Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Vol 17, 1992

k-NETWORKS, AND COVERING PROPERTIES OF CW-COMPLEXES

YOSHIO TANAKA

Dedicated to Professor Akihiro Okuyama on his 60th birthday

1. INTRODUCTION.

Let X be a CW-complex, and let $\{e_\lambda; \lambda\}$ be the cells of X. We characterize certain covering properties of the collection $\{e_\lambda; \lambda\}$ by means of k-networks, etc.

First of all, we shall give some main definitions used in this paper.

Let X be a space, and let $\mathcal{C}$ be a cover of X. Then X *is determined by* $\mathcal{C}$ [5] (or X has the weak topology with respect to $\mathcal{C}$ in the usual sense), if $F \subset X$ is closed in X if and only if $F \cap C$ is closed in C for every $C \in \mathcal{C}$. Here, we can replace "closed" by "open". Every space is determined by an open cover. X is *is dominated by* $\mathcal{C}$ [6] (or X has the weak topology with respect to $\mathcal{C}$ in the sense of [9]), if the union of any subcollection $\mathcal{C}'$ of $\mathcal{C}$ is closed in X, and the union is determined by $\mathcal{C}'$. Clearly, if X is dominated by $\mathcal{C}$, then X is determined by $\mathcal{C}$. If the cover $\mathcal{C}$ is increasing, countable and closed, then the converse holds.

Let X be a space, and let $\mathcal{P}$ be a cover of X. Then $\mathcal{P}$ is a *k-network* if whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. If we replace "compact" by "single point", then such a cover is called "net(or network)". k-networks have played a role in $\aleph_0$-*spaces* [6] (i.e., regular spaces with a countable k-network) and $\aleph$-*spaces* [10] (i.e., regular spaces with a σ-locally finite k-network).

We assume that all spaces are T_2 in this paper.

Let $\mathcal{A} = \{A_\alpha; \alpha \in A\}$ be a collection of subsets of a space X. Then $\mathcal{A}$ is *closure-preserving* if $\overline{\cup\{A_\alpha; \alpha \in B\}} = \cup\{\bar{A}_\alpha; \alpha \in B\}$ for any $B \subset A$. $\mathcal{A}$ is *hereditarily closure-preserving* if $\overline{\cup\{B_\alpha; \alpha \in B\}} = \cup\{\bar{B}_\alpha; \alpha \in B\}$ whenever $B \subset A$ and $B_\alpha \subset A_\alpha$ for each $\alpha \in B$. Every space is dominated by a hereditarily closure-preserving closed cover. A *σ-hereditarily closure-preserving* collection is the union of countably many hereditarily closure-preserving collections, etc. We shall use "σ-CP (resp. σ-HCP)" instead of "σ-closure-preserving (resp. σ-hereditarily closure-preserving ". $\mathcal{A}$ is *point-finite* (resp. *point-countable*) if every $x \in X$ is in at most finitely (resp. countably) many element of $\mathcal{A}$.

The concept of CW-complexes due to J.H. Whitehead [18] is well-known. A space X is a CW-complex, if it is a complex with cells $\{e_\lambda; \lambda\}$ satisfying (a) and (b) below.

(a) Each cell e_λ is contained in a finite subcomplex of X.

(b) X is determined by the closed cover $\{\bar{e}_\lambda; \lambda\}$ of X.

We note that every $\bar{e}_\lambda$ is not a subcomplex. A Whitehead complex due to C.H. Dowker [3] is a CW-complex such that each closure of cell is a subcomplex.

As is well-known, every CW-complex X is dominated by the cover of all finite subcomplexes of X, hence X is dominated by a cover of compact metric subsets of X.

Let $\{e_\lambda; \lambda\}$ be the cells of a CW-complex X. We shall say that $\{e_\lambda; \lambda\}$ is $(\sigma-)$ locally finite; $(\sigma-)$ HCP, etc., if so is respectively the collection $\{e_\lambda; \lambda\}$ of subsets of X. We note that the collection $\{e_\lambda; \lambda\}$ is $(\sigma-)$ locally finite; $(\sigma-)CP$; $(\sigma-)HCP$ if and only if so is respectively $\{\bar{e}_\lambda; \lambda\}$.

Now let X be a CW-complex with cells $\{e_\lambda; \lambda\}$. Then the following hold. (a) is well-known, and (b) is due to [2]. (c) is shown in this paper.

(a) X is a paracompact, and σ-space (i.e., X has a σ-locally finite net).

(b) X is an M_1-space (in the sense of [2]), hence X has a σ-CP k-network.

(c) X has a point-countable k-network.

However, every CW-complex is not a metric space (not even a Fréchet space, nor an $\aleph$-space). We have the following characterizations of X. Recall that a space if *Fréchet* (= Fréchet-Uryshon), if whenever $x \in \bar{A}$, there exists a sequence in A converging to the point x. (A) is well-known, and (B) is due to [17]. (C) $\sim$ (F) is proved in this paper.

(A) X is a metric space if and only if $\{e_\lambda; \lambda\}$ is locally finite.

(B) X is a Fréchet space if and only if $\{e_\lambda; \lambda\}$ is HCP.

(C) X is an $\aleph-$space if and only if $\{e_\lambda; \lambda\}$ is σ-locally finite.

(D) X has a σ-HCP k-network if and only if $\{e_\lambda; \lambda\}$ is σ-HCP.

(E) X is a symmetric space (in the sense of [1]) if and only if $\{\bar{e}_\lambda; \lambda\}$ is point-finite.

(F) X has a point-countable closed k-network if and only if $\{\bar{e}_\lambda; \lambda\}$ is point-countable.

Results. Recall that a cover $\mathcal{C}$ of X is *star-countable* if each member of $\mathcal{C}$ meets only countably many members. The following lemma follows from the proof of Theorem 1 in [15].

Lemma 1. *Let X be determined by a star-countable cover $\mathcal{C}$. Then X is the topological sum of $\{X_\alpha; \alpha \in A\}$ such that each X_α is determined by a countable subcollection $\mathcal{A}_\alpha$ of $\mathcal{C}$, and $\mathcal{C} = \cup\{\mathcal{A}_\alpha; \alpha \in A\}$.*

Lemma 2. *Let X be dominated by a cover $\mathcal{C} = \{X_\alpha; \alpha \in A\}$ of compact metric subsets. Suppose that X has a σ-HCP (resp. σ-locally finite) closed k-network $\mathcal{F} = \cup\{\mathcal{F}_n; n \in N\}$ with $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Then X has a σ-HCP (resp. σ-locally finite) k-network consisting of a compact metric subsets.*

Proof: Let $p \in X$, and let A be a sequence in X converging to the point p with $A \not\ni p$. Let $\mathcal{F}_A = \{F \in \mathcal{F}; F \cap A \text{ is infinite}\}$. Since each $\mathcal{F}_n$ is HCP, $\mathcal{F}_A \cap \mathcal{F}_n$ is at most finite. Then $\mathcal{F}_A$ is at most countable. Let $\mathcal{A} = \{A_n; n \in N\}$ be the collection of all finite unions of elements of $\mathcal{F}_A$ such that each $A - A_n$ is at most finite ($\mathcal{A}$ is not empty, for $\mathcal{F}$ is a k-network). For each $n \in N$, let $B_n = \cap\{A_i; i \leq n\}$. Then some B_k is contained in a finite union of elements of $\mathcal{C}$, thus B_k is compact metric. Indeed, suppose that any B_n is not contained in any finite union of elements of $\mathcal{C}$. Then there exist a sequence $B = \{x_n; n \in N\}$ in X and a subcollection $\{X_{\alpha(n)}; n \in N\}$ of $\mathcal{C}$ such that $x_n \in (B_n \cap X_{\alpha(n)}) - \cup\{X_{\alpha(j)}; j \leq n-1\}$. Since each $B \cap X_{\alpha(n)}$ is finite, B is discrete in X. But each neighborhood of the point p contains some B_ℓ, because $\mathcal{F}$ is a k-network for X, and the sequence A converges to p not in A. Then, since $x_n \in B_n$ for each $n \in N$, B has an accumulation point in X. This is a contradiction. Then some B_k is compact metric. Thus each neighborhood of the point p contains some compact metric subset B_m. But $A - B_m$ is at most finite, and B_m can be expressed as a union of finitely many closed sets F_{mn} such that each F_{mn} is an intersection of finitely many elements of $\mathcal{F}$. Consequently, each neighborhood U of the point p contains some finite intersection F_U of elements of $\mathcal{F}$ such that F_U is compact metric, and F_U contains a subsequence of the sequence A, hence the point p.

Now, for each $m, n \in N$, let $\mathcal{F}_{mn} = \{F_1 \cap F_2 \cap \ldots \cap F_m;\ F_i \in \mathcal{F}_n, i \leq m\}$. Since each $\mathcal{F}_n$ is HCP, it is routinely verified that each $\mathcal{F}_{nm}$ is HCP. Let $\mathcal{K} = \cup\{\mathcal{F}_{nm}; m, n \in N\}$, and let $\mathcal{L} = \{\{x\}; \{x\} \in \mathcal{F}\}$. Let $\mathcal{P} = \{K \in \mathcal{K}; K \text{ is compact metric }\} \cup \mathcal{L}$. Then $\mathcal{P}$ is a σ-HCP. Moreover, $\mathcal{P}$ satisfies (a) and (b) below. Indeed, since $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for each $n \in N$, the above argument suggests that (a) holds in general. For (b), if the point x is isolated in $X, x \in \{x\} \subset U$ with $\{x\} \in \mathcal{P}$. If x is not isolated in $X, X - \{x\}$ is not closed in X. Since X is determined by metric subsets, there exists a sequence C converging to the point x with $C \not\ni x$. Then (b) holds by (a).

(a) Let $x \in X$, and let C be a sequence converging to the point x with $C \not\ni x$. Then for each neighborhood U of x, there exists $P \in \mathcal{P}$ such that $P \subset U$, and P contains a subsequence of C, hence the point x.

(b) Let $x \in X$. Then for each neighborhood U of x, there exists $P \in \mathcal{P}$ such that $x \in P \subset U$.

We shall show that $\mathcal{P}$ is a k-network for X. Let K be compact, and U be a neighborhood of K. Let $\mathcal{P} = \cup\{\mathcal{P}_n;\ n \in N\}$, where each $\mathcal{P}_n$ is HCP with $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. For each $n \in N$, let $U_n = \cup\{P \in \mathcal{P}_n; P \subset U\}$. Then $K \subset U_m$ for some $m \in N$. Indeed, suppose that $K \not\subset U_n$ for any $n \in N$. Then there exists a sequence $L = \{x_n; n \in N\}$ in K with $x_n \notin U_n$. But by (b), $K \subset \cup\{U_n; n \in N\}$ with $U_n \subset U_{n+1}$. Then L is infinite. But X has a σ-CP closed net. Then each point of K is a G_δ-set in X, hence in K. Thus the compact set K is sequentially compact. Thus there exists a subsequence M of L converging to a point q not in M. Since $q \in K \subset U$, by (a) some U_m contains a subsequence of M. This is a contradiction. Hence $K \subset U_m$ for some $m \in N$. But $\mathcal{P}_m$ is HCP and K is compact. Then K is covered by some finite subcollection of $\mathcal{P}_m$. This shows that $\mathcal{P}$ is a k-network for X. Hence $\mathcal{P}$ is a σ-HCP k-network of compact metric subsets.

For the parenthetic part, let each $\mathcal{F}_n$ be a locally finite collection which is closed under finite intersections. Let K be a compact subset of X, and let $\mathcal{F}_K = \{F \in \mathcal{F}; F \cap K \neq \emptyset\}$. Then $\mathcal{F}_K$ is at most countable. Thus by a similar way as in the first half of the above proof, we can show that $\{F \in \mathcal{F};\ F$ is compact metric $\}$ is a σ-locally finite k-network for X. Thus the parenthetic part holds.

We recall the following basic properties of a CW-complex; see [18], etc.

Lemma 3. *Let X be a CW-complex with cells $\{e_\lambda; \lambda\}$.*

(1) Let each $X_\lambda (\lambda \in \Lambda)$ be a subcomplex, and let $C = \cup\{X_\lambda; \lambda \in \Lambda\}$. Then C is a CW-complex with cells $\{e_\lambda; e_\lambda \subset C\}$.

(2) Each compact subset K of X meets only finitely many e_λ, hence K is contained in a finite union of e_λ's.

The following lemma is easily proved.

Lemma 4. *Let $\mathcal{F} = \{X_\lambda; \lambda \in \Lambda\}$ be a closed cover of a space X. For each $\lambda \in \Lambda$, let $\mathcal{P}_\lambda$ be a k-network for X_λ. If each compact subset of X is contained in a finite union of elements of $\mathcal{F}$, then $\cup\{\mathcal{P}_\lambda; \lambda \in \Lambda\}$ is a k-network for X.*

Theorem 5. *Let X be a CW-complex with cells $\{e_\lambda; \lambda\}$. Then the following are equivalent.*

(a) $\{e_\lambda; \lambda\}$ is σ-discrete.
(b) $\{e_\lambda; \lambda\}$ is σ-locally finite.
(c) $\{e_\lambda; \lambda\}$ is σ-locally countable.
(d) $\{e_\lambda; \lambda\}$ is locally countable.
(e) X is an $\aleph$-space.

Proof: The implication (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (d). X is determined by a star-countable cover $\{\bar{e}_\lambda; \lambda\}$. Then if follows from Lemma 1 that $\{\bar{e}_\lambda; \lambda\}$. is locally countable, hence so is $\{e_\lambda; \lambda\}$.

(d) $\Rightarrow$ (e). X is determined by a star-countable cover $\{\bar{e}_\lambda; \lambda\}$. Then by Lemma 1, X is the topological sum of space $X_\alpha(\alpha \in A)$, where each X_α is covered by a countable subcollection of $\{\bar{e}_\lambda; \lambda\}$. Then each X_α is an $\aleph_0$-space by Lemmas 3(2) & 4. Thus X is an $\aleph$-space by Lemma 4.

(e) $\Rightarrow$ (a). X is dominated by compact metric subsets. Then by Lemma 2, X has a σ-locally finite k-network $\mathcal{P}$ consisting of compact metric subsets. Since $\mathcal{P}$ is star-countable, by Lemma 1, X is the topological sum of σ-compact spaces $X_\alpha(\alpha \in A)$. Since each e_λ is connected, it is contained in some X_α. But by Lemma 3(2), each σ-compact space X_α is a countable union of cells e_λ. Then we see that $\{e_\lambda; \lambda\}$.is a σ-discrete.

In the proof of (e) $\Rightarrow$ (a) of the previous theorem, the X_α are CW-complexes in X. Indeed, if $e_\lambda \subset X_\alpha$, then $\bar{e}_\lambda \subset X_\alpha$. Then the X_α are subcomplexes of X, hence CW-complexes.

Then we have the following corollary in view of the proof of the previous theorem.

Corollary 6. *A CW-complex is an $\aleph$-space if and only if it is the topological sum of countable CW-complexes.*

Theorem 7. *Let X be a CW-complex with cells $\{e_\lambda; \lambda\}$. Then $\{e_\lambda; \lambda\}$ is σ-HCP if and only if X has a σ-HCP k-network.*

Proof: "If": We recall that, among regular spaces, if $\mathcal{A}$ is HCP, then so is $\{\bar{A}; A \in \mathcal{A}\}$ (see [8; Lemma 5.5], etc.). Then we can assume that X has a σ-HCP closed k-network. But X is dominated by a cover of compact metric subsets. Thus, by Lemma 2, X has a σ-HCP k-network $\mathcal{P} = \cup\{\mathcal{P}_n; n \in N\}, \mathcal{P}_n \subset \mathcal{P}_{n+1}$, consisting of compact metric subsets. Fore each $n \in N$, let $X_n = \cup\mathcal{P}_n$, and let $C_n = \cup\{C;\ C$ is a finite subcomplex with $C \subset X_n\}$. Then each C_n is a CW-complex with cells $\{e_\lambda; e_\lambda \subset C_n\}$ by Lemma 3(1). On the other hand, each X_n has a HCP cover $\mathcal{P}_n$ of closed metric subsets. Hence each X_n is Fréchet, then so is each CW-complex C_n. Thus each $\{e_\lambda; e_\lambda \subset C_n\}$ is HCP by (B) in Introduction. But, each finite subcomplex is contained in some X_n, because $\mathcal{P}$ is a k-network for X. Then each cell e_λ in X is contained in some C_n. Hence $\{e_\lambda; \lambda\}$. is σ-HCP.

"Only if": Since $\{\bar{e}_\lambda; \lambda\}$. is σ-HCP, put $\{\bar{e}_\lambda; \lambda\}. = \cup\{\mathcal{F}_n; n \in N\}$, where each $\mathcal{F}_n$ is HCP with $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Each $X_n = \cup\mathcal{F}_n$ has a HCP cover of compact metric subsets. Since $\mathcal{F}_n$ is HCP, each compact subset of X_n is contained in a finite union of elements of $\mathcal{F}_n$. Hence each X_n has a σ-HCP k-network by Lemma 4. But, by Lemma 3(2), each compact subset of X is contained in some X_n. Hence X has a σ-HCP k-network by Lemma 4.

For $\alpha \geq \omega$, let S_α be the quotient space obtained from the topological sum of α convergent sequences by identifying all the limit points. The following lemma is due to [14; Lemma 2.2].

Lemma 8. *Let X be a CW-complex with cells $\{e_\lambda; \lambda\}$. Then $\{\bar{e}_\lambda; \lambda\}$ is point-finite (resp. point-countable) if and only if X contains no closed copy of S_ω (resp. S_{ω_1}).*

A space X is *symmetric* [1], if there exists a real valued, non-negative function d defined on $X \times X$ satisfying the following conditions:

$d(x,y) = 0 \Leftrightarrow x = y, d(x,y) = d(y,x)$, and $F \subset X$ is closed in $X \Leftrightarrow d(x,F) > 0$ for any $x \notin F$. If X is a σ-space (in particular, CW-complex), "X is symmetric" is equivalent to "X satisfies the weak first axiom of countability in the sense of [1] (or X is g-first countable in the sense of [11])"; see [1].

Theorem 9. *Let X be a CW-complex with cells $\{e_\lambda; \lambda\}$.*

(1) $\{\bar{e}_\lambda; \lambda\}$ is point-finite if and only if X is symmetric.

(2) $\{\bar{e}_\lambda; \lambda\}$ is point-countable if and only if X has a point-countable closed k-network.

Proof: (1) For the "if" part, we note that a symmetric space X with symmetric d contains no closed copy of S_ω. Indeed, suppose X contains a closed copy $\cup\{C_n; n \in N\} \cup \{\infty\}$ of S_ω, where each C_n is a sequence converging to ∞. For each $m \in N$, $F_m = \cup\{C_n : n \geq m\} \cup \{\infty\}$ is closed in X, but $F_m - \{\infty\}$ is not closed in X. Then there exists a sequence $A = \{x_k; k \in N\}$ in X such that $x_k \in C_{n(k)}$, and $d(x_k, \infty) < 1/k$. Then the sequence A converges to ∞. This is a contradiction. Then X contains no closed copy of S_ω. Thus this part follows from Lemma 8. For the "only" part, note that X is determined by a point-finite cover $\{\bar{e}_\lambda; \lambda\}$ of metric subsets. Then X is a symmetric space [12; Theorem 3.3].

(2) For the "if" part, we note that a space with a point-countable closed k-network contains no closed copy of S_{ω_1} in view of [13; Proposition 1]. Thus this part follows from Lemma 8 . For the "only if " part, X is covered by a point-countable closed cover $\{\bar{e}_\lambda; \lambda\}$ of metric subsets. Then, by Lemma 3(2) and Lemma 4, X has a point-countable closed k-network.

Remark 10. (1) In (1) of the previous theorem, unlike (2), we can not replace "X is symmetric" by "X has a point-finite closed k-network". Indeed, an infinite convergent sequence together with the limit point has no point-finite closed k-networks, then neither does a finite CW-complex I, where I is the closed unit interval [0,1].

(2) In (2) of the previous theorem, we can not omit the closedness of the k-network. Indeed, any CW-complex with cells $\{e_\lambda; \lambda\}$ has a point-countable k-network $\{V_{n\lambda} \cap e_\lambda; n, \lambda\}$, where $\{V_{n\lambda}; n\}$ is a countable base for $\bar{e}_\lambda$.

A space X is *g-metrizable* [11] if X is a regular space having a σ-locally finite weak base. For the definition of weak base, see [1; p. 129]. A Fréchet, g-metrizable space is metrizable [11]. Combining [4; Theorem 2.4] with [1; Theorem 2.8], we see that every g-metrizable space is precisely a symmetric, $\aleph$-space. Thus we have the following corollary by Theorems 5 and 9(1).

Corollary 11. *Let X be a CW-complex with cells $\{e_\lambda; \lambda\}$. Then $\{\bar{e}_\lambda; \lambda\}$ is point-finite and locally countable if and only if X if g-metrizable.*

We shall give some examples and a question.

Example 12. *Let X be a CW-complex with cells $\{e_\lambda; \lambda\}$.*

(1) The property "$\{\bar{e}_\lambda; \lambda\}$ is HCP" need not imply that X has a point-countable closed K-network, and not imply that $\{\bar{e}_\lambda; \lambda\}$ is point-countable.

(2) The property "$\{e_\lambda; \lambda\}$ is CP" need not imply that X has a CP or σ-HCP k-network, and not imply that $\{e_\lambda; \lambda\}$ is σ-HCP.

(3) The property "X is a symmetric space with a σ-CP k-network" need not imply that X has a σ-HCP k-network, not imply that $\{e_\lambda; \lambda\}$ is σ-CP.

Proof: (1) Let X be the quotient space obtained from the topological sum of uncountable many closed unit intervals by identifying all zero points. Then X is a CW-complex having the obvious cells $\{e_\lambda; \lambda\}$ such that $\{\bar{e}_\lambda; \lambda\}$ is HCP, but is not point-countable at the point 0. Then by Theorem 9(2), X has no point-countable closed k-networks.

(2) Let X be CW-complex obtained from the topological sum of uncountably many triangles $T_\lambda = \Delta a_\lambda b_\lambda c_\lambda$ by identifying all of segments $\overline{a_\lambda b_\lambda}$ with the closed unit interval I. Obviously, the set I has no CP-networks, hence has no CP k-networks. Then X has no CP k-networks. The collection $\{T_\lambda; \lambda\}$ is CP, but it is not σ-HCP. Indeed, suppose that $\{T_\lambda; \lambda\}$ is σ-HCP. Then some countable $\{T_{\lambda_n};\ n \in N\} \subset \{T_\lambda; \lambda\}$ is HCP. Let $Y = \cup\{T_{\lambda_n};\ n \in N\}$. Then Y is Fréchet. For each $n \in N$, let L_n be a segment from a point $1/n \in I$ to the vertex c_λ of T_λ. Let $A = \cup\{L_n; n \in N\} - \{1/n; n \in N\}$. Then $0 \in \bar{A}$ in Y. But there exist no sequences in A converging to the point 0. Hence Y is not Fréchet. This is a contradiction. Then $\{T_\lambda; \lambda\}$ is not σ-HCP, hence neither does the obvious cells of X. Thus by Theorem 7, X has no σ-HCP k-networks.

(3) Let I be the closed unit interval. For each $\alpha \in I$, let S_α be a 2-sphere. Let X be the quotient space obtained from the topological sum of $\{I, S_\alpha; \alpha \in I\}$ by identifying each $\alpha \in I$ with a point p_α of S_α. Then X is a CW-complex with the cells $\mathcal{A} = \{\{0\}, \{1\}, (0,1), T_\alpha; \alpha \in 1\}$, where $T_\alpha = S_\alpha - \{p_\alpha\}$. Since $\{\bar{T}; T \in \mathcal{A}\}$ is point-finite, X is symmetric by Theorem 9(1). Also, since X is a CW-complex, X has a σ-CP k-network by (b) in Introduction. But $\mathcal{A}$ is not σ-CP. Indeed, suppose that $\mathcal{A}$ is σ-CP. Then $\{\bar{T}_\alpha; \alpha \in I\}$ is a σ-CP closed collection. Then $I = \cup\{\bar{T}_\alpha; \alpha \in I\} \cap I$ is a countable union of discrete closed subsets of I . Hence the compact set I is at most countable. This is a contradiction. Thus the cells $\mathcal{A}$ is not σ-CP. Then X has no σ-HCP k-networks by Theorem 7.

In view of Theorem 7 and Example 12(2) & (3), we have the following question.

Question 13. Let X be a CW-complex with the cells $\{e_\lambda; \lambda\}$. Characterize "$\{e_\lambda; \lambda\}$ is CP (or σ-CP)" by means of a nice topological property of X.

Finally, we shall consider spaces dominated by compact metric subsets.

Let X be a space. Suppose that X is dominated by a cover $\{X_\lambda; \lambda < \alpha\}$. For each $\lambda < \alpha$, let $E_0 = X_0$, $E_\lambda = E_\lambda - \cup\{X_\mu; \mu < \lambda\}$. We will use this notation. The following property $(*)$ is due to [16].

$(*)$ Let $x \in X$. For each $\lambda < \alpha$, let A_λ be any subset of E_λ such that $\{x\} \cup A_\lambda$ is closed in X. Then $S = \{x\} \cup \{A_\lambda; \lambda < \alpha\}$ is closed in X. In particular, if each A_λ is finite, then S is closed and discrete in X.

Lemma 14. *Let X be domintaed by a cover $\{X_\lambda; \lambda < \alpha\}$. Then the following hold.*

(1) X is determined by $\{\bar{E}_\lambda; \lambda\}$.

(2) Each compact subset of X meets only finitely many E_λ.

(3) Let each E_λ be Fréchet. If X contains no closed copy of S_ω(resp. S_{ω_1}). then $\{\bar{E}_\lambda; \lambda\}$ is point-finite (reap. point-countable).

(4) Let F be a closed subset of X. If F is first countable (resp. Fréchet), then $\{E_\lambda; E_\lambda \subset F\}$ is locally finite (resp. HCP) in X.

Proof: (1) is due to [16]. (2) and (3) follow from the property $(*)$. For (4), note that F is dominated by $\{X_\lambda \cap F; \lambda\}$. When F is first countable for $x \in F$, let $\{V_n; n \in N\}$ be a decreasing local base at x in F. In view of $(*)$, some V_n meets only finitely many $F_\lambda = E_\lambda \cap F$. This implies that $\{E_\lambda; E_\lambda \subset F\}$ is locally finite in X. When F is Frécht, $\{F_\lambda; \lambda\}$ is HCP in F in view of Lemma 1.1 in [17]. Hence $\{E_\lambda; E_\lambda \subset F\}$ is HCP in X.

Concerning spaces dominated by compact metric subsets, similarly to CW-complexes the following analogue can be proved by means of Lemmas 2, 4, and 14.

Theorem 15. *Let X be a space dominated by a cover $\{X_\lambda;\lambda\}$ with each $\bar{E}_\lambda$ compact metric. Then it is possible to replace $\{e_\lambda;\lambda\}$ (or $\{\bar{e}_\lambda;\lambda\}$) by $\{E_\lambda;\lambda\}$ (or $\{\bar{E}_\lambda;\lambda\}$) in (A) $\sim$ (F) in Introduction.*

REFERENCES

1. A. V. Arhangel'skii, *Mappings and spaces*, Russian Math. Surveys, **21** (1966), 115-162.
2. J.G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math., **4** (1961), 105-125.
3. C. H. Dowker, *Topology of metric complexes*, Amer. J. Math., **74** (1952), 555-577.
4. L. Foged, *On g-metrizability*, Pacific J. Math., **98** (1982), 327-332.
5. G. Gruenhage, E. Michael and Y. Tanaka, *Spaces deteermined by point-countable covers*, Pacific J. Math., **113** (1984), 303-332.
6. E. Michael, *Continuous selections I*, Ann. Math., **63** (1956), 361-382.
7. ——, *$\aleph_0$-spaces*, J. Math. Mech., **15** (1966), 893-1002.
8. E. Michael, R. C. Olson and F. Siwiec, *A-spaces and countable bi-quotient maps*, Dissertationes Mathematicas (Warszawa), **133** (1976), 4-43.
9. K. Morita, *On spaces having the weak topology with respect to closed coverings*, Proc. Japan Acad., **29** (1953), 537-543.
10. P. O'Meara, *On paracompactness in function spaces with the compact-open topology*, Proc. Amer. Math. Soc., **29** (1971), 183-189.
11. F. Siwiec, *On defining a space by a weak base*, Pacific J. Math., **52** (1974), 233-245.
12. Y. Tanaka, *On symmetric spaces*, Proc. Japan Acad., **49** (1973), 106-111.
13. ——, *Closed maps on metric spaces*, Topology and Appl., **11** (1980), 87-92.
14. ——, *Products of spaces of countable tightness*, Topology Proceedings, **6** (1981), 115-133.
15. ——, *Point-countable k-systems and products of k-spaces*, Pacific J. Math., **101** (1982), 199-208.
16. ——, *Necessary and sufficient conditions for products of k-spaces*, Topology Proceedings, **14** (1989), 281-313.
17. Y. Tanaka and Zhou Hao-xuan, *Spaces dominated by metric subsets*, Topology Proceedings, **9** (1984), 149-163.
18. J. H. C. Whitehead, *Combinatorial homotopy I*, Bull. Amer. Math. Soc., **55** (1949), 213-245.

Tokyo Gakugei University
Koganei-shi, Tokyo (184) Japan