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PRODUCTS OF TOPOLOGICAL GROUPS WHICH SATISFY AN OPEN MAPPING THEOREM

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ABSTRACT. It is shown that the product of a $B(\mathcal{A})$ [resp., $B_r(\mathcal{A})$] topological group with a totally sup-complete, totally minimal [resp., sup-complete, minimal] topological group is a $B(\mathcal{A})$ [resp., $B_r(\mathcal{A})$] topological group.

All topological groups will be assumed Hausdorff unless otherwise stated. Recall, for example from [C, p. 1248] that a topological group G is called *minimal* if no strictly coarser (Hausdorff) topology makes G into a topological group, *totally minimal* if G/H is minimal for every closed normal subgroup H of G. Equivalently, G is totally minimal (resp., minimal) if and only if every continuous (resp., and injective) homomorphism from G onto a Hausdorff group is open. For an exhaustive treatment, see [DPS].

Let Γ denote topological closure, with appropriate subscripting when required to distinguish spaces or topologies. Let $\mathcal{V}(G)$ denote the filter of unit neighbourhoods of a topological group G. A homomorphism $f : G \to K$ of Hausdorff groups is *almost open* if $\Gamma_K f(V) \in \mathcal{V}(G)$ for every $V \in \mathcal{V}(G)$. After [H, p. 89–90], we say that G is a $B(\mathcal{A})$ (resp., $B_r(\mathcal{A})$) group if every continuous, almost open (resp., and injective) homomorphism from G onto any Hausdorff group is open. (In categorical terms, G is a $B_r(\mathcal{A})$ (resp., $B(\mathcal{A})$) group if and

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only if G is minimal (resp., totally minimal) in the category of Hausdorff topological groups with morphisms consisting of continuous, almost open homomorphisms. For relations to the open mapping and closed graph theorem for topological groups, see Chapter V of [H] and [G1].)

Clearly, every compact group is totally minimal, every totally minimal group is a $B(\mathcal{A})$ group, and every minimal group is a $B_r(\mathcal{A})$ group. Results of Stephenson [St], of Sulley [Su] and of the author [G1] can be combined to show that minimality and the $B_r(\mathcal{A})$ property are equivalent for totally bounded groups, as are total minimality and the $B(\mathcal{A})$ property. It was originally observed by Sunyach [Sy] that the group U of complex roots of unity, the torsion subgroup of the circle group, is a $B(\mathcal{A})$ group which is not complete and so, of course, not compact; since it is totally bounded, it is also totally minimal. Locally compact and complete metrizable groups are $B(\mathcal{A})$ groups [H], as are Čech–complete groups [B]. However, a locally compact Abelian group is minimal if and only if it is compact [St].

If s, t are Hausdorff group topologies on a group G with $s \subseteq t$, let $s(t)$ denote the group topology whose unit neighbourhood filter is given by $\{\Gamma_s V : V \in \mathcal{V}(t)\}$. Further, if N is a normal subgroup of (G, t), let $t|N$ and t/N denote, respectively, the relative topology of t on N and the natural quotient topology induced by t on G/N.

Proposition 1. *(i)* $s \subseteq s(t) \subseteq t$;
(ii) (G, t) *is a* $B_r(\mathcal{A})$ *group if and only if* $s = t$ *whenever* $s \subset t$, s *is Hausdorff, and* $t \subseteq s(t)$;
(iii) (G, t) *is a* $B(\mathcal{A})$ *group if and only if* G/H *is a* $B_r(\mathcal{A})$ *group for every closed normal subgroup* H *of* G.

Parts (i) and (ii) of the above proposition are Proposition 31.8 and Theorem 31.4 of [H], respectively, while part (iii) is a special case of Lemma 1.1 of [G1]. The latter paper also has a topological characterization of $B(\mathcal{A})$ groups.

Doichinov [D] showed that the product of a minimal topological group with a compact group is minimal. Eberhardt, S. Dierolf and Schwanengel [EDS] defined a topological group to be *sup–complete* if every net in G that is simultaneously a left and right Cauchy net is convergent in G, and to be *totally sup–complete* if every Hausdorff quotient of G is sup–complete. They then generalized Doichinov's result by showing that the product of a minimal group with a sup–complete group is minimal, and that the product of a totally minimal group with a totally sup–complete group is totally minimal. See Chapter 6 of [DPS] for a wide variety of related results.

The question of whether the $B_r(\mathcal{A})$ and $B(\mathcal{A})$ properties are preserved in products has remained unsettled. The author showed [G1] that the product of U with the real line (or, in fact, with the discrete group of integers) is not even a $B_r(\mathcal{A})$ group, even though both factors are $B(\mathcal{A})$ groups. However, any power of U is totally minimal and hence a $B(\mathcal{A})$ group [G2].

To show that results strongly analagous to those in [EDS] hold in this case, we make extensive use of the following result of Merzon [M], which was also exploited in [EDS].

Lemma 2 *[M]. Two comparable group topologies s, t on the group G are equal if there exists a subgroup N of G such that s and t induce the same topology on N and on the left coset space G/N.*

Theorem 3. *The product of a $B_r(\mathcal{A})$ group with a minimal, sup–complete group is a $B_r(\mathcal{A})$ group.*

Proof: Let X be a $B_r(\mathcal{A})$ group, Y a minimal and sup–complete topological group, t the product topology on $X \times Y$, s a Hausdorff topology on $X \times Y$ such that $s \subseteq t$ and $s(t) = s$, $N = \{e\} \times Y$. We employ Merzon's result (cf. Lemma 2, or Lemma 1 of [EDS]) to show that $s = t$ by establishing that $s|N = t|N$ and $s/N = t/N$.

Since N is homeomorphic to Y and so is minimal, $s|N \subseteq t|N$, and so $s|N = t|N$.

Since N is sup–complete, N is s–closed, and so s/N is a Hausdorff topology on $(X \times Y)/N$. Clearly, $s/N \subseteq t/N$. Applying the definition preceding Proposition 1, we let $(s/N)(t/N)$ denote the group topology on $(X \times Y)/N$ whose unit neighbourhood filter is given by $\{\Gamma_{s/N}W : W \in \mathcal{V}(t/N)\}$. It then follows from Proposition 1(i) that $s/N \subseteq (s/N)(t/N) \subseteq t/N$. Conversely, recall that N is (algebraically) normal in $X \times Y$. Then, for any $T \in \mathcal{V}(t)$, $(\Gamma_s T)N = (\cap\{TS : S \in \mathcal{V}(s)\})N \subseteq \cap\{TSN : S \in \mathcal{V}(s)\} = \cap\{(TN)(SN) : S \in \mathcal{V}(s)\} = \Gamma_{s/N}TN$. Therefore, $(s/N)(t/N) \subseteq s(t)/N = s/N$.

Now, G/N is topologically isomorphic to X, which is a $B_r(\mathcal{A})$ group, so it follows that $s/N = t/N$. Therefore, $s = t$, and $X \times Y$ is a $B_r(\mathcal{A})$ group. □

Theorem 4. *The product of a $B(\mathcal{A})$ group with a totally minimal, totally sup–complete group is a $B(\mathcal{A})$ group.*

Proof: Let X be a $B(\mathcal{A})$ group, Y totally minimal and totally sup– complete, Q a closed normal subgroup of $X \times Y$, t the Hausdorff quotient topology on $(X \times Y)/Q = G$, s a Hausdorff group topology on G such that $s \subseteq t$ and $s(t) = s$. We apply Merzon's criterion (Lemma 2) with $N = q(\{e\} \times Y)$.

Now N is sup–complete, and so is s-closed. Also, N is minimal, so $s|N = t|N$. As in Theorem 3, we have that $(s/N)(t/N) = s/N$. Now, G/N is topologically isomorphic to $(X \times Y)/q^{-1}q(\{e\} \times Y)$, which is in turn isomorphic with

$$[(X \times Y)/(\{e\} \times Y)]/[q^{-1}q(\{e\} \times Y)/(\{e\} \times Y)]$$

which is a quotient of X, and so a $B(\mathcal{A})$ group. It then follows that $s/N = t/N$, and the proof is complete. □

Corollary 5. *The product of a $B_r(\mathcal{A})$ (resp., $B(\mathcal{A})$) group with a compact group is a $B_r(\mathcal{A})$ (resp., $B(\mathcal{A})$) group.*

Let us conclude by observing that the group U, mentioned above, is a $B(\mathcal{A})$ group (in fact, totally minimal), but is not sup–complete. However, it is a torsion group, and so every element is contained in a finite, hence sup–complete, subgroup. However, $R \times U$ is not a $B_r(\mathcal{A})$ group. Therefore, condition (**) of [EDS] is not sufficient to produce an analogue of either Theorem 3 or 4.

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