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AUTOHOMEOMORPHISM GROUPS OF OSTASZEWSKI SPACES

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ABSTRACT. Assume $\diamond$. Let G be a countable group. Then there is an Ostaszewski space whose group of non-trivial homeomorphisms is G. Furthermore, many uncountable groups are also non-trivial homeomorphism groups of Ostaszewski spaces.

0. INTRODUCTION.

Ostaszewski spaces are well-known sources of counterexamples. They are also important canonical examples in the study of NAC (normal almost compact) spaces, which have been intensively studied by Fleissner, Kulesza, and Levy. Fleissner asked the author if an Ostaszewski space can have an infinite nontrivial autohomeomorphism group. The answer is emphatically yes. In fact, under e.g. $\diamond$, every countable group and many uncountable groups are the non-trivial homeomorphism groups of Ostaszewski spaces.

Preliminaries.

NAC's. A NAC is a regular Hausdorff space in which the collection of closed non-compact sets forms a filterbase. (Equivalently, a Hausdorff space X is a NAC iff X is normal and $\beta X\setminus X$ has at most one element, hence the terminology "normal almost compact.")

Ostaszewski spaces. An Ostaszewski space is a scattered, countable compact, locally compact, 0-dimensional, hereditarily separable NAC.

Scattered spaces. Every scattered space X can be written as an increasing continuous union $X = \cup_{\alpha<\delta} X_\alpha$ where $X_{\alpha+1} \backslash X_\alpha = X(\alpha)$ is a dense set of isolated points in $X^\alpha = X \backslash X_\alpha$, $X_0 = \emptyset$, and each X_α is open in X. For $x \in X$ we say $ht(x) = \alpha$ iff $x \in X(\alpha)$. We define $ht(X)$ to be the least α with $X(\alpha) = \emptyset$. Notice that if α is a limit, then the topology on X_α is the direct limit of the topologies on all X_β, where $\beta < \alpha$.

Thin-tall spaces. A thin-tall space is a regular Hausdorff locally compact scattered space of Cantor-Bendixson width ω and Cantor-Bendixson height ω_1, i.e. $X = \cup_{\alpha<\omega_1} X_\alpha$ and each $X(\alpha)$ is a countable dense set of isolated points in X^α. In particular, Ostaszewski spaces are thin-tall.

There is a canonical way of constructing thin-tall spaces. Suppose we know X_α, and want to construct $X_{\alpha+1}$. To each point $x \in X(\alpha)$ we assign a countable collection $\{u_{n,x} : n < \omega\}$ of compact clopen sets in X_α. Let $U_x = \cup_{n<\omega} u_{n,x}$. Then an open neighborhood base for x consists of all $\{x\} \cup U_x \backslash K$, where K is compact in X_α. We say that U_x determines the topology on x. U_x will have extra properties depending on the space we're trying to construct: e.g., to get a Hausdorff space, if $x \neq y$ then $U_x \cap U_y$ must be compact.

Non-trivial autohomeomorphisms. An autohomeomorphism f of a scattered space is trivial iff there is some infinite X^α on which f is the identity (equivalently, there is some infinite $X(\alpha)$ on which f is the identity); otherwise it is non-trivial. The group $N(X)$ of trivial autohomeomorphisms is normal; $G(X)$, the group of non-trivial autohomeomorphisms, is defined as $\mathrm{Aut}(X)/N(X)$.

Every autohomeomorphism of a scattered space X determines a permutation of $X(0)$; given a permutation of $X(0)$, we want to know when it extends to an autohomeomorphism of X. If it does extend, there is a uniquely defined extension: given f a permutation of $X(0)$ we define its extension $\hat{f}$ to X as follows: $\hat{f}(x)$ is the unique y so that for all $Z \subset X(0)$, $y \in \mathrm{cl}\ f[Z]$ iff $x \in \mathrm{cl}\ Z$. Note that $\hat{f}$ need not be defined everywhere.

Results. Dow and Simon proved that every countable group is some $G(X)$; the author has proved that several uncountable groups also are $G(X)$ for some thin-tall X. The spaces constructed in those results tend to be no prettier than they have to be: e.g. not countably compact, not hereditarily separable, and not NAC's. Can we extend these results to Ostaszewski or related spaces? Ostaszewski-like constructions need some sort of set-theoretic hypotheses, generally either $\diamond$ or

(*) the universe is a forcing extension by the partial order $Fn(\omega_1, 2)$.

Our results hold under $\diamond$ and (*), but I do not know whether, for example, $\diamond$ can be weakened to CH, or (*) can be weakened to

(**) there is a cofinal ω_1-sequence of Cohen reals.

For (*) the reader is referred to [Roitman$_1$], [Juhasz] and [Koszmider]. Proofs are stated for $\diamond$ only; the $\diamond$-like principle implied by (*) is mentioned at the end, as is a sketch of the adaptation of the $\diamond$ proof (*).

Theorem 1. *Assume $\diamond$ or (*). Let G be a countable group. Then there is an Ostaszewski space whose group of non-trivial autohomeomorphisms is G.*

Theorem 2. *Assume $\diamond$ or (*). There are Ostaszewski spaces whose non-trivial autohomeomporhism groups are uncountable; in particular, the free (Abelian) group on ω_1 generators is such a group.*

In the next two sections, we assume $\diamond$.

1. Proof of Theorem 1.

We begin with a claim of more general application.

Claim 3. *Let X be a countable, locally compact, scattered Hausdorff space. Let $x \in X$, and suppose E is a finite set of autohomeomorphisms of X so that $g(x) \neq h(x)$ for g, h distinct elements of E. Then there is a compact open neighborhood u of x such that $\{g[u] : g \in E\}$ is pairwise disjoint.*

Proof: Since X is Hausdorff, for all distinct $h, g \in E$ there are compact open neighborhoods v of $g(x)$, w of $h(x)$ with $v \cap w = \emptyset$. Let $u_{g,h} = g^{-1}[v] \cap h^{-1}[w]$. Let $u = \bigcap\{u_{g,h} : g, h$ are distinct elements of $E\}$.

To prove theorem 1, we first consider the case where G is infinite, indicating at the end of this section the necessary modifications for G finite.

Let $X(0) = G \times \{0\}$. Since G is countable it embeds as a countable subgroup of $X(0)!$ by the map $g(h, 0) = (gh, 0)$. Note that for all $x \in X(0)$ and all $h, g \in G$, if $hx = gx$ then $h = g$.

We will construct a thin-tall topology on $X = \bigcup_{\alpha<\omega_1} X_\alpha = \bigcup_{\alpha<\omega_1} X(\alpha)$ where $X(\alpha) = G \times \{\alpha\}$. The topology on X will be constructed inductively, where at stage α we construct neighborhood bases for points in $X(\alpha)$. Fix an enumeration $\{f_\alpha : \alpha < \omega_1\}$ of $X(0)!$ There are seven induction hypotheses, listed forthwith:

IH1. Each X_α is countable, locally compact, 0-dimensional, Hausdorff, not compact.

IH2. For each $g \in G$, its natural extension $\hat{g}$ is a non-trivial homeomorphism of each X_α, and, for all β, $\hat{g}(h, \beta) = (gh, \beta)$. (We notationally confuse $\hat{g}$ with g.)

IH3. For all $x \in X_\alpha$, and all $g, h \in G$, if $gx = hx$ then $g = h$. (This follows from IH2.)

IH4. Either $\hat{f}_\alpha | X_{\alpha+1}$ is not an autohomeomorphism of X_α, or $\hat{f}_\alpha | X(\alpha) = g | X(\alpha)$ for some $g \in G$.

Let $\{S_\alpha : \alpha \leq \omega_1\}$ be a diamond sequence for $X = G \times \omega_1$. By this we mean that each $S_\alpha \subset X_\alpha$ and that for any $Y \subset X$ $\{\alpha : S_\alpha = Y \cap X_\alpha\}$ is stationary. We will also require.

IH5. If S_α does not have compact closure in X_α then

(a) $X^\alpha \subset \text{cl } S_\alpha$.

(b) for each $\beta > \alpha$ some infinite subset of S_α is closed discrete in X_β.

Now let $\{(R_\alpha, T_\alpha) : \alpha < \omega_1\}$ be a double diamond sequence for $[\mathcal{P}(X)]^2$, i.e. each $R_\alpha, T_\alpha \subset X_\alpha$ and if $Y, Z \subset X$ then $\{\alpha : R_\alpha = Y \cap X_\alpha \text{ and } T_\alpha = Z \cap X_\alpha\}$ is stationary. We require

IH6. If R_α and T_α do not have compact closure in X_α then their closures are not disjoint in $X_{\alpha+1}$.

Let $\{W_\alpha : \omega \leq \alpha < \omega_1\}$ enumerate $[X]^\omega$ so that each $W_\alpha \subset X_\alpha$. Our final requirement is

IH7. If W_α does not have compact closure in X_α then it has an accumulation point in $X_{\alpha+1}$.

Suppose we can construct X to satisfy all these requirements. Then IH2 guarantees that every element of G is non-trivial; IH4 guarantees any other non-trivial autohomeomorphism equals some element of G mod trivial; IH5 guarantees that the space is hereditarily separable; IH6 guarantees that it is a NAC; and IH7 guarantees that it is countably compact.

Suppose we know the topology on X_α. The key lemmas are the following:

Lemma 4. *If Y is a subset of X_α whose closure is not compact then Y has an infinite closed discrete subspace in the topology on X_α.*

Proof: X_α is countable, so a subset Y of X_α is compact iff it is closed and countably compact. Suppose $Y \subset X_\alpha$ and cl Y is not compact. Since X_α is countable, we can find $\{u_n : n < \omega\}$ a collection of pairwise disjoint compact open sets covering all of X_α such that no finite subcollection covers Y. Let Z be an infinite subset of Y so $| Z \cap u_n | \leq 1$ for all n. Then Z is the desired set.

Lemma 5. *Suppose induction hypotheses 1 through 3 hold at α, that $\mathcal{Y} = \{Y_n : n < \omega\}$ is a pairwise disjoint family of infi-*

nite closed discrete subsets of X_α, and f is an autohomeomorphism of X_α so that for all $g \in G$, for all $\beta < \alpha, f|(X_\alpha)^\beta \neq g|(X_\alpha)^\beta$. Then we can construct $X_{\alpha+1}$ so that in $X_{\alpha+1}$ each Y_n has both a convergent subsequence and an infinite closed discrete subset, so that properties 1 through 3 hold, so that $cl_{X_{\alpha+1}} Y_i \cap cl_{X_{\alpha+1}} Y_j \neq \emptyset$ for all $i, j < \omega$, and so that f does not extend to an autohomeomorphism of $X_{\alpha+1}$.

Proof: Fix an enumeration $\{g_n : n < \omega\}$ of G where $g_0 =$id. We will construct $\{u_n : n < \omega\}$ a sequence of compact clopen sets so that

i. $\{g_i[u_n] : i \leq n < \omega\}$ is a pairwise disjoint cover of X_α

ii. for each $Y \in \mathcal{Y}$ and each $g \in G\{n : Y \cap g[u_n] \neq \emptyset\}$ is infinite.

iii. for all $g \in G\{n : f[u_n] \cap g[u_n] = \emptyset\}$ is infinite.

To construct the u_n's: Let $\{x_n : n < \omega\}$ list X_α, let $\{y_{i,m} : m < \omega\}$ list Y_i, let $\varphi : \omega \to \omega^2$ be 1-1 onto so that if $\varphi(r) = (s,t)$ then $s, t \leq r$. Suppose we have constructed $\{g_i[u_k] : i \leq k < n\}$. Let $v_n = \cup_{i \leq k < n} u_k$

a. If $n = 0$ mod 3, let j be the first number so $x_j \notin v_n$. Let $x^* = x_j$.

b. If $n = 1$ mod 3, $n = 3r + 1$, and $\varphi(r) = (s,t)$, let j be least with $y_{s,j} \notin v_n$. Let $x^* = g_t^{-1}(y_s, j)$.

c. If $n = 2$ mod 3, $n = 3r + 2$, and $\varphi(r) = (s,t)$, let j be the first number with $x_j \notin v_n$ and $f(x_j) \neq g_r(x_j)$. Let $x^* = x_j$.

By claim 3, we can find in each case a compact open neighborhood u of x^* with $\cup_{i \leq n} g_i[u] \cap v_n = \emptyset$, and, in case c., $f[u] \cap g_r[u] = \emptyset$. Let $u_n = u$. Property i. is ensured by a., property ii. by b., and property iii. by c..

By properties i., ii., iii., we can construct E an infinite subset of ω so that for each n and each g there are infinitely many $k \in E$ and infinitely many $s \notin E$ with

iv. $Y_n \cap g[u_k] \neq \emptyset$ and

v. $Y_n \cap g[u_s] \neq \emptyset$,

and for each g there are

vi. infinitely many $k \in E$ with $g[u_k] \cap g[u_k] = \emptyset$.

For each $y = (g_r, \alpha)$ let $V_y = \bigcup_{n \geq r} g_r[u_n]$. The V_y's are pairwise disjoint. Let $x = (\mathrm{id}, \alpha)$. Let $U_x = \bigcup_{k \in E} u_k$. For each $y = (g, \alpha)$ let $U_y = V_y \cap g[U_x]$ and let U_y determine y for all such y. Since the V_y's are pairwise disjoint, $X_{\alpha+1}$ is Hausdorff. IH1 through IH3 are immediate. By construction, for each $n, Y_n \setminus \bigcup_{y \in X(\alpha)} U_y$ is infinite and, for each $n, y, Y_n \cap U_y$ is infinite. We need to check the other properties of the lemma.

If f were to extend to a homeomorphism $\bar{f}$ of $X_{\alpha+1}$, what would $\bar{f}(x)$ be? By vi. it could not be any point (g, α) so f does not extend.

By property iv., for each n and each $y \in X(\alpha), y \in \mathrm{cl}_{X_{\alpha+1}} Y_n$. So each Y_n has many convergent subsequences, and no two distinct Y_n's have disjoint closures.

By property v., for each n, and each $g_r = g$, pick $s > r$ with $s \notin E$ and $Y_n \cap g[u_s] \neq \emptyset$. Let $x_r \in Y_n \cap g[u_s]$. Then $\{x_r : r < \omega\}$ is closed discrete in $X_{\alpha+1}$, since it avoids all U_y's and $\bigcup_{y \in X(\alpha)} U_y$ is clopen. So each Y_n has a closed discrete subset in $X_{\alpha+1}$.

Now we are ready to prove theorem 1. Suppose we know the topology on X_α and are ready to construct the topology on $X_{\alpha+1}$. For each $\beta < \alpha$ so that S_β does not have compact closure in X_β, we let Y_β be an infinite subset of S_β which is closed discrete in X_α. We can do this by IH5. If S_α does not have compact closure in X_α then by lemma 4 we let Y_α be an infinite subset of S_α which is closed discrete in X_α. If both R_α and T_α do not have compact closures in C_α then we let R be an infinite subset of R_α which is closed discrete in X_α and let T be an infinite subset of T_α which is closed discrete in X_α. If W_α does not have compact closure in X_α then we let W be an infinite subset of W_α which is closed discrete in X_α. Let $\mathcal{Y} = \{R < T, W\} \cup \{Y_\beta : \beta \leq \alpha\}$ and S_β does not have compact closure in $X_\alpha\}$. Applying lemma 5 completes the proof.

If G is finite, we modify the proof as follows: Set each $X(\alpha) = G \times \omega \times \{\alpha\}$. Again, G embeds as a countable subgroup of $X(0)!$ so that for all $x \in X(0)$ and all $h, g \in G$, if $hx = gx$ then $h = g$, and this property will be extended to all levels: we will have $\hat{g}(h, n, \alpha) = (gh, n, \alpha)$. Lemma 5 adapts as follows:

Let V_r be as before. Let E be as before. Partition E into infinite sets $\{P_n : n < \omega\}$. Let $U_n = \cup_{k \in P_n} u_k$. Let $V_r \cap g_r[U_n]$ determine (g_r, n, α).

2. Proof of theorem 2.

First, some notation. Given the ordinal α, $\alpha - 1 = \beta$ if $\alpha = \beta + 1$; if α is a limit, $\alpha - 1 = \alpha$. If the group G is generated by $\{g_\beta : \beta < \omega_1\}$ we let G_α be the group generated by $\{g_\beta : \beta < \alpha\}$.

Let's first show how to get $G(X)$ of size ω_1. Our final group G will be generated by a sequence $\{g_\alpha : \alpha < \omega_1\}$, where we construct G inductively along with X. We define $X(\alpha) = G_\alpha \times \{\alpha\}$ and add requirement

IH8. $G_{\alpha-1}$ is a group of autohomeomorphisms of X_α, for all α.

We drop IH4. Given $G_{\beta-1}, X_\beta$ for $\beta \leq \alpha$, construct $X(\alpha)$ as in section 1. Since every countable locally compact scattered space has uncountable many autohomeomorphisms, once we've constructed $X(\alpha)$, we let g_α be any non-trivial autohomeomorphism of $X_{\alpha+1}$ which is not in G_α, thus continuing the construction.

To get theorem 2 takes a little more work. For K a group, we say $\varphi : K \to Y!$ is an almost embedding iff φ is 1-1 and for every $g, h \in K$, $\{y : \varphi g \varphi h(y) \neq \varphi gh(y)\}$ is finite; we say K almost embeds in $Y!$ via φ.

Suppose G is a free group generated by $\{g_\beta : \beta < \omega_1\}$. Revise IH8 to

IH8′. G_α almost embeds in $X(\alpha)!$ via φ_α, for all α, where

(a) $\varphi_\alpha : G_\alpha \to \mathrm{Aut}(X_{\alpha+1})$,

(b) if $\beta < \alpha$ and $x \in X_{\beta+1}, g \in G_\beta$, then $\varphi_\alpha g(x) = \varphi_\beta g(x)$.

Revise IH4 as follows:

IH4′. If $f_\alpha \notin \varphi_\alpha[G_\alpha]$ then either f_α is not an autohomeomorphism of $X_{\alpha+1}$ or, for some $g \in G_\alpha, \{x \in X_{\alpha+1} : f(x) \neq \varphi_\alpha g(x)\}$ is compact.

Change IH3 to

IH3′. For every distinct $g, h \in \varphi_\alpha[G_\alpha]$, $\{x \in X(\alpha) : gx = hx\}$ is finite.

Recall that for α a successor the point-set of $X(\alpha)$ is $G_{\alpha-1} \times \{\alpha\}$. Given $G_{\beta-1}, X_\beta$ for $\beta \leq \alpha$, construct $X(\alpha)$ as before.

Define f' so that the group generated by $G_{\alpha+1}$ almost embeds in $X(\alpha)!$ via the map $\varphi' \supset \varphi_\alpha$, where $\varphi'(g_\alpha) = f$. (See e.g. [Roitman$_1$] on how to do this.) Note that for all γ every permutation of $X(\gamma)$ extends to an autohomeomorphism of $X_{\gamma+1}$, and use this to define$\varphi_{\alpha+1}$.

Theorem 2 generalizes as follows (the definitions are from [Roitman$_2$]):

Definition 9. If K is an infinite countable group, then the natural embedding from K into $K!$ is the function $\varphi_K : K \to K!$ where $\varphi_K(g)(h) = gh$ for all $h \in K$. Note that this is an almost embedding of G into $G!$.

Definition 10. G is stepwise expandable iff $G = \bigcup_{\alpha<\omega_1} G_\alpha$ is continuous increasing union, each G_α is a countable subgroup of G, G_0 is infinite, and for every α there is a function $\varphi_\alpha : G_{\alpha+1} \to G_\alpha!$ extending φ_{G_α} so that for every $g, h \in G_{\alpha+1}$ $\{k \in G_\alpha : \varphi_\alpha(gh)(k) \neq \varphi_\alpha(g)\varphi_\alpha(h)(k)\}$ is finite.

I.e. G is stepwise expandable iff the natural embedding of each G_α extends into an embedding (modulo finite) of $G_{\alpha+1}$ into $G_\alpha!$ (this is the stepwise expansion)

Theorem 11. *Assume $\diamond$ or (*). If G is stepwise expandable and $|G| = \omega_1$ then G is the non-trivial autohomeomorphism group of a thin-tall locally compact countably compact NAC.*

Many groups are stepwise expandable, in particular all groups of size ω_1 which are free, free Abelian, free idempotent, etc.. I don't know if there is any group of size ω_1 which is not.

3. A NOTE ON (*).

The principle implied by (*) which substitutes for $\diamond$ is:

(***) $H(\omega_1) = \cup_{\alpha<\omega_1} M_\alpha$ where if $Y \subset \omega_1$ then $\{\alpha : Y \cap \alpha \in M_\alpha\}$ is stationary, and for every α there is some $x_\alpha \in M_{\alpha+1}$ which is a Cohen real over M_α, where $H(\omega_1)$ is the collection of sets whose transitive closure is countable, and $\{M_\alpha : \alpha < \omega_1\}$ is a continuous sequence of submodels of $H(\omega_1)$.

The $\diamond$ proof adapts as follows:

In lemma 5, at stage $\alpha + 1$ we destroy all $f \in M_\alpha \setminus G$. The u_n's are constructed not by induction but by a countable forcing partial order, i.e. by adding a Cohen real. Genericity automatically takes care of condition iii; we don't need case c..

The reason we need (*) and not the weaker (**) is to make the space Ostaszewski; it is not known if there is an Ostaszewski space under CH or (**).

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