\mathcal{B}$-property, all properties in Figure 2 (resp. Figure 3) are equivalent.

(5) In the Tychonoff space with the $\mathcal{B}$-property pseudocompactness (resp, pseudo-Lindelöfness[D]) implies compactness (Lindelöfness) since a Tychonoff space is pseudocompact (resp. pseudo-Lindelöf) iff it is ω-starcompact (resp. ω-star-Lindelöf[D]).

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