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CONSTRAINTS ON FUNDAMENTAL GROUPS OF MIXED SPACEFORMS

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ABSTRACT. Mixed spaceform problems in topology ask for characterizations of the fundamental groups of manifolds whose universal cover is homotopy finite, especially if the universal cover is a product $(N^n \times \mathbf{R}^k)/\Gamma$. Existential results and some characterization theorems are presented. Fundamental groups Γ of closed mixed spaceforms are found to be virtual Poincaré duality groups if some finiteness conditions are satisfied.

1. INTRODUCTION

This is a report on investigations into common properties of a large number of geometrically pleasant manifolds M. We are particularly interested in generalizing from these important examples.

Example 1.1 Homogeneous spaces $\Gamma\backslash G$, where G is a connected Lie group and Γ is a discrete subgroup of G, have universal covering spaces $\widetilde{M} = \widetilde{G} = K_0 \times \mathbf{R}^k$, where K_0 is a maximal compact subgroup of $\widetilde{G}$.

Example 1.2 The universal covering space of any complete Riemannian manifold of nonpositive sectional curvature is diffeomorphic to a Euclidean space.

Example 1.3 M. Davis [6], [7] has constructed examples of closed manifolds whose universal covering spaces are contractible but not homeomorphic to Euclidean space.

Example 1.4 (a) Discrete subgroups Γ in a connected, noncompact Lie group G are most often studied through the action of Γ on the contractible homogeneous space G/K, where K is a maximal compact subgroup of G. For example, this is the setting of the Clifford–Klein spaceform problems of geometry, concerning manifolds of constant sectional curvature [33].
(b) "Semiclassical spaceform problems" in geometry concern actions of discrete subgroups of G on G/H, where H is a closed, connected subgroup of G, not necessarily compact [18].

C. T. C. Wall posed topological counterparts of the Clifford–Klein spaceform problems [29] which motivated much subsequent research on manifolds whose universal covering space is homeomorphic to S^n or $\mathbf{R}^n$. These problems of Wall are usually known as the "topological spherical spaceform problem" and the "topological Euclidean spaceform problem."

Example 1.5 A mix of constructive arguments as well as methods adapted from the topological spherical spaceform problem and the topological Euclidean spaceform problem has been used to produce manifolds whose universal covering spaces are products of a sphere with a Euclidean space, a class of spaces sometimes called "spherical–Euclidean spaceforms." See [5], [8], [12], [15], [22].

Terminology for the next class of manifolds to be considered is still unsettled, but we suggest the following provisional definitions, which are meant to capture the common features of the examples above.

Definition 1.6 Let M be a connected manifold.
(a) M is a *weak mixed spaceform* if and only if the universal covering space of M is homotopy equivalent to a finite complex.

(b) M is a *strong mixed spaceform* if and only if the universal covering space of M is a product, $\widetilde{M} = N^n \times \mathbf{R}^k$, where N^n is a closed, simply connected manifold

The most essential problems in this setting concern the fundamental group. Can we characterize the class of fundamental groups of mixed spaceforms (in either the weak or strong case), especially the fundamental groups of closed mixed spaceforms?

These questions in manifold topology turn out to be equivalent via regular neighborhoods to similar questions on complexes.

Proposition 1.7. *Let Γ be a countable group.*
(a) There is a finite–dimensional, locally finite, connected simplicial complex X with fundamental group Γ such that the universal cover $\widetilde{X}$ is homotopy equivalent to a finite complex if and only if there exists an open PL manifold with fundamental group Γ and homotopy–finite universal cover.
(b) There is a finite, connected simplicial complex X with fundamental group Γ such that the universal cover $\widetilde{X}$ is homotopy equivalent to a finite complex if and only if there exists a compact PL manifold with boundary whose fundamental group is Γ and whose universal cover is homotopy–finite.
(c) There is a Poincaré complex X with fundamental group Γ and homotopy–finite universal cover if and only if there is a closed PL manifold with fundamental group Γ and homotopy–finite universal cover.

Proof: Statements (a) and (b) are easily established by taking a regular neighborhood of a proper imbedding of X in a high–dimensional Euclidean space.

To prove (c), recall that a Poincaré complex X is a finitely dominated complex satisfying Poincaré duality with respect to an orientation character $w\colon \Gamma \to \mathbf{Z}/2$ [28], [30]. The product of X with an odd–dimensional sphere S^{2r+1} ($r > 0$) is then homotopy equivalent to a finite complex Y with fundamental group Γ, with homotopy–finite universal cover, and satisfying

Poincaré duality with respect to w [10]. Imbed Y in a high-dimensional Euclidean space, take a compact regular neighborhood N of Y with respect to this imbedding and consider the inclusion $\partial N \hookrightarrow N \simeq Y$. This map is the Spivak normal fibration of Y, and its homotopy fiber is homotopy equivalent to a sphere S^d. On taking universal covers, we have a ladder with fibrations as rows

$$\begin{array}{ccccc} S^d & \longrightarrow & \widetilde{\partial N} & \longrightarrow & \widetilde{N} \simeq \widetilde{Y} \\ \| & & \downarrow & & \downarrow \\ S^d & \longrightarrow & \partial N & \longrightarrow & N \simeq Y, \end{array}$$

implying that $\widetilde{\partial N}$ is at least finitely dominated, so that if $r > 0$ then $S^{2r+1} \times \partial N$ is a closed PL manifold with fundamental group Γ and homotopy–finite universal cover. $\square$

The Proposition above, some of the results in Section 4, and the summary tables in Section 5 are new. Most of the other results in this paper are taken from [8], [23], [24], [25], [26].

2. Open manifolds

The following realization theorem indicates that open manifolds which are strong mixed spaceforms are plentiful and varied:

Theorem 2.1. *If Γ is a countable group of finite virtual cohomological dimension then there is a simply connected closed manifold N^n such that Γ acts as a group of covering transformations on $N^n \times \mathbf{R}^k$ for sufficiently large k.*

Proof: This result appears in [23] and similar constructions are made by Johnson in [14] (see also [29]) and [15]. We sketch the argument here and refer to [23] for details; parts of this line of thought reappear below in Theorem 4.1.

A group Γ is said to have finite virtual cohomological dimension if and only if there is an extension $1 \to \Gamma_0 \to \Gamma \to Q \to 1$ in which Q is a finite group and Γ_0 has finite cohomological dimension; "virtual cohomological dimension" is frequently shortened to "VCD." Following [14], we know that Γ_0 is the fundamental group of an open manifold V^r whose universal cover is homeomorphic to $\mathbf{R}^r$. (Take a proper imbedding of a finite–dimensional $B\Gamma_0$ complex in a high dimensional Euclidean space, let W be an open regular neighborhood of this imbedded $B\Gamma_0$, and let $V = W \times \mathbf{R}$. Stallings' recognition theorem for Euclidean spaces [21] shows that the universal covering space $\widetilde{V}$ is piecewise–linearly homeomorphic to a Euclidean space.

Two tools are used to pass from the covering action of Γ_0 on a Euclidean space $\widetilde{V} = \mathbf{R}^s$ to the desired action of Γ. First, there is an induced action of Γ on $\widetilde{V}^Q$; this is most quickly described as the natural left action of Γ on the sections of $\Gamma \times_{\Gamma_0} \widetilde{V} \to \Gamma/\Gamma_0$, $(\gamma \cdot \sigma)(x) = \gamma \cdot (\sigma(\gamma^{-1}x))$ ($\gamma \in \Gamma$, σ a section of $\Gamma \times_{\Gamma_0} \widetilde{V} \to \Gamma/\Gamma_0$, and $x \in \Gamma/\Gamma_0$). Secondly, there is a free action of the finite group Q on a simply connected closed manifold N^n; such a manifold may be obtained through a special unitary representation of Q or through another regular neighborhood construction. The diagonal action of Γ on $\widetilde{V}^Q \times N^n$ is then an action by covering transformations on a simply connected manifold homeomorphic to $\mathbf{R}^{s|Q|} \times N^n$. $\square$

The argument sketched above is robust enough that one might wonder whether the dimension condition is unnecessary: perhaps every countable group is the fundamental group of an open mixed spaceform. The next few results establish a growth constraint on groups of infinite virtual cohomological dimension which are fundamental groups of CW complexes whose universal covers are homotopy finite. Details are found in [23].

If $\mathbf{F}$ is a field then we let $\beta_i(—;\mathbf{F})$ denote the i–th Betti number for homology with coefficients in $\mathbf{F}$.

Lemma 2.2. *Let* $\mathbf{F}$ *be a field. Suppose that* X *is a connected,* *finite-dimensional CW complex such that the universal cover $\widetilde{X}$ is homotopy equivalent to a finite complex, and suppose that $\Gamma = \pi_1 X$ acts trivially on $H_*(X; \mathbf{F})$. If all the homology groups $H_i(\Gamma; \mathbf{F})$ are finite-dimensional $\mathbf{F}$-vector spaces then there are positive constants A and B so that $\beta_k(\Gamma; \mathbf{F}) \leq A \cdot B^k$ for all $k \geq 0$.*

Theorem 2.3. *There are countable groups Γ which can not be the fundamental group of a connected, finite-dimensional CW complex X such that $\widetilde{X}$ is homotopy equivalent to a finite complex.*

Proof: It suffices to describe a family of countable groups Γ such that all the homology groups $H_i(\Gamma; \mathbf{Z})$ are finitely generated and such that the sequence $\operatorname{rank}(H_i(\Gamma; \mathbf{Z}))$ satisfies no exponential bound. (If Γ is the fundamental group of a complex X as above and if $\mathbf{F}$ is a finite field, then a finite-index subgroup Γ_0 of Γ fixes the finite-dimensional $\mathbf{F}$-vector space $H_*(\widetilde{X}; \mathbf{F})$. Γ_0 will share the super-exponential Betti number growth of Γ over the integers, implying that $\beta_i(\Gamma_0; \mathbf{F})$ grows super-exponentially, contradicting the lemma.)

I know of two methods for producing such a countable group. One may apply the Kan-Thurston theorem [17], [1] to produce a group with specified homology. A more geometrical construction takes the fundamental group of a one-point union of odd-dimensional closed Riemannian flat manifolds which are $\mathbf{Z}[1/2]$-homology spheres [13], [27]. □

3. Closed manifolds and finite complexes

Algebraic finiteness properties are important constraints on fundamental groups of mixed spaceforms. A countable group Γ is said to be of *type* FP(∞) if and only if there exists a resolution of the trivial Γ-module $\mathbf{Z}$ by finitely generated projective $\mathbf{Z}\Gamma$-modules. This property is the key to universal coefficient theorems [3], [4] and is an algebraization of a familiar topological property: if Γ finitely presented then Γ is of type FP(∞)

if and only if there is a $K(\Gamma, 1)$ complex of finite type (every skeleton is a finite complex).

An algebraic argument with resolutions modeled on trees [24], [25] has the following topological consequence:

Theorem 3.1. *Let X be a connected CW complex of finite type. If Γ is the group of covering transformations of a regular covering projection $p\colon W \to X$, where W is homotopy equivalent to a complex of finite type, then Γ is of type* $\mathrm{FP}(\infty)$.

In particular, if Γ is the fundamental group of a mixed spaceform M which is homotopy equivalent to a finite complex then Γ must be of type $\mathrm{FP}(\infty)$. (Note that this implies that all the Betti numbers of Γ are finite, so the growth condition of Lemma 2.2 applies if the cohomological dimension of Γ is infinite.) If Γ is also of finite virtual cohomological dimension then stringent consequences follow [26]:

Theorem 3.2. *Let Γ be a finitely presented group of finite virtual cohomological dimension. Γ is a virtual Poincaré duality group if and only if there exists a closed PL manifold M with fundamental group Γ and homotopy finite universal cover $\widetilde{M}$.*

Recall that a Poincaré duality group is a group Γ of type FP such that for some $n > 0$ $H^i(\Gamma; \mathbf{Z}\Gamma) = 0$ for $i \neq n$ and $H^n(\Gamma; \mathbf{Z}\Gamma)$ is an infinite cyclic group [2], [16], [4]. These conditions are equivalent to the topological condition that the Eilenberg–MacLane space $K(\Gamma, 1)$ is a Poincaré complex, and mimic the homological properties of the fundamental group of a closed aspherical manifold, such as the manifolds of examples 1.2 and 1.3 (and many of the manifolds in example 1.4).

4. Extensions and groups of infinite VCD

The class of fundamental groups of mixed spaceforms is closed under the formation of extensions with finite quotients or passing to subgroups of finite index:

Theorem 4.1. *Suppose that* $1 \to \Delta \to \Gamma \to Q \to 1$ *is an exact sequence of countable groups in which* Q *is finite.*
(a) Δ *is the fundamental group of a mixed spaceform if* Γ *is the fundamental group of a mixed spaceform. Moreover, if* Γ *is the fundamental group of a weak, strong, or closed mixed spaceform, then* Δ *is the fundamental group of a mixed spaceform of the same type.*
(b) Γ *is the fundamental group of an open mixed spaceform if* Δ *is the fundamental group of a mixed spaceform. In addition, if* Δ *is the fundamental group of a weak or strong mixed spaceform then* Γ *is the fundamental group of a mixed spaceform of the same type.*

Proof: (a) If Γ is the fundamental group of a mixed spaceform X (of either type) then Δ is the fundamental group of a finite-sheeted covering space of X, which is a mixed spaceform of the same type (weak or strong) as X. This covering space is a closed manifold if and only if X is a closed manifold, so if Γ is the fundamental group of a closed spaceform of either type, so is Δ.

(b) Suppose that $\Delta = \pi_1(X)$, where X is a mixed spaceform. Take a simply connected closed manifold V on which Q acts as a group of covering transformations and let Γ act on Q through the surjection $\Gamma \to Q$. Γ acts on $(\widetilde{X})^Q$ through the induced action discussed in the proof of Theorem 2.1, and the diagonal action of Γ on $(\widetilde{X})^Q \times V$ is free and properly discontinuous. The manifold $M = (\widetilde{X})^Q \times_\Gamma V$ is a mixed spaceform of the same type (weak or strong) as X, but is necessarily open if Q is nontrivial. $\square$

Most of the topologists interested in spherical–Euclidean spaceforms have concentrated on questions suggested by periodicity results for Farrell–Tate cohomology [31], [32] which were most naturally posed for groups of finite virtual cohomological dimension [5], [12], [15], [22]. This circumstance, perhaps enhanced by the naiveté of our expectations, led us to ignore groups of infinite VCD until Prassidis [19] produced open spherical–Euclidean spaceforms whose fundamental groups are of infinite VCD (although they are periodic groups, in an extended sense of the term).

The infinite virtual cohomological dimension of the groups studied by Prassidis may be ascribed to their structure as extensions with finite central kernels. Farrell and I found that a similar phenomenon seen in some cocompact discrete subgroups of Lie groups leads to closed spherical–Euclidean spaceforms whose fundamental groups have infinite VCD [8].

Theorem 4.2. *For each $n \geq 2$ and each $k \geq n(n+1)$ there are smooth closed manifolds with universal covering space $S^{2n-1} \times \mathbf{R}^k$ and fundamental group of infinite virtual cohomological dimension.*

The theorem depends upon results of Raghunathan [20] which show that for all $n > 1$ any lattice $\tilde{\Gamma}$ in the universal cover $\tilde{G}$ of the symplectic group $\mathrm{Sp}(2n, \mathbf{R}) = \mathrm{Sp}(\mathbf{R}^{2n})$ contains $8\pi_1$, where $\pi_1 = \pi_1(G)$ is the infinite cyclic kernel of the projection homomorphism $\tilde{G} \to \mathrm{Sp}(2n, \mathbf{R})$. (Recall that a lattice Γ in a Lie group G is a discrete subgroup such that the Haar measure of the quotient $\Gamma \backslash G$ is finite.) From this result in $\tilde{G}$, Raghunathan concludes that if G is a covering group of $\mathrm{Sp}(2n, \mathbf{R})$ of finite degree not dividing 8, then every lattice $\Gamma < G$ contains elements of finite order and is thus of infinite cohomological dimension; since this claim is established for every lattice, the VCD of these lattices is infinite as well.

In [8] we show that for well-chosen covering groups $G \to \mathrm{Sp}(2n, \mathbf{R})$ of degree not dividing 8, there is a closed subgroup H such that the preimage Γ in G of any cocompact lattice $\Gamma_0 < \mathrm{Sp}(2n, \mathbf{R})$ acts freely on G/H and $G/H \cong S^{2n-1} \times \mathbf{R}^{n(n+1)}$. Raghunathan's results now yield the claim.

As in Prassidis' examples, these groups Γ are extensions with finite central kernels, and one is led to wonder whether every central extension $0 \to A \to \Gamma \to \Delta \to 1$ in which Δ is a mixed spaceform group is also a mixed spaceform group. More generally, one would like to know whether the class of mixed spaceform groups is closed under the formation of extensions with kernels and quotients in this class.

5. Summary

Tables 1 and 2 give a telegraphic account of the state of our knowledge concerning fundamental groups of weak and strong mixed spaceforms. The tables' columns classify the groups by virtual cohomological dimension, while the rows correspond to open and closed manifolds.

	VCD(Γ)$< \infty$	VCD(Γ)$= \infty$
M open:	All Γ are realized.	Subexponential Betti number growth
M closed:	Γ is realized $\iff$ Γ is a virtual Poincaré duality group.	Γ is of type FP(∞).

Table 1. Weak spaceforms: $\tilde{M} \simeq$ (finite comples), $\Gamma = \pi_1 M$

In both cases the column of the table corresponding to groups of finite virtual cohomological dimension is more satisfactory than the second column: for weak spaceforms we have sharp results in both rows of the first column and for strong spaceforms we have a sharp result for open manifolds. The result alluded to in Table 2, Row 2, Column 1 is not presented in this paper but appears in [23]: the ends of many virtual Poincaré duality groups prohibit their acting as a cocompact group of covering transformations on any $N^n \times \mathbf{R}^k$, where N^n is a simply connected closed manifold. However, we do not know whether every virtual Poincaré duality group with an appropriate end is realized by a closed manifold which is a strong mixed spaceform, and this remains one of the most interesting open questions in the subject.

	VCD(Γ) < ∞	VCD(Γ) = ∞
M open:	All Γ are realized.	Examples - Prassidis
M closed:	End obstructions	Examples - Farrell-Stark

Table 2. Strong spaceforms: $\tilde{M} = N^n \times R^k, \Gamma = \pi_1 M$

In both tables the second column (groups of infinite virtual cohomological dimension) is more problematic than the first. Necessary conditions in the weak case are provided by Theorems 2.3 and 3.1 above. We do not seem to know much about additional necessary conditions satisfied by a group of infinite VCD which is the fundamental group of a strong mixed spaceform, although pseudoproper methods in the sense of [9] give a bit of information. On this account the second column of Table 2 emphasizes examples rather than necessary conditions, although one would hope eventually to know more about fundamental groups of closed strong mixed spaceforms; perhaps asymptotic or end-related properties will be the key to this class of groups.

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