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MEAGER-NOWHERE DENSE GAMES (III): REMAINDER STRATEGIES.

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ABSTRACT. Player ONE chooses a meager set and TWO, a nowhere dense set per inning. They play ω innings. ONE's consecutive choices must form a (weakly) increasing sequence. TWO wins if the union of the chosen nowhere dense sets covers the union of the chosen meager sets. A strategy of TWO which depends on knowing only the uncovered part of the most recently chosen meager set is said to be a remainder strategy. TWO has a winning remainder strategy for this game played on the real line with its usual topology.

1. INTRODUCTION

A variety of topological games from the class of meager-nowhere dense games were introduced in the papers [B-J-S], [S1] and [S2]. The existence of winning strategies which use only the most recent move of either player (so-called coding strategies) and the existence of winning strategies which use only a bounded number of moves of the opponent as information (so-called k-tactics) are studied there and in [K] and [S3]. These studies are continued here for yet another fairly natural type of strategy, the so-called *remainder strategy.*

The symbol $J_{\mathbb{R}}$ denotes the ideal of nowhere dense subsets of the real line (with its usual topology), while the symbol "$\subset$" is used exclusively to denote " is a proper subset of ". Let (S, τ) be a T_1-space without isolated points, and let J be its

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ideal of nowhere dense subsets. The symbol $\langle J \rangle$ denotes the collection of meager subsets of the space. For Y a subset of S, the symbol $J\lceil_Y$ denotes the set $\{T \in J : T \subseteq Y\}$.

The game $WMEG(J)$ (defined in [S2]) proceeds as follows: In the first inning ONE chooses a meager set M_1, and TWO responds with a nowhere dense set N_1. In the second inning ONE chooses a meager set M_2, subject to the rule that $M_1 \subseteq M_2$; TWO responds with a nowhere dense set N_2, and so on. The players play an inning for each positive integer, thus constructing a play $(M_1, N_1, \ldots, M_k, N_k, \ldots)$ of $WMEG(J)$. TWO wins such a play if $\bigcup_{k=1}^{\infty} M_k = \bigcup_{k=1}^{\infty} N_k$. A strategy of TWO of the form $N_1 = F(M_1)$ and $N_{k+1} = F(M_{k+1} \setminus (\bigcup_{j=1}^{k} N_j))$ for all k is said to be a *remainder strategy.*

It is clear that TWO has a winning remainder strategy in $WMEG(J)$ if $J = \langle J \rangle$. The situation when $J \subset \langle J \rangle \subseteq \mathcal{P}(S)$, studied in Section 2, is not so easy. We prove among other things Theorem 1, which implies that TWO has a winning remainder strategy in $WMEG(J_{\mathbb{R}})$.

The game $WMG(J)$ proceeds just like $WMEG(J)$; only now the winning condition for TWO is relaxed so that TWO wins if $\bigcup_{n=1}^{\infty} M_n \subseteq \bigcup_{n=1}^{\infty} N_n$. In Section 3 we study remainder strategies for this game. In Section 4 we discuss the game $SMG(J)$. In Section 5 we attend to the version $VSG(J)$.

For convenience we also consider the "random equal game on J", denoted $REG(J)$. It is played as follows: $(M_1, N_1, \ldots, M_k, N_k, \ldots)$ is a play of $REG(J)$ if $M_k \in \langle J \rangle$ and $N_k \in J$ for each k. TWO is declared the winner of this play if $\cup_{k=1}^{\infty} M_k = \cup_{k=1}^{\infty} N_k$. We shall use the fact that TWO has a winning perfect information strategy in $REG(J)$.

Theorem 8 is due to Winfried Just, while Theorem 14 is due to Fred Galvin. I thank Professors Galvin and Just for kindly permitting me to present their results here and for fruitful conversations and correspondence.

2. The weakly monotonic equal game, $WMEG(J)$.

When defining a remainder strategy F for TWO in $WMEG(J)$, we shall take care that $F(A) \subseteq A$ and $F(A) \neq \emptyset$ if (and only if) $A \neq \emptyset$, for each $A \in \langle J \rangle$. Otherwise, the strategy F is sure not to be a winning remainder strategy for TWO in $WMEG(J)$. We shall also use the fact that if $(M_1, N_1, \ldots, M_k, N_k, \ldots)$ is a play of $WMEG(J)$, then $M_k \setminus M_{k+1} = N_k$ for each k, without further mention.

Theorem 1. *If* $(\forall X \in \langle J \rangle \setminus J)(cof(\langle J \rangle, \subset) \leq |J\lceil_X|)$, *then T-WO has a winning remainder strategy in* $WMEG(J)$.

Theorem 1 follows from the next two lemmas.

Lemma 2. *If* $cof(\langle J \rangle, \subset)$ *is infinite and* $(\forall X \in \langle J \rangle \setminus J)$ $(cof(\langle J \rangle, \subset) \leq |J\lceil_X|)$, *then TWO has a winning remainder strategy in* $WMEG(J)$.

Proof: Let $\mathcal{A} \subset \langle J \rangle \setminus J$ be a cofinal family of minimal cardinality. Then $|\mathcal{A}| \leq |\mathcal{P}(X)|$ for each $X \in \langle J \rangle \setminus J$.

For each $Y \in J$ such that $|\mathcal{A}| \leq |\mathcal{P}(Y)|$ the set Y is infinite: Write $Y = \bigcup_{n=1}^{\infty} Y_n$ where $\{Y_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection such that $|Y_n| = |Y|$ for each n. Choose for each n a surjection $\Psi_n^Y : \mathcal{P}(Y_n) \setminus \{\emptyset, Y_n\} \rightarrow {}^{<\omega}\mathcal{A}$.

If for $X \in \langle J \rangle \setminus J$ there is no $Y \in J\lceil_X$ such that $|\mathcal{A}| \leq |\mathcal{P}(Y)|$, then $|Y| < |X|$ for each $Y \in J\lceil_X$: we fix a decomposition $X = \bigcup_{n=1}^{\infty} X_n$ where $\{X_n : n \in \mathbb{N}\}$ is a disjoint collection of sets from $\langle J \rangle \setminus J$. For each such X_n we further fix a representation $X_n = \bigcup_{m=1}^{\infty} X_{n,m}$ where $X_{n,1} \subseteq X_{n,2} \subseteq \ldots$ are from J, and a surjection $\Theta_n^X : J\lceil_{X_n} \rightarrow {}^{<\omega}\mathcal{A}$.

Let U and V be sets in $\langle J \rangle$ such that we have chosen a decomposition $U = \cup_{n=1}^{\infty} U_n$ as above. The notation $U \subseteq^* V$ denotes that there is an m such that $U_n \subseteq V$ for each $n \geq m$; we say that m *witnesses that* $U \subseteq^* V$.

Fix a well-ordering $\prec$ of $\langle J \rangle$. For $X \in \langle J \rangle$ we define:

(1) $\Theta(X)$: the $\prec$-first element A of $\mathcal{A}$ such that $X \subseteq A$,
(2) $\Phi(X)$: the $\prec$-first element Z of $\langle J \rangle \setminus J$ such that $Z \subseteq^* X$ whenever this is defined, and the empty set otherwise,

(3) $k(X)$: the smallest natural number which witnesses that $\Phi(X) \subseteq^* X$ whenever $\Phi(X) \neq \emptyset$, and 0 otherwise,
(4) $\Gamma(X)$: the $\prec$-first $Y \in J$ such that $|J\lceil_X| \leq |\mathcal{P}(Y)|$ and $Y \subseteq^* X$ whenever this is defined, and the empty set otherwise, and
(5) $m(X)$: the smallest natural number which witnesses that $\Gamma(X) \subseteq^* X$ whenever $\Gamma(X) \neq \emptyset$, and 0 otherwise.

Let G be a winning perfect information strategy for TWO in $REG(J)$. We are now ready to define TWO's remainder strategy $F : \langle J \rangle \to J$. Let $B \in \langle J \rangle$ be given.
$\underline{B \in J}$: Then we define $F(B) = B$.
$\underline{B \notin J}$: Then $k(B) \geq 1$. We distinguish between two cases:
Case 1: $\Gamma(B) \neq \emptyset$. Then $m(B) \geq 1$.
Write Y for $\Gamma(B)$ and n for $m(B)$. For $1 \leq j \leq n$ define σ_j so that

$$\sigma_j = \begin{cases} \Psi_j^Y(Y_j \backslash B) & \text{if } Y_j \backslash B \notin \{\emptyset, Y_j\} \\ \emptyset & \text{otherwise} \end{cases}$$

Let τ be $\sigma_1 \frown \cdots \frown \sigma_n \frown \langle \Theta(B) \rangle$, the concatenation of these finite sequences, and choose $V \in \mathcal{P}(Y_{n+1}) \backslash \{\emptyset, Y_{n+1}\}$ so that $\Psi_{n+1}^Y(V) = \tau$. Then define $F(B) = B \cap [Y_1 \cup \cdots \cup Y_n \cup V \cup ((\cup\{G(\sigma) : \sigma \subseteq \tau\}) \backslash Y)]$.
Case 2: $\Gamma(B) = \emptyset$.
Write X for $\Phi(B)$ and n for $k(B)$. For $1 \leq j \leq n$ define σ_j so that

$$\sigma_j = \begin{cases} \Theta_j^X(X_j \backslash B) & \text{if } X_j \backslash B \in J \\ \emptyset & \text{otherwise} \end{cases}$$

Let τ be $\sigma_1 \frown \cdots \frown \sigma_n \frown \langle \Theta(B) \rangle$, and choose $V \in J\lceil_{X_{n+1}}$ such that $\Theta_{n+1}^X(V) = \tau$. Then define $F(B) = B \cap [X_{1,n+1} \cup \cdots \cup X_{n,n+1} \cup V \cup ((\cup\{G(\sigma) : \sigma \subseteq \tau\}) \backslash X)]$.

This defines $F(B)$. From its definition it is clear that $F(B) \subseteq B$ for each $B \in \langle J \rangle$. To see that F is a winning remainder strategy for TWO in $WMEG(J)$, consider a play $(M_1, N_1, \ldots, M_k, N_k, \ldots)$ during which TWO followed the strategy F. To facilitate the exposition we write:

(1) B_1 for M_1 and B_{j+1} for $M_{j+1} \backslash \cup_{i=1}^{j} N_i$,

(2) Y^j for $\Gamma(B_j)$,
(3) X^j for $\Phi(B_j)$,
(4) A^j for $\Theta(B_j)$,
(5) k_j for $k(B_j)$ and
(6) m_j for $m(B_j)$.

We must show that $\cup_{j=1}^{\infty} B_j \subseteq \cup_{j=1}^{\infty} N_j$. We may assume that $B_j \notin J$ for each j.

Suppose that $Y^{j+1} \neq \emptyset$ for some j. Then N_{j+1}, defined by Case 1, is of the form $B_{j+1} \cap [Y_1^{j+1} \cup \cdots \cup Y_{m_{j+1}}^{j+1} \cup V_{j+1} \cup ((\cup\{G(\sigma) : \sigma \subseteq \tau_{j+1}\}) \backslash Y^{j+1})]$ where V_{j+1} and τ_{j+1} have the obvious meanings. Thus $Y^{j+1} \subseteq^* B_{j+2}$ is a candidate for Y^{j+2}, and $Y^{j+2} \neq \emptyset$, so that N_{j+2} is also defined by Case 1.

We conclude that if $Y^j \neq \emptyset$ for some j, then $Y^i \neq \emptyset$ and $Y^{i+1} \preceq Y^i$ for each $i \geq j$. Since $\prec$ is a well-order, there is a fixed k such that $Y^i = Y^k$ for all $i \geq k$. Let Y be this common value of $Y^i, i \geq k$. An inductive computation shows that $(A^k, \ldots, A^j) \subseteq \tau_j$ for each $j \geq k$. But then $B_j \cap [(G(A^k) \cup \cdots \cup G(A^k, \ldots, A^j)) \backslash Y] \subseteq N_j$ for each $j \geq k$, so that $\cup_{j=k}^{\infty} B_j \backslash Y \subseteq \cup_{j=k}^{\infty} N_j$. It is also clear that $Y \cap (\cup_{j=1}^{\infty} B_j) \subseteq \cup_{j=k}^{\infty} N_j$. The monotonicity of the sequence of M_j-s implies that TWO has won this play.

The other case to consider is that $Y^{j+1} = \emptyset$ for all j. In this case, $X^{j+1} \neq \emptyset$ for each j. Then N_{j+1}, defined by Case 2, is of the form:

$$B_{j+1} \cap [X_{1,k_{j+1}}^{j+1} \cup \cdots \cup X_{k_{j+1},k_{j+1}}^{j+1} \cup V_{j+1} \cup \\ ((\cup\{G(\sigma) : \sigma \subseteq \tau_{j+1}\}) \backslash X^{j+1})],$$

where V_{j+1} and τ_{j+1} have the obvious meaning. Now $X^{j+1} \subseteq^* B_{j+2}$, and X^{j+1} is a candidate for X^{j+2}. It follows that $X^{j+2} \preceq X^{j+1}$ for each $j < \omega$. Since $\prec$ is a well-order we once again fix k such that $X^j = X^k$ for all $j \geq k$. Let X denote X^k. As before, $(A^k, \ldots, A^j) \subseteq \tau_j$ for each such j, and it follows that TWO also won these plays. $\square$

Lemma 3. *If $\langle J \rangle = \mathcal{P}(S)$, then TWO has a winning remainder strategy in $WMEG(J)$.*

Proof: Let $\prec$ be a well-order of $\mathcal{P}(S)$, and write $S = \bigcup_{n=1}^{\infty} S_n$ such that $S_n \in J\backslash\{\emptyset\}$ for each n, and the S_n-s are pairwise disjoint. For each countably infinite $Y \in J$ write $Y = \bigcup_{n=1}^{\infty} Y_n$ so that $\{Y_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of nonempty finite sets. For X and Y in $\langle J \rangle$ write $Y \subseteq^* X$ if $Y\backslash X$ is finite.

For each $X \in \langle J \rangle\backslash J$, either there is an infinite $Y \in J\lceil_X$, or else X is countably infinite.

In the first of these cases, let $\Phi(X)$ be the $\prec$-first countably infinite element Y of J such that $Y \subseteq^* X$, and let $m(X)$ be the smallest n such that $Y_m \subseteq X$ for all $m \geq n$.

In the second case, let $\Phi(X)$ be the $\prec$-least element Y of $\langle J \rangle\backslash J$ such that $Y \subseteq^* X$, and let $m(X)$ be the minimal n such that $\Phi(X) \cap S_m \subseteq X$ for all $m \geq n$. Also write $\Phi(X)_j$ for $\Phi(X) \cap S_j$ for each j, in this case.

Then define $F(X)$ so that

(1) $F(X) = X$ if $X \in J$, and
(2) $F(X) = X \cap [(S_1 \cup \cdots \cup S_{m(X)})\backslash\Phi(X)) \cup (\Phi(X)_1 \cup \cdots \cup \Phi(X)_{m(X)})]$

Then F is a winning remainder strategy for TWO. □

Corollary 4. *Player TWO has a winning remainder strategy in* $WMEG(J_{\mathbb{R}})$.

We shall later see that the sufficient condition for the existence of a winning coding strategy given in Theorem 1 is to some extent necessary (Theorems 8 and 14). However, this condition is not absolutely necessary. First, note that for any decomposition $S = \bigcup_{j=1}^{k} S_k$, the following statements are equivalent:

(1) TWO has a winning remainder strategy in $WMEG(J)$,
(2) For each j, TWO has a winning remainder strategy in $WMEG(J\lceil_{S_j})$.

Now let S be the disjoint union of the real line and a countable set S^*. Define $X \in J$ if $X \cap S^*$ is finite and $X \cap \mathbb{R} \in J_{\mathbb{R}}$. Then $S^* \in \langle J \rangle$, and $J\lceil_{S^*}$ is a countable set, while $cof(\langle J \rangle, \subset)$

is uncountable. According to Corollary 4 and Lemma 3, TWO has a winning remainder strategy in $WMEG(J)$.

Let λ be an infinite cardinal of countable cofinality. For $\kappa \geq \lambda$, declare a subset of κ to be open if it is either empty, or else has a complement of cardinality less than λ. With this topology, $J = [\kappa]^{<\lambda}$.

Corollary 5. *Let λ be a cardinal of countable cofinality, and let $\kappa > \lambda$ be a cardinal number. If $cof([\kappa]^{\lambda}, \subset) \leq \lambda^{<\lambda}$, then TWO has a winning remainder strategy in $WMEG([\kappa]^{<\lambda})$.*

Recall (from [S2]) that G is a coding strategy for TWO if: $N_1 = G(\emptyset, M_1)$ and $N_{k+1} = G(N_k, M_{k+1})$ for each k.

If F is a winning remainder strategy for TWO in $WMEG(J)$, then the function G which is defined so that $G(W, B) = W \cup F(B \setminus W)$ is a winning coding strategy for TWO in $WMEG(J)$. Thus, Corollary 4 solves Problem 2 of [S2] positively. Also, Theorem 6 of [S2] implies that TWO does not have a winning coding strategy in $WMEG([\omega_1]^{<\aleph_0})$.

Let $\mathcal{A}$ be a subset of $\langle J \rangle$. The game $WMEG(\mathcal{A}, J)$ is played like $WMEG(J)$, except that ONE is confined to choosing meager sets which are in $\mathcal{A}$ only. Thus, $WMEG(J)$ is the special case of $WMEG(\mathcal{A}, J)$ for which $\mathcal{A} = \langle J \rangle$.

For cofinal families $\mathcal{A} \subset \langle J \rangle$ which have the special property that $A \neq B \Leftrightarrow A \Delta B \notin J$, there is an equivalence between the existence of winning coding strategies and winning remainder strategies in $WMEG(\mathcal{A}, J)$.

Proposition 6. *Let $\mathcal{A} \subset \langle J \rangle$ be a cofinal family such that for A and B elements of $\mathcal{A}$, $A \neq B \Leftrightarrow A \Delta B \notin J$. Then the following statements are equivalent:*

(1) *TWO has a winning coding strategy in $WMEG(\mathcal{A}, J)$.*
(2) *TWO has a winning remainder strategy in $WMEG(\mathcal{A}, J)$.*

Proof: We must verify that 1 implies 2. Thus, let F be a winning coding strategy for TWO in $WMEG(\mathcal{A}, J)$. We define a remainder strategy G. Let X be given. If $X \in \mathcal{A}$ we define $G(X) = F(\emptyset, X)$. If $X \notin \mathcal{A}$ but there is an $A \in \mathcal{A}$

such that $X \subset A$ and $A \backslash X \in J$, then by the property of $\mathcal{A}$ there is a unique such A and we set $T = A\backslash X (\in J)$. In this case define $G(X) = F(T, A)$. In all other cases we put $G(X) = \emptyset$. Then G is a winning remainder strategy for TWO in $WMEG(\mathcal{A}, J)$. $\square$

It is not always the case that there is a cofinal $\mathcal{A} \subset \langle J \rangle$ which satisfies the hypothesis of Proposition 6. For example, let $J \subset \mathcal{P}(\omega_2)$ be defined so that $X \in J$ if, and only if, $X \cap \omega$ is finite and $X \cap (\omega_2 \backslash \omega)$ has cardinality at most $\aleph_1$. Let $\{S_\alpha : \alpha < \omega_2\}$ be a cofinal family. Choose $\alpha \neq \beta \in \omega_2$ such that:

(1) $\omega \subset (S_\alpha \cap S_\beta)$ and
(2) $S_\alpha \neq S_\beta$.

Then $S_\alpha \Delta S_\beta \in J$.

Proposition 6 together with the proof of Theorem 6 of [S2] show that if $\mathcal{A}$ is any stationary subset of ω_1, then TWO does not have a winning remainder strategy in $WMEG(\mathcal{A}, [\omega_1]^{<\aleph_0})$. This result is strengthened in Theorem 8 below.

Though there may be cofinal families $\mathcal{A}$ such that TWO does not have a winning remainder strategy in $WMEG(\mathcal{A}, J)$, there may for this very same J also be cofinal families $\mathcal{B} \subset \langle J \rangle$ such that TWO does have a winning remainder strategy in $WMEG(\mathcal{B}, J)$.

Theorem 7. *Let λ be an infinite cardinal number of countable cofinality. If $\kappa > \lambda$ is a cardinal for which $cof([\kappa]^\lambda, \subset) = \kappa$, then there is a cofinal family $\mathcal{A} \subset [\kappa]^\lambda$ such that TWO has a winning remainder strategy in $WMEG(\mathcal{A}, [\kappa]^{<\lambda})$.*

Proof: Let $(B_\alpha : \alpha < \kappa)$ bijectively enumerate a cofinal subfamily of $[\kappa]^\lambda$. Write $\kappa = \cup_{\alpha<\kappa} S_\alpha$ where $\{S_\alpha : \alpha < \kappa\} \subset [\kappa]^\lambda$ is a pairwise disjoint family.

Define: $A_\alpha = \{\alpha\} \cup (\cup_{x \in B_\alpha} S_x)$ for each $\alpha < \kappa$, and put $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$. Then $\mathcal{A}$ is a cofinal subset of $[\kappa]^\lambda$. Also let $\Psi : \mathcal{A} \to \kappa$ be such that $\Psi(A_\alpha) = \alpha$ for each $\alpha \in \kappa$.

Choose a sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \ldots$ of cardinal numbers converging to λ. For each $A \in \mathcal{A}$ we write $A =$

$\cup_{n=1}^{\infty} A^n$ where $A^1 \subset A^2 \subset \dots$ are such that $|A^n| = \lambda_n$ for each n.

Now define TWO's remainder strategy F as follows:

(1) $F(A) = \{\Psi(A)\} \cup A^1$ for $A \in \mathcal{A}$,
(2) $F(A) = \{\Psi(B)\} \cup (\cup(\{C^{m+1} : \Psi(C) \in \Gamma(A)\}) \cap B) \cup B^{m+1}$ if $A \notin \mathcal{A}$ but $A \subset B$ and $|B \backslash A| < \lambda$ for some $B \in \mathcal{A}$. Observe that this B is unique. In this definition, $\Gamma(A) = B \backslash A$, and m is minimal such that $|\Gamma(A)| \leq \lambda_m$.
(3) $F(A) = \emptyset$ in all other cases.

Observe that $|F(A)| < \lambda$ for each A, so that F is a legitimate strategy for TWO. To see that F is a winning remainder strategy for TWO, consider a play $(M_1, N_1, \dots, M_k, N_k, \dots)$ of $WMEG(\mathcal{A}, [\kappa]^{<\lambda})$ during which TWO used F.

Write $M_i = A_{\alpha_i}$ for each i. By the rules of the game we have: $A_{\alpha_1} \subseteq A_{\alpha_2} \subseteq \dots$. Also, $N_1 = \{\alpha_1\} \cup A^1_{\alpha_1}$ and n_1 is minimal such that $|N_1| \leq \lambda_{n_1}$. An inductive computation shows that $N_{k+1} = F(M_{k+1} \backslash (\cup_{j=1}^k N_j))$ is the set

$$([\{\alpha_{k+1}\} \cup (\cup\{A_\gamma^{n_k+1} : \gamma \in N_k\}) \cap A_{\alpha+k+1}) \cup A_{\alpha_{k+1}}^{n_k+1}$$

from which it follows that:

(1) $N_1 \subseteq N_2 \subseteq \dots \subseteq N_k \subseteq \dots$,
(2) $n_1 < n_2 < \dots < n_k < \dots$,
(3) $\alpha_j \in N_k$ whenever $j \leq k$, and thus
(4) $A^p_{\alpha_j} \subseteq N_k$ for $j \leq k$ and $p \leq n_{k-1}$.

The result follows from these remarks. □

3. The weakly monotonic game $WMG(J)$.

It is clear that if TWO has a winning remainder strategy in $WMEG(J)$, then TWO has a winning remainder strategy in $WMG(J)$.

Problem 1. *Is it true that if TWO has a winning remainder strategy in $WMG(J)$, then TWO has a winning remainder strategy in $WMEG(J)$?*

As with $WMEG(J)$, a winning remainder strategy for TWO in $WMG(J)$ gives rise to the existence of a winning coding strategy for TWO. In general, the statement that TWO has a winning remainder strategy in $WMG(J)$ is stronger than the statement that TWO has a winning coding strategy. To see this, recall that TWO has a winning coding strategy in $WMG([\omega_1]^{<\aleph_0})$ (see Theorem 2 of [S2]) while, according to the next theorem, TWO does not have a winning remainder strategy in $WMG([\omega_1]^{<\aleph_0})$.

Theorem 8 (Just) *If $\kappa \geq \aleph_1$, then TWO does not have a winning remainder strategy in $WMG([\kappa]^{<\aleph_0})$.*

Proof: Let F be a remainder strategy for TWO. For each $\alpha < \omega_1$ we put

$$\Phi(\alpha) = \sup(\cup\{F(\alpha\backslash T) : T \in [\alpha]^{<\aleph_0}\} \cup \alpha).$$

Then $\Phi(\alpha) \geq \alpha$ for each such α. Choose a closed, unbounded set $C \subset \omega_1$ such that:

(1) $\Phi(\gamma) < \alpha$ whenever $\gamma < \alpha$ are elements of C, and
(2) each element of C is a limit ordinal.

Then, by repeated use of Fodor's pressing down lemma, we inductively define a sequence $((\phi_1, S_1, T_1), \dots, (\phi_n, S_n, T_n), \dots)$ such that:

(1) $C \supset S_1 \supset \dots \supset S_n \supset \dots$ are stationary subsets of ω_1,
(2) $F(\alpha) \cap \alpha = T_1$ for each $\alpha \in S_1$, and
(3) $F(\alpha\backslash(T_1 \cup \dots \cup T_n)) = T_{n+1}$ for each n and each $\alpha \in S_n$.

Put $\xi = \sup(\cup_{n=1}^{\infty} T_n) + \omega$. Choose $\alpha_n \in S_n$ so that $\xi \leq \alpha_1 < \alpha_2 < \dots < \alpha_n < \dots$. By the construction we have: $F(\alpha_1) \cap \xi = T_1$ and $F(\alpha_{n+1}\backslash(T_1 \cup \dots \cup T_n)) \cap \xi = T_{n+1}$ for each n.

Then $(\cup_{n=1}^{\infty} T_n) \cap \xi \subset \xi = (\cup_{n=1}^{\infty} \alpha_n) \cap \xi$, and TWO lost this play of $WMG([\omega_1]^{<\aleph_0})$. □

For a cofinal family $\mathcal{A} \subseteq \langle J \rangle$, $WMG(\mathcal{A}, J)$ proceeds just like $WMG(J)$, except that ONE must now choose meager sets from $\mathcal{A}$ only. The proof of Theorem 8 shows that for every stationary set $\mathcal{A} \subseteq \omega_1$ TWO does not have a winning remainder strategy in $WMG(\mathcal{A}, [\omega_1]^{<\aleph_0})$. This should be contrasted with Theorem 7, which implies that there are many uncountable cardinals κ such that for some cofinal family $\mathcal{A} \subset [\kappa]^{\aleph_0}$, TWO has a winning remainder strategy in $WMG(\mathcal{A}, [\kappa]^{<\aleph_0})$.

4. The strongly monotonic game $SMG(J)$.

A sequence $(M_1, N_1, \ldots, M_k, N_k, \ldots)$ is a play of the strongly monotonic game if: $M_k \cup N_k \subseteq M_{k+1} \in \langle J \rangle$, and $N_k \in J$ for each k. Player TWO wins such a play if $\cup_{j=1}^{\infty} M_j = \cup_{j=1}^{\infty} N_j$. These rules give TWO more control over how ONE's meager sets increase as the game progresses. This game was studied in [B-J-S] and [S1]. It is clear that if TWO has a winning remainder strategy in $WMG(J)$, then TWO has a winning remainder strategy in $SMG(J)$. The converse is also true:

Lemma 9. *If TWO has a winning remainder strategy in $SMG(J)$, then TWO has a winning remainder strategy in $WMG(J)$.*

Proof: Let F be a winning remainder strategy for TWO in $SMG(J)$. We show that it is also a winning remainder strategy for TWO in $WMG(J)$.

Let $(M_1, N_1, \ldots, M_k, N_k, \ldots)$ be a play of $WMG(J)$ during which TWO used F as a remainder strategy. Put $M_1^* = M_1$ and $M_{k+1}^* = M_{k+1} \cup (N_1 \cup \cdots \cup N_k)$ for each k. Then $(M_1^*, N_1, \ldots, M_k^*, N_k, \ldots)$ is a play of $SMG(J)$ during which TWO used the winning remainder strategy F. It follows that $\cup_{k=1}^{\infty} M_k \subseteq \cup_{k=1}^{\infty} N_k$, so that TWO won the F-play of $WMG(J)$. □

The additional strategic value to TWO of the rules of the strongly monotonic game is revealed by considering the games $SMG(\mathcal{A}, J)$ for cofinal $\mathcal{A} \subseteq \langle J \rangle$.

Lemma 10. *If TWO has a winning coding strategy in $WMG(J)$ and if $\mathcal{A} \subset \langle J \rangle$ is a cofinal family such that $A \Delta B \notin J$ whenever $A \neq B$ are in $\mathcal{A}$, then TWO has a winning remainder strategy in $SMG(\mathcal{A}, J)$.*

Proof: One can show that if TWO has a winning coding strategy in $WMG(J)$, then TWO has a winning coding strategy F which has the property that $N \subseteq F(N,M)$ for all $(N,M) \in J \times \langle J \rangle$ – see [S4]. Let F be such a winning coding strategy for TWO in $WMG(J)$. Also let $\mathcal{A}$ be a cofinal family as in the hypotheses. If B is not in $\mathcal{A}$, but there is an $A \in \mathcal{A}$ such that $B \subset A$ and $A \backslash B \in J$, then there is a unique such A. Let $\Psi(B) \in \mathcal{A}$ denote such an A when this happens.

Define a remainder strategy G for TWO as follows. Let $B \in \langle J \rangle$ be given:

$$G(B) = \begin{cases} F(\emptyset, B) & \text{if } B \in \mathcal{A} \\ F(\Psi(B)\backslash B, \Psi(B)) & \text{if } B \notin \mathcal{A}, \text{ but } \Psi(B) \text{ is defined} \\ \emptyset & \text{otherwise} \end{cases}$$

Then G is a winning remainder strategy for TWO. □

Corollary 11. *Let λ be a cardinal number of countable cofinality. For each $\kappa \geq \lambda$, there is a cofinal family $\mathcal{A} \subset [\kappa]^\lambda$ such that TWO has a winning remainder strategy in $SMG(\mathcal{A}, J)$.*

Proof: Write $\kappa = \cup_{\alpha<\kappa} S_\alpha$ where $\{S_\alpha : \alpha < \kappa\}$ is a disjoint collection of sets, each of cardinality λ. For each $A \in [\kappa]^\lambda$, put $A^* = \cup_{\alpha \in A} S_\alpha$. Then $\mathcal{A} = \{A^* : A \in [\kappa]^\lambda\}$ is a cofinal subset of $[\kappa]^{<\lambda}$ which has the properties required in Theorem 10. The result now follows from that theorem and the fact that TWO has a winning coding strategy in $WMG([\kappa]^{<\lambda})$ – see [S4].

Corollary 12. *There is a cofinal $\mathcal{A} \subset [\omega_1]^{\aleph_0}$ such that TWO has a winning remainder strategy in $SMG(\mathcal{A}, [\omega_1]^{<\aleph_0})$, but no winning remainder strategy in $WMG(\mathcal{A}, [\omega_1]^{<\aleph_0})$.*

Proof: Put $\mathcal{A} = \{\alpha < \omega_1 : cof(\alpha) = \omega\}$. □

5. The very strong game, $VSG(J)$.

Moves by player TWO in the game $VSG(J)$ (introduced in [B-J-S]) consist of pairs of the form $(S,T) \in \langle J \rangle \times J$, while those of ONE are elements of $\langle J \rangle$. A sequence $(O_1, (S_1, T_1), O_2,$

$(S_2, T_2), \dots)$ is a play of $VSG(J)$ if: $O_{n+1} \supseteq S_n \cup T_n$, and $O_n, S_n \in \langle J \rangle$ and $T_n \in J$ for each n.

TWO wins such a play if $\cup_{n=1}^{\infty} O_n \subseteq \cup_{n=1}^{\infty} T_n$. A strategy F is a remainder strategy for TWO in $VSG(J)$ if

$$(S_{n+1}, T_{n+1}) = F(O_{n+1} \setminus (\cup_{j=1}^{n} T_n))$$

for each n.

For $X \in \langle J \rangle$ we write $F(X) = (F_1(X), F_2(X))$ when F is a remainder strategy for TWO in $VSG(J)$. When F is a winning remainder strategy for TWO, we may assume that it has the following properties:

(1) $F_1(X) \cap F_2(X) = \emptyset$; for G is a winning remainder strategy if $G_1(X) = F_1(X) \setminus F_2(X)$ and $G_2(X) = F_2(X)$ for each X.
(2) $X \setminus F_2(X) \subseteq F_1(X)$; for G is a winning remainder strategy if $G_1(X) = (X \cup F_1(X)) \setminus F_2(X)$ and $G_2(X) = F_2(X)$ for each X.

Lemma 13. *If $J \subset \langle J \rangle \subset \mathcal{P}(S)$ and if F is a winning remainder strategy for TWO in the game $VSG(J)$, then: For each $x \in S$ there exist a $C_x \in \langle J \rangle$ and a $D_x \in J$ such that:*

(1) $C_x \cap D_x = \emptyset$ and
(2) $x \in F_2(B)$ for each $B \in \langle J \rangle$ such that $C_x \subseteq B$ and $D_x \cap B = \emptyset$.

Proof: Let F be a remainder strategy of TWO, but assume the negation of the conclusion of the lemma. We also assume that for each $X \in \langle J \rangle$, $X \setminus F_2(X) \subseteq F_1(X)$ and $F_1(X) \cap F_2(X) = \emptyset$.

Choose an $x \in S$ witnessing this negation. Then there is for each $C \in \langle J \rangle$ and for each $D \in J$ with $x \in C$ and $C \cap D = \emptyset$ a $B \in \langle J \rangle$ such that $B \cap D = \emptyset$, $C \subseteq B$ and $x \notin F_2(B)$. We now construct a sequence $\langle (B_k, C_k, D_k, M_k, S_k, N_k) : k \in \mathbb{N} \rangle$ as follows:

Put $C_1 = \{x\}$ and $D_1 = \emptyset$. Choose $B_1 \in \langle J \rangle$ such that $C_1 \subseteq B_1$ and $x \notin F_2(B_1)$. Put $M_1 = B_1$ and $(S_1, N_1) = F(M_1)$. This defines $(B_1, C_1, D_1, M_1, S_1, N_1)$.

Put $C_2 = S_1$ and $D_2 = N_1$. Choose $B_2 \in \langle J \rangle$ such that $C_2 \subseteq B_2$, $D_2 \cap B_2 = \emptyset$, and $x \notin F_2(B_2)$. Put $M_2 = B_2 \cup D_2$ and $(S_2, N_2) = F(M_2 \backslash N_1)$. This defines $(B_2, C_2, D_2, M_2, S_2, N_2)$.

Put $D_3 = (N_1 \cup N_2)$ and $C_3 = S_2 \backslash D_3$. Choose $B_3 \in \langle J \rangle$ such that $C_3 \subseteq B_3$, $D_3 \cap B_3 = \emptyset$, and $x \notin F_2(B_3)$. Put $M_3 = B_3 \cup D_3$ and $(S_3, N_3) = F(M_3 \backslash D_3)$. This defines $(B_3, C_3, D_3, M_3, S_3, N_3)$. Continuing like this we construct $(B_1, C_1, D_1, M_1, S_1, N_1), \ldots, (B_k, C_k, D_k, M_k, S_k, N_k), \ldots$, so that:

(1) $D_{j+1} = (N_1 \cup \cdots \cup N_j) \in J$,
(2) $C_{j+1} = S_j \backslash D_{j+1} \in \langle J \rangle$ and $x \in C_{j+1}$ for all j,
(3) $C_j \subseteq B_j$, while $x \notin F_2(B_j)$ and $B_j \cap D_j = \emptyset$, and
(4) $M_j = B_j \cup D_j$ and
(5) $(S_j, N_j) = F(M_j \backslash D_j)$ for all j, and
(6) $(B_1, C_1, D_1, M_1, S_1, N_1)$ and $(B_2, C_2, D_2, M_2, S_2, N_2)$ are as above.

Then $(M_1, (S_1, N_1), \ldots, M_k, (S_k, N_k), \ldots)$ is a play of $VSG(J)$ during which player TWO used the remainder strategy F and lost. □

Theorem 14 (Galvin) *For $\kappa > \aleph_1$, TWO does not have a winning remainder strategy in $VSG([\kappa]^{<\aleph_0})$.*

Proof: Let F be a remainder strategy for TWO. If it were winning, choose for each $x \in \kappa$ a $D_x \in [\kappa]^{<\aleph_0}$ and a $C_x \in [\kappa]^{\leq\aleph_0}$ such that:

(1) $C_x \cap D_x = \emptyset$,
(2) $x \in C_x$ and
(3) $x \in F_2(B)$ for each $B \in [\kappa]^{\leq\aleph_0}$ such that $B \cap D_x = \emptyset$ and $C_x \subseteq B$.

Now $(D_x : x \in \kappa)$ is a family of finite sets. By the Δ-system lemma we find an $S \in [\kappa]^\kappa$ and a finite set R such that $(D_x : x \in S)$ is a Δ-system with root R. For $x \in S$ define:

$$f(x) = \{y \in S : D_y \cap C_x \neq \emptyset\}.$$

Then $f(x)$ is a countable set and $x \notin f(x)$ for each $x \in S$. By Hajnal's set-mapping theorem (see §44 of [E-H-M-R]) we find $T \in [S]^\kappa$ such that $C_x \cap D_y = \emptyset$ for all $x, y \in T$. Let $K \in [T]^{\aleph_0}$ be given, and put $B = \cup_{x \in K} C_x$. Then $K \subseteq F_2(B)$, a contradiction. □

Using similar ideas but with the appropriate cardinality assumption to ensure that the corresponding versions of the Δ-system lemma and the set-mapping theorems are true, one obtains also:

Theorem 15. *Let λ be a cardinal of countable cofinality. If $\kappa > 2^\lambda$, then TWO does not have a winning remainder strategy in $VSG([\kappa]^{<\lambda})$.*

Since for every cardinal λ of countable cofinality, and for each cardinal κ player TWO has a winning coding strategy in $WMG([\kappa]^{<\lambda})$ (see for example [S4]), Theorems 14 and 15 also show that the existence of a winning remainder strategy for TWO in $VSG([\kappa]^{<\lambda})$ is a stronger statement than the existence of a winning coding strategy for TWO in $WMG([\kappa]^{<\lambda})$.

Problem 2. *Let λ be an uncountable cardinal of countable cofinality. Let κ be a cardinal number such that $\lambda^{<\lambda} < cof([\kappa]^\lambda, \subset) \leq 2^\lambda$. Does TWO have a winning remainder strategy in any of $WMEG([\kappa]^{<\lambda})$, $WMG([\kappa]^{<\lambda})$ or $VSG([\kappa]^{<\lambda})$?*

Theorem 16. *If $cof(\langle J \rangle, \subset) = \aleph_1$, then TWO has a winning remainder strategy in $VSG(J)$.*

Proof: We may assume that there is for each $X \in \langle J \rangle \backslash J$, a $Y \in \langle J \rangle \backslash J$ such that $X \cap Y = \emptyset$ (else, TWO has an easy winning remainder strategy even in $WMEG(J)$). Let $\prec$ be a well-ordering of S, the underlying set of our topological space. Choose two ω_1-sequences $(C_\alpha : \alpha < \omega_1)$ and $(x_\alpha : \alpha < \omega_1)$ such that:

(1) $C_\alpha \subset C_\beta \in \langle J \rangle$,
(2) $x_\alpha \in C_\beta$,
(3) $x_\beta \notin C_\beta$,
(4) $x_\alpha \prec x_\beta$ and
(5) $C_\beta \backslash C_\alpha \notin J$ for all $\alpha < \beta < \omega_1$, and

(6) $\{C_\alpha : \alpha < \omega_1\}$ is cofinal in $\langle J\rangle$.

For each $X \in \langle J\rangle$ we write $\beta(X)$ for $\min\{\alpha < \omega_1 : X \subseteq C_\alpha\}$. Put $\mathfrak{X} = \{x_\alpha : \alpha < \omega_1\}$. Write Ω for $\omega_1\backslash\omega$. Let F be a winning perfect information strategy for TWO in $REG(J)$, and let G be a winning perfect information strategy for TWO in $REG([\{x_\delta : \delta \in \Omega\}]^{<\aleph_0})$. We may assume that if σ is a sequence of length r of subsets of Ω, at least one of which is infinite, then $|G(\sigma)| \geq r$. We also define: $K_\beta = \{x_\gamma : \gamma \leq \beta\}$ for each $\beta \in \Omega$.

We define a remainder strategy H for TWO in $VSG(J)$. Let $B \in \langle J\rangle$ be given.

(1) If $B \in J$: Then put $H(B) = (C_{\beta(B)+\omega}, \{x_0, x_{\beta(B)}\})$

(2) If $B \notin J$:

(a) If $\{n < \omega : x_n \notin B\} = \{0, 1, \ldots, k\}$:
Let T be $\{x_{\beta(B)}\}$ together with the first $\leq k+1$ elements of $\{x_\alpha : \alpha \in \Omega\}\backslash B$. Put $S = T \cup (\cup\{G(\sigma) : \sigma \in {}^{\leq k+2}\{K_\delta : x_\delta \in T\}\})$, a set in $[\{x_\delta : \delta \in \Omega\}]^{<\aleph_0}$. Let p be the cardinality of S. Then define

$$\overline{S} = \{x_0, \ldots, x_p\} \cup S \cup ((\cup\{F(\sigma) : \sigma \in {}^{\leq p}\{C_\alpha : x_\alpha \in S\}\})\backslash\mathfrak{X}).$$

Put $H(B) = (C_{\beta(B)+\omega}, \overline{S})$.

(b) If $\{n < \omega : x_n \notin B\}$ is not a finite initial segment of ω: Then we put $H(B) = (C_{\beta(B)+\omega}, \{x_0, x_{\beta(B)}\})$.

To see that H is a winning remainder strategy for TWO, consider a play

$$(O_1, (S_1, T_1), \ldots, O_n, (S_n, T_n), \ldots)$$

where $(S_1, T_1) = H(O_1)$ and $(S_{n+1}, T_{n+1}) = H(O_{n+1}\backslash(\cup_{j=1}^n S_j)$ for each n.

For convenience we put $W_0 = T_0 = \emptyset$ and $W_{n+1} = W_n \cup T_{n+1}$, $B_n = O_n\backslash W_n$, $\beta_n = \beta(B_n)$ and $\alpha_n = \beta_n + \omega$ for each n.

Note that if B_j is such that $\{n \in \omega : x_n \notin B_j\}(= \{0, 1, \ldots, k_j\}$ say) is a finite initial segment of ω, then the same is true for B_{j+1}. Thus (S_j, T_j) is defined by Case 2(a) for each $j > 1$, and

$(k_j : j \in \mathbb{N})$ is an increasing sequence. Further, $\{x_{\beta_1}, \ldots, x_{\beta_j}\} \subset T_j$ for these j. This in turn implies that $\bigcup_{j=1}^{\infty} F(C_{\beta_1}, \ldots, C_{\beta_j}) \setminus \mathfrak{X} \subseteq \bigcup_{n=1}^{\infty} T_n$, and $\bigcup_{j=1}^{\infty} G(K_{\beta_1}, \ldots, K_{\beta_j}) \subseteq \bigcup_{n=1}^{\infty} T_n$.

But then $\bigcup_{n=1}^{\infty} O_n \subseteq \bigcup_{n=1}^{\infty} T_n$. □

Corollary 17. *TWO has a winning remainder strategy in* $VSG([\omega_1]^{<\aleph_0})$

Using the methods of this paper we can also show that if $J \subset \mathcal{P}(S)$ is a free ideal such that there is an $A \in \langle J \rangle$ such that $cof(\langle J \rangle, \subset) \leq |J\lceil_A|$, then TWO has a winning remainder strategy in $VSG(J)$.

Corollary 18. *For every T_1-topology on ω_1, without isolated points, TWO has a winning remainder strategy in $VSG(J)$.*

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