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SPACES DETERMINED BY GENERALIZED METRIC SUBSPACES

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Dedicated to Professor Akihiro Okuyama on his 60th birthday

INTRODUCTION

First, we shall give some definitions which will be used in this paper.

Let X be a space. Let $d : X \times X \to R$ be a non-negative, real valued function such that $d(x, y) = 0$ if and only if $x = y$. We shall consider the following conditions:

(a) $G \subset X$ is open if and only if for each $x \in G$, there exists $S_n(x) \subset G$, where $S_n(x) = \{y \in X;\ d(x, y) < 1/n\}$ $(n \in N)$.

(b) For $x \in X$ and $n \in N$, $S_n(x)$ is open in X.

(c) For $x \in X$ and $n \in N$, $\text{int } S_n(x) \ni x$.

Then d is called an *o-metric* [16] if it satisfies (a). An o-metric d is called a *generalized metric* [12] if it satisfies (b); equivalently, for each $x \in X$, $\{S_n(x);\ n \in N\}$ is a base at x.

A space X is called *o-metric* [16] if it has an o-metric d. Every o-metric space is a sequential space, hence a k-space.

We note that a space X is weakly first countable ($= X$ satisfies the weak first axiom of countability in the sense of [1]) if and only if X is o-metric; and that a space X is first countable if and only if it has an o-metric staisfying (b) (or (c)); cf. [16].

Let X be a space. Let $d : X \times X \to R$ be a non-negative, real valued function. Let us consider following conditions as a generalization of metric functions.

(1) $d(x, y) = d(y, x)$.

(2) $d(x,z) \leq d(x,y) + d(y,z)$.
(3) $d(x,z) \leq \max \{d(x,y), d(y,z)\}$.
(4) For any compact set K and closed set F with $K \cap F = \emptyset$, $\inf \{d(x,y);\ x \in K,\ y \in F\} > 0$;

A space X is called *symmetric* if it has an o-metric d satisfying (1), and such a funtion d is called *symmetric* for X.

A space X is called *semi-metric* if it has an o-metric d satisfying (1) and (c).

A space X is called *quasi-metric* (= Δ-metric in the sense of ([16]) if it has a generalized metric d satisfying (2). Here we can replace "generalized metric" by "o-metric".

A space X is called *non-archimedian quasi-metric* (simply, n.a.-quasi-metric) if it has a generalized metric d satisfying (3). Here we can replace "generalized metric" by "o-metric".

A space X is called *γ-metric* (= γ-space) if it has a generalized metric staisfying (4).

In this paper, we shall use "X is symmetric; (n.a-) quasi-metric, etc" instead of "X is symmetrizable; (n.a.-) quasi-metrizable; etc".

(N.a.-) quasi-metric spaces; γ-metric spaces are characterized by means of g-functions, interior-preserving covers, quasi-uniformities, or sequences of neighbornets, etc., and they are investigated or surveyed in [5], [6], [12], [16], etc.

Concerning symmetric, (n.a-) quasi-metric, or γ-metric spaces, etc., the following diagram is known; see [6], for example. A space is *Fréchet* if whenever $x \in \bar{A}$, then there exists a sequence in A converging to the point x. For the definition of semi-stratifiable spaces; see [3], and for (a), see [10]; and [4].

Diagram. For a space, the following implications hold.

(a) o-metric and semi-stratifiable $\Rightarrow$ symmetric. But, symmetric $\not\Rightarrow$ closed sets are G_δ-sets.

(b) developable $\Rightarrow$ semi-metric $\Leftrightarrow$ Fréchet and symmetric $\Leftrightarrow$ first countable and semi-stratifiable. But, semi-metric $\not\Rightarrow$ σ-space.

(c) metacompact and developable $\Rightarrow$ n.a.-quasi-metric $\Rightarrow$ quasi-metric $\Rightarrow$ γ-metric $\Rightarrow$ first countable. But, n.a.-quasi-metric $\not\Rightarrow$ closed sets are G_δ-sets.

(d) symmetric and γ-metric $\Leftrightarrow$ developable and quasi-metric. But, developable $\not\Rightarrow$ γ-metric.

Let X be a space, and let $\mathcal{C}$ be a cover (not necessarily closed or open) of X. Then X *is determined by* $\mathcal{C}$ [7] (= X has the weak topology with respect to $\mathcal{C}$ in the usual sense), if $F \subset X$ is closed in X if and only if $F \cap C$ is closed in C for every $C \in \mathcal{C}$. Here, we can replace "closed" by "open". Every space is detemined by an open cover. If a space X is determined by a countable and increasing cover $\{X_n;\ n \in N\}$, then X is called the *inductive limit* of $\{X_n;\ n \in N\}$, and denoted by $X = \varinjlim X_n$.

We recall that a space X is *sequential* if X is determined by the cover of all (compact) metric subspaces.

Let X be a space, and let $\mathcal{F}$ be a closed cover of X. Then X *is dominated by* $\mathcal{F}$ [14] (=X has the weak topology with respect to $\mathcal{F}$ in the sense of [15]), if the union of any subcollection $\mathcal{F}'$ of $\mathcal{F}$ is closed in X, and the union is determined by $\mathcal{F}'$. Every space is dominated by a hereditarily closure-preserving closed cover. As is well-known, every CW-complex is dominated by a cover of compact metric subspaces.

We recall canonical quotient spaces S_ω and S_2, which is called the *sequential fan* and the *Arens' space* respectively.

S_ω is the quotient space obtained from the topological sum of countably many convergent sequences by identifying all the limit points.

$S_2 = (N \times N) \cup N \cup \{\infty\}$ is the space with each point of $(N \times N)$ isolated. A basic neighborhood of $n \in N$ consists of all sets of the form $\{n\} \cup \{(m,n);\ m \geq k\}$. And U is a neighborhood of ∞ if and only if $\infty \in U$ and U is a neighborhood of all but finitely many $n \in N$.

The spaces S_ω and S_2 are dominated by an increasing countable cover of compact metric subsets. But, S_ω nor S_2 is first countable. Then S_ω is not semi-metric, not (n.a.-) quasi-metric, not γ-metric, and neither is S_2. Then the following question in [12; Question 3] is negative.

Let X be a space dominated by a cover of quasi-metric; n.a.-quasi-metric; or γ-metric subsets. Then is X so respectively?

In this paper, we give a characterization for the above space X to be quasi-metric; n.a.-quasi-metric; or γ-metric respectively. We also give some analogous characterizations when spaces are determined by certain covers of these generalized subspaces, or semi-metric subspaces, etc.

We assume that all spaces are regular and T_1.

1. Spaces determined by countable covers.

For each $n \in N$, let Y_n be homeomorphic to the product X^n of a space X. First, we shall consider the inductive limit of $\{Y_n;\ n \in N\}$.

Lemma 1.1. *Let X be a sequential space, and let $x \in X$. For each $n \in N$, let $Y_n = X^n \times \{x\} \times \{x\} \times \dots$. Let $Y = \varinjlim Y_n$. If the point x is not isolated in X, then Y contains a closed copy of S_ω, and a closed copy of S_2.*

Proof: Since X is sequential, there exists a sequence $\{x_n;\ n \in N\}$ in X converging to x with $x_n \neq x$. Let $p = (x, x, \dots)$, let $p_{m\ n} = (x_n, x_n, \dots x_n, x, x, \dots) \in Y_m$ for each $m,\ n \in N$. Let $S = \cup\{p_{m\ n};\ m,\ n \in N\} \cup \{p\}$. Since each $S \cap Y_n$ is closed in Y_n, S is closed in Y. For each $m \in N$, let $k(m) \in N$, and let $F = \cup\{p_{m\ n};\ m \in N,\ n \leq k(m)\}$. Then each $F \cap Y_n$ is finite, hence closed in Y_n. Thus F is closed in Y. This implies that S is a copy of S_ω. Then Y contains a closed copy of S_ω. Next, for each $m \in N$, let $q_m = (x_m, x, \dots)$. And, for each $m,\ n \in N$, let $q_{m\ n} = (x_m, x, \dots x, x_n, x, \dots)$, where x_n is the m-th coordinate. Let $T = \{q_{m\ n};\ m,\ n \in N\} \cup \{q_m;\ m \in N\} \cup \{p\}$. Similarly T is closed in Y, and T is a copy of S_2.

Theorem 1.2. *Let X be a space, and $x \in X$. For each $n \in N$, let $Y_n = X^n \times \{x\} \times \{x\} \times \ldots$. Let $Y = \varinjlim Y_n$. Then (1) and (2) below hold.*

(1) Suppose that X is symmetric. Then the following are equivalent.

(a) Y is symmetric.

(b) Y is a sequential space which contains no closed copy of S_ω.

(c) Y is a sequential space, and the point x is isolated in X.

(2) Suppose that X is metric; semi-metric; quasi-metric; n.a-quasi-metric; γ-metric; or developable. Then the following are equivalent.

(a) Y is so respectively.

(b) Y contains no closed copy of S_ω.

(c) Y contains no closed copy of S_2.

(d) The point x is isolated in X.

Proof: (1) For (a) $\Rightarrow$ (b), suppose that Y contains a closed subset $\cup\{L_n;\ n \in N\} \cup \{p\}$, with $L_n \to p$, which is a copy S_ω. Then each $L_n \cup \{p\}$ is closed, but L_n is not closed in Y. Thus there exists a point $y_n \in L_n$ with $y_n \in S_n(p)$ for each $n \in N$. Then the sequence $\{y_n;\ n \in N\}$ converges to the point ∞. This is a contradiction. Hence Y contains no closed copy of S_ω. (b) $\Rightarrow$ (c) follows form Lemma 1.1. If (c) holds, then each Y_n is closed and open in Y. Then Y is the topological sum of $\{Y_n - Y_{n-1};\ n \in N\}$. But each Y_n is sequential with X symmetric. Then each Y_n is symmetric by [19; Theorem 4.1], hence so is each $Y_n - Y_{n-1}$. Thus Y is symmetric. Hence (a) holds.

(2) (a) $\Rightarrow$ (b) or (c) is easy, because Y is first countable. (b) or (c) $\Rightarrow$ (d) follows from Lemma 1.1. For (d) $\Rightarrow$ (a), Y is the topological sum of $\{Y_n - Y_{n-1};\ n \in N\}$. But each Y_n is so respectively, then so is each $Y_n - Y_{n-1}$. Thus Y is so respectively.

Theorem 1.3. *Let X be a non-discrete, sequential space. Suppose that X is homogeneous; that is, for each $p, q \in X$, there exists a homeomorphism of X onto X taking p to q. For each $n \in N$, let us consider X^n as a subspace of X^{n+1}; that is, as a subspace $X^n \times \{x\} \times \{x\} \times \ldots$ of X^ω for $x \in X$. For each $n \in N$, let $Y_n = X^n$. Let $Y = \underrightarrow{\lim} Y_n$. Then Y contains a closed copy of S_ω and a closed copy of S_2; hence, Y is not o-metric nor Fréchet.*

Proof: Since X is non-discrete and homogeneous, any point of X is not isolated. Then Y contains a closed copy of S_ω and a closed copy of S_2 by Lemma 1.1. Thus Y is not Fréchet, for Y contains a copy of S_2. Also, Y is not o-metric, for Y contains a closed copy of S_ω.

Let I; R be the closed interval; the real line respcetively. For all $n \in N$, let $X_n = I^n$ (or R^n). The previous theorem shows that $X = \underrightarrow{\lim} X_n$ is not even o-metric nor Fréchet, hence X is not symmetric, not quasi-metric, nor γ-metric, etc.

The following lemma is due to [5]. Unlike this, every space which is a countable union of open metric subsets is neither semi-metric nor symmetric; see, Example 1.9.

Lemma 1.4. *Let $\{G_\alpha; \alpha\}$ be a σ-point-finite open cover of X. If the G_α are quasi-metric; n.a.-quasi-metric; or γ-metric, then so is X respectively.*

A space X is *strongly Fréchet* [17], if whenever $\{A_n, n \in N\}$ is a decreasing sequence in X with $x \in \bar{A}_n$ for any $n \in N$, then there exists a sequence $\{x_n; n \in N\}$ in X converging to the point x with $x_n \in A_n$. The following lemma is due to [20].

Lemma 1.5. *Let X be a space determined by a countable cover $\mathcal{C}$ such that each finite union of elements of $\mathcal{C}$ is first countable. If X contains no closed copy of S_ω and no S_2, then X is strongly Fréchet.*

Every space $\varinjlim X_n$ with each X_n compact metric need not be quasi-metric, nor γ-metric even if X is symmetric (or Fréchet), as is seen by the space S_2 (or S_ω). But we have the following theorem (cf. Theorem 1.2).

Theorem 1.6. *Let $X = \varinjlim X_n$. Suppose that the X_n are quasi-metric; n.a.-quasi-metric; or γ-metric. Then the following are equivalent.*

(a) X is so respectively.

(b) X is first countable.

(c) X contains no closed copy of S_ω and no S_2.

Proof: (a) $\Rightarrow$ (b) is clear, and (b) $\Rightarrow$ (c) is obvious. So, we prove (c) $\Rightarrow$ (a). Since X is strongly Fréchet by Lemma 1.5, for each $x \in X$, $x \in \text{int } X_m$ for some $m \in N$. Indeed, suppose not. Then $x \in \overline{X - X_n}$ for any $n \in N$. Then there exists a sequence $K = \{p_n;\ n \in N\}$ in X converging to the point x such that $p_n \in X - X_n$, and $p_n \neq x$ for any $n \in N$. But, each $K \cap X_n$ is finite, hence closed in X_n. Thus K is closed in X, hence $K \ni x$. This is a contradiction. Thus, $x \in \text{int } X_m$ for some $m \in N$. This implies that $\{\text{int } X_n;\ n \in N\}$ is a countable open cover of X. But, the int X_n are quasi-metric; n.a.-quasi-metric; γ-metric respectively. Thus, X is so respectively by Lemma 1.4.

A space X is *submetacompact* (= θ-refinable) if for each open cover $\mathcal{U}$ of X there exists a sequence $\{\mathcal{U}_n;\ n \in N\}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$ there exists an open cover $\mathcal{U}_n$ which is finite at x. As is well-known, metacompact spaces, and subparacompact spaces are submetacompact.

Every semi-metric space is semi-stratifiable, hence submetacompact. But, every n.a.-quasi-metric space is not submetacompact; see, Example 1.9.

The following lemma is due to [18]. But, unlike this, every submetacompact and locally metric space is neither quasi-metric nor γ-metric; see, Example 2.5(2).

Lemma 1.7. *Let X be a submetacompact space.*

(1) If X is locally developable, then X is developable.

(2) If X is locally semi-metric, then X is semi-metric.

Corollary 1.8. *Let $X = \varinjlim X_n$. Suppose that the X_n are semi-metric; or developable. Then the following are equivalent.*

(a) X is so respectively.

(b) X is first countable, and every closed subset of X is a G_σ-set.

(c) X is first countable, and submetacompact.

Proof: Every semi-metric space is a submetacompact space in which every closed subset is a G_δ-set. Thus (a) $\Rightarrow$ (b) and (c) holds. Suppose that (b) holds. Since X is first countable. X is a countable union of open semi-metric subsets by the proof of Theorem 1.6. But each open subset of X is an F_σ-set, then X is a countable union of closed semi-stratifiable subsets. Thus X is a semi-stratifiable space. Then X is semi-metric by (b) in Diagram. Thus (b) $\Rightarrow$ (a) holds. Suppose that (c) holds. Since X is first countable, X is locally semi-metric by the proof of Theorem 1.6. But X is submetacompact, then X is semi-metric by (2) of Lemma 1.7. Thus (c) $\Rightarrow$ (a) holds. For the developable case, (b) implies that X is a semi-stratifiable space which is locally developable. Since X is submetacompact, X is developable by (1) of Lemma 1.7. Thus (b) (or (c)) $\Rightarrow$ (a) also holds in this case.

Concerning symmetric spaces and semi-metric spaces, Theorem 1.6 does not hold by the following example. Hence, the additional condition of X in (b) or (c) of Corollary 1.8 is essential.

Example 1.9. *A n.a.-quasi-metric (hence first countable) space $X = \varinjlim X_n$ such that the X_n are semi-metric open subsets. But X is not symmetric (nor submetacompact).*

Proof: Let X be the space Z in Example 3.3 in [4], where Z is not submetacompact and has a closed subset which is not a G_δ-set. As is seen there, Z has the σ-locally countable base $\mathcal{B} = \cup\{\mathcal{B}_n;\ n \in N\}$, which is also σ-disjoint. But, it follows that each member of $\mathcal{B}$, which is the basic nbd $B(x_1, x_2, \dots x_k)$ or $V_k(\alpha) = \{\alpha\} \cup \{\cup B_\alpha(n);\ n \geq k\}$ defined there, is clopen in Z, and metrizable (the $V_k(\alpha)$ has a σ-locally finite base, hence is metrizable). Let $G_n = \cup\mathcal{B}_n$ for each $n \in N$. Then each G_n has a locally finite closed cover $\mathcal{B}_n$ in G_n, hence G_n is metrizable. Thus X has a countable open cover $\{G_n;\ n \in N\}$ of metric subsets. Hence X is n.a.-quasi-metric by Lemma 1.4. For each $n \in N$, let $X_n = \cup\{G_m;\ m \leq n\}$. Then each X_n is an open subset of X which is developable, hence semi-metric. Then X is determined by a countable, increasing open cover $\{X_n;\ n \in N\}$ of semi-metric subsets. But X is a first countable space which is not semi-stratifiable. Then X is not symmetric by (b) in Diagram.

2. Spaces determined by point-finite covers.

The space S_2 is a symmetric space determined by a point-finite, countable cover of compact metric subspaces. But S_2 is neither semi-metric nor quasi-metric. Concerning spaces determined by point-finite covers of certain gereralized metric subspaces, we have the following theorem.

Theorem 2.1. *Let X be a space determined by a point-finite cover $\mathcal{C} = \{X_\alpha\}$.*

(1) If the X_α are o-metric; or symmetric, then so is X respectively.

(2) Suppose that the X_α are semi-metric. Then X is semi-metric if and only if X is first countable (or Fréchet).

(3) Suppose that the X_α are metacompact developable. Then the following are equivalent.

(a) X is (metacompact) developable.

(b) X is (n.a.-) quasi-metric.

(c) X is γ-metric.

(d) *X is semi-metric.*
(e) *X is first countable.*
(f) *X is Fréchet.*

Proof: (1) Let the X_α be o-metric. Then, for $x \in X_\alpha$, one can associate a sequence $\{S_{\alpha\ n}(x);\ n \in N\}$ of subsets of X_α such that $x \in S_{\alpha\ n+1}(x) \subset S_{\alpha\ n}(x)$; and $U \subset X_\alpha$ is open in X_α if and only if for each $x \in U$ there exists $n \in N$ with $S_{\alpha\ n}(x) \subset U$. For $x \in X$, let $Q_n(x) = \cup\{S_{\alpha\ n}(x);\ x \in X_\alpha\}$. Since X is determined by $\mathcal{C}$, for $x \in X$, the sequence $\{Q_n(x);\ n \in N\}$ of subsets of X satisfies the above conditions with respect to X. For $x,\ y \in X$, let $d(x,y) = 1/n$, when $n =$Max $\{m;\ y \in Q_m(x)\}$. Then d is o-metric for X. Hence X is o-metric. When X_α are symmetric, similarly we show that X is symmetric. (2) follows form (1) and (a) in Diagram. For (3), we shall prove only (f) $\Rightarrow$ (a). First, we prove that for each $x \in X$, $x \in$ int $\mathrm{St}(x,\mathcal{C})$.

To show this, suppose not. Then $x \in \overline{X - St(x,\mathcal{C})}$. Thus there exists a sequence $K = \{x_n;\ n \in N\}$ in $X-$ $\mathrm{St}(x,\mathcal{C})$ converging to the point x. Let $C \in \mathcal{C}$. If $C \not\ni x$, $K \cap C$ is closed in C. If $C \ni x$, $K \cap C = \emptyset$. Thus $K \cap C$ is closed in C. Then K is closed in X. Hence $K \ni x$. This is a contradiction. Then $x \in$ int $\mathrm{St}(x,\mathcal{C})$ for each $x \in X$. Now, to show that X is metacompact, let $\mathcal{U}$ be an open cover of X. Since each X_α is metacompact, there exists a point-finite open refinement $\mathcal{U}_\alpha$ of $\{U \cap X_\alpha;\ U \in \mathcal{U}\}$ in X_α. Let $\mathcal{V} = \cup\{\mathcal{U}_\alpha; \alpha\}$. But X is determined by a cover $\{X_\alpha\}$, and each X_α is determined by the open cover $\mathcal{U}_\alpha$. Then X is determined by $\mathcal{V}$. Thus, as is seen in the above, for each $x \in X$, $x \in$ int $\mathrm{St}(x,\ \mathcal{V}\ )$. Then X has a point-finite refinement $\mathcal{V}$ of $\mathcal{U}$ such that $x \in$ int $\mathrm{St}(x,\mathcal{V}\ )$ for each $x \in X$. Then X is metacompact by [9; Theorem 2.2]. Finally, we show that X is developable. Since X is semi-metric by (2), by [8; Theorem 1] it suffices to show that X has a point-countable base. Since each X_α is metacompact developable, it has a development $\{\mathcal{G}_{\alpha\ n};\ n \in N\}$ such that each $\mathcal{G}_{\alpha\ n}$ is point-finite, and $\mathcal{G}_{\alpha\ n+1}$ is a refinement of $\mathcal{G}_{\alpha\ n}$. For each $n \in N$, let $\mathcal{G}_n = \cup\{\mathcal{G}_{\alpha\ n}; \alpha\}$, and let $\mathcal{G} = \cup\{\mathcal{G}_n;\ n \in N\}$. Then, X is determined by a point-finite cover $\mathcal{G}_n(n \in N)$. Let $x \in U$ with U open in X. Then there exists $n \in N$ such that $x \in \mathrm{St}(x, \mathcal{G}_n) \subset U$. But, as is seen in the above, $x \in \mathrm{int}\ \mathrm{St}(x, \mathcal{G}_n)$. This shows that X has a point-countable cover $\mathcal{G}$ such that for any $x \in X$ and any nbd U of x, there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ with $x \in \mathrm{int} \cup \mathcal{G}', \cup \mathcal{G}' \subset U$. But X is Fréchet. Then X has a point-countable base by [2; Theorem 6.2].

Corollary 2.2. *Let X be determined by a point-finite closed cover of developable subspaces. Then X is developable if and only if X is first countable (or Fréchet.)*

Proof: For the "if" part, by (2) in Theorem 2.1, X is semi-metric, hence submetacompact. On the other hand, in view of the proof of (f) $\Rightarrow$ (a) in Theorem 2.1(3), X is locally developable. Hence X is developable by (1) of Lemma 1.7.

In view of Theorem 2.1 and Corollary 2.2, we have the following question.

Question 2.3. Let X be a first countable space determined by a point-finite cover $\{x_\alpha\}$. If the X_α are quasi-metric; n.a.-quasi-metric; γ-metric; or developable, then so is X respectively?

Remark 2.4. In the previous question, if the X_α are metacompact and closed in X, then the question is affirmative. Indeed, the proof of (3) in Theorem 2.1 suggests that X is a metacompact space, and each point has a nbd which is quasi-metric; n.a.-quasi-metric; or γ-metric respectively. Hence, X has a point-finite open cover of quasi-metric; n.a.-quasi-metric; or γ-metric subspaces respectively. Then so is X respectively by Lemma 1.4.

Concerning the metrizability; or quasi-metrizability of a space determined by a point-finite; or point-countable cover of metric subspaces, Theorem 2.1 is not valid by (1); or (2) of Example 2.5 below respectively. Also, see Example 1.9.

Example 2.5. (1) There exists a n.a.-quasi-metric (hence first countable) space X determined by a point-finite clopen cover of metric subspaces, but X is not metric.

(2) There exzists a semi-metric (hence, first countable) space X determined by a point-countable clopen cover of metric subspaces, but X is neither quasi-metric nor γ-metric.

Proof: (1) Let X be an upper half plane. Let a basic nbd of (x,y) with $y > 0$ be $\{(x,y)\}$, and let a basic nbd of $(r,0)$ be $\{(x,y);\ y = \mid x - r \mid < 1/n\}$, $n \in N$. Then X is metacompact and developable, hence n.a.-quasi-metric. For $(r,0) \in X$, let $X_r = \{(x,y);\ y = \mid x - r \mid\}$. Then X is determined by a point-finite clopen cover $\{X_r;\ (r,0) \in X\}$ of metric subspaces. But, by the R. Baire's Category theorem, X is not normal, hence not metric.

(2) Let X be the developable space Y constructed in [13; Example 2], where Y is not quasi-metric. That is; let $A = R \times \{0\}$, and $B = \{(x,y); x,y$ are rationals with $y > 0\}$. For each $p \in A$ and $n \in N$, let $T(p,1/n)$ denote the set of all points in B that belong to the interior of the isosceles right triangle above A having vertex p and hypothenuse of length $2/n$ parallel to A. For each $q \in B$ and $n \in N$, let $C(q,1/n)$ denote the intersection with B of the circle of radius $1/n$ and center q. Let $\mathcal{U}$ be the collection of all countable infinite subsets of A. Let $Y = A \cup (B \times \mathcal{U})$. Let a basic nbd of $p \in A$ be $V_n(p) = \{p\} \cup T(p,1/n) \times \mathcal{U}(p)$, where $\mathcal{U}(p) = \{\alpha \in \mathcal{U};\ p \in \alpha\}$, and let a basic nbd of $(q,\alpha) \in B \times \mathcal{U}$ be $V_n(q,\alpha) = C(q,1/n) \times \{\alpha\}$. Then the basic nbds $V_n(p)$ and $V_n(q,\alpha)$ are metric (indeed, the $V_n(p)$ has a σ-locally finite base, hence it is metric). Obviously $\mathcal{B} = \{V_n(p),\ V_n(q,\alpha);\ p \in A,\ (q,\alpha) \in B \times \mathcal{U},\ n \in N\}$ is a point-countable base for Y. But, we can assume that each element of $\mathcal{B}$ is clopen in Y (Y is zero-dimensional). Therefore, Y has a point-countable base consisting of clopen and metric subspaces. But Y is developable, hence symmetric. Thus Y is

not γ-metric by (d) in Diagram.

3. Spaces dominated by covers.

The following lemma is due to [5] (for the symmetric case, see [19]; and for the smi-metric or developable case, see [18]).

Lemma 3.1. *Let $\{F_\alpha; \alpha\}$ be a locally finite closed cover of a space X. If the F_α are symmetric; semi-metric; quasi-metric; n.a.-quasi-metric; γ-metric; or developable, then so is X respectively.*

Let X be a space dominated by a cover $\{X_\alpha; \alpha < \lambda\}$. For each $\alpha < \lambda$, let $L_0 = X_0$, $L_\alpha = X_\alpha - \cup\{X_\beta;\ \beta < \alpha\}$, and let $F_\alpha = \bar{L}_\alpha$. Then we have

Lemma 3.2. *Let X be a space dominated by a cover $\{X_\alpha;\ \alpha < \lambda\}$. Then (1) and (2) hold. If the X_α are Fréchet, then (3) $\sim$ (5) hold.*

(1) X is determined by $\{F_\alpha;\ \alpha < \lambda\}$.

(2) Let $x \in X$. For each $\alpha < \lambda$, let A_α be any subset of L_α such that $A_\alpha \cup \{x\}$ is closed in X. Then $S = \cup\{A_\alpha; \alpha < \lambda\} \cup \{x\}$ is closed in X.

(3) If X contains no closed copy of S_ω, then $\{F_\alpha;\ \alpha < \lambda\}$ is point-finite in X.

(4) If X contains no closed copy of S_2, then $\{F_\alpha; \alpha < \lambda\}$ is hereditarily closure-preserving in X.

(5) If X contains no closed copy of S_ω and no S_2, then $\{F_\alpha;\ \alpha < \lambda\}$ is locally finite in X.

Proof: (1) and (2) are due to [21; Lemma 2.5]. For (3), suppose that $\{F_\alpha;\ \alpha < \lambda\}$ is not point-finite. Since each F_α is Fréchet, it follows from (2) that X contains a closed copy of S_ω. This is a contradiction. Then $\{F_\alpha;\ \alpha < \lambda\}$ is point-finite. For (4), suppose that $\{F_\alpha;\ \alpha < \lambda\}$ is not hereditarily closure-preserving. Then for each $\alpha < \lambda$, there exists a closed subset C_α of F_α such that $A = \cup\{C_\alpha;\ \alpha < \lambda\}$ is not closed in X. By (1), $A \cap F_\alpha$ is not closed in some Fréchet space F_α. Thus there exist a point $x \notin A$, a sequence $\{x_m;\ m \in N\}$ in A, and an infinite subset $\{\alpha(m);\ m \in N\}$ of $\{\alpha;\ \alpha < \lambda\}$ such that $x_m \rightarrow x$, $x_m \in C_{\alpha(m)}$. But each $F_{\alpha(m)}$ is Fréchet. Then for each $m \in N$, there exists a sequence $\{x_{m\ n};\ n \in N\}$ in $L_{\alpha(m)}$ such that $x_{m\ n} \rightarrow x_m$. Let $T = \{x\} \cup \{x_m;\ m \in N\} \cup \{x_{m\ n}; m, n \in N\}$. Then, it follows from (2) that T is a closed copy of S_2. Thus X contains a closed copy of S_2. This is a contradiction. Then $\{F_\alpha;\ \alpha < \lambda\}$ is hereditarily closure-preserving. (5) follows form (1) and (2).

Theorem 3.3 *Let X be a space dominated by $\{X_\alpha\}$. Then (1) and (2) below hold.*

(1) Let each X_α be first countable. Then X is an o-metric if and only if X contains no closed copy of S_ω.

(2) Let each X_α be Fréchet. Then X is Fréchet if and only if X contains no closed copy of S_2.

Proof: (1) We prove only the "if" part. By (1) and (2) in Lemma 3.2, X is determined by a point-finite cover $\{F_\alpha;\ \alpha < \lambda\}$. But each F_α is an o-metric. Then X is an o-metric by Theorem 2.1.

(2) For the "if" part by Lemma 3.2(4), X has a hereditarily closure-preserving cover $\{F_\alpha;\ \alpha < \lambda\}$. Since each F_α is Fréchet, so is X. The "only if" part follows from the easy fact that any Fréchet space contains no copy of S_2.

Theorem 3.4. *Let X be a space dominated by $\{X_\alpha\}$.*

(1) Suppose that the X_α are semi-metric. Then the following are equivalent.

(a) X is symmetric.

(b) X is o-metric.

(c) X contains no closed copy of S_ω.

(2) Suppose that the X_α are metric; semi-metric; quasi-metric; n.a.-quasi-metric; γ-metric; or developable. Then the following are equivalent.

(a) X is so respectively.

(b) X is first countable.

(c) X contains no closed copy of S_ω and no S_2.

Proof: (1) holds in view of (1) in Theorem 2.1 & 3.3 and Lemma 3.2. (2) follows from Lemma 3.1 and Lemma 3.2(5). (For the metric case, (a) $\Leftrightarrow$ (c) is due to [22; Theorem 1.5].)

In view of (1) in Theorem 3.3 and 3.4, we have a question below. If (2) is affirmative, then so is (1). When the X_α are semi-stratifiable, (1) is affirmative. Indeed, by [18; Theorem 4.5], X is semi-stratifiable. While, X is o-metric. Then X is symmetric by (a) in Diagram.

Question 3.5. Let X be a space dominated by $\{X_\alpha\}$. Suppose that the X_α are symmetric.

(1) If X is o-metric, then X is symmetric ?

(2) If X contains no closed copy of S_ω, then X is symmetric?

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