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A CARDINAL FUNCTION RELATED TO ALMOST κ-COMPACTNESS

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1. Introduction

The motivation for this note is the observation that a number of proofs involving compactness do not use its full strength. Instead of using the fact that any centered collection of closed sets has non-empty intersection, they only require that $\bigcap_{A\in\mathcal{A}}\overline{A}\neq\emptyset$, where $\mathcal{A}$ is a particular centered collection of open sets. For example, see [DvM 2.1, Ku 2.2.22]. Prompted by this, we examine the first instance where this fails:

Definition 1.1. *For a topological space X, the* almost initial compactness number *of X is defined to be*

$$aic(X) = min\ \{|\mathcal{U}| : \mathcal{U} \textit{ is a centered collection}$$

$$\textit{of open subsets of } X \textit{ such that } \bigcap\overline{\mathcal{U}} = \emptyset\}.$$

Of course, in a compact space, every centered collection of closed sets has non-empty intersection. So $aic(X)$ is undefined for compact X. Furthermore, recall that a Hausdorff space is *H-closed*[1] if every filter with a base of open sets has a cluster point (equivalently if it is a closed subspace of every Hausdorff space in which it is contained [En]). Therefore, for Hausdorff spaces at least, $aic(X)$ is defined if and only if X is not H-closed. In this note, all spaces are assumed to be Hausdorff

[1] H-closed spaces with no separation are called H(i)[SS].

and not H-closed, unless specified otherwise, so that the almost initial compactness number is always defined.

It is clear from the definition that $aic(X)$ is defined if and only if X has an open cover having no finite subfamily with dense union. In fact, equivalent to Definition 1.1, we have

Definition 1.1'. *For a topological space X, the almost initial compactness number of X is defined to be*

$$aic(X) = min\{|\mathcal{V}| : \mathcal{V} \textit{ is an open cover of } X \textit{ with}$$
$$\textit{no finite subset having dense union }\}.$$

and we shall use whichever of 1.1 or 1.1' seems most convenient. In [Fr], Frolík calls a space *almost κ-compact* if every open cover of cardinality $\leqslant \kappa$ has a finite subset with dense union. This definition provides our terminology. Unfortunately, 'almost initial compactness number' is a somewhat unwieldy name. The author informally thinks of the centered family of open sets as collapsing to nothing; so, for brevity, we shall write this cardinal as $aic(X)$, but refer to the almost initial compactness number as 'the collapse of X'.

As well as pointing out various elementary properties of $aic(X)$, we establish connections between it, other cardinal functions and some 'small' cardinals. In particular, we show that for any infinite cardinal κ, there is a space X with $aic(X) = \kappa$, and that for every first countable space X that is not countably compact, $aic(X) \leqslant \mathfrak{d}$. Moreover, by studying collapse, we obtain some easy proofs of seemingly unrelated results. For instance, we show that there can be no Hausdorff extension of ψ with character smaller than $\mathfrak{p}$ and that if X is compact then its cellularity can be no greater than its local cardinality.

The reader is directed to [Ku] for definitions of set-theoretic notions used herein and to [En] for topological definitions. Information about cardinal functions can be found in [Ho, Ju]. We denote the set of natural numbers by ω, the first uncountable ordinal by ω_1 and the cardinality of the continuum by $\mathfrak{c}$. We refer to [vD] for the definitions of $\mathfrak{a}, \mathfrak{d}, \mathfrak{p}$ and $\mathfrak{u}$; a detailed discussion of these cardinals can be found in [vD, Va2]. We simply note here that their values lie somewhere between ω_1 and $\mathfrak{c}$. For notational ease, if $\mathcal{A}$ is a collection of sets, we denote $\bigcap_{A\in\mathcal{A}}\overline{A}$ by $\bigcap\overline{\mathcal{A}}$.

2. The collapse of $X, aic(X)$

The collapse of a space $X, aic(X)$, is defined by either of the two equivalent definitions in the Introduction. From 1.1, it may be regarded as a measure of completeness: if $aic(X) = \kappa$ then every filter on X has an accumulation point whenever it has a base of open sets with cardinality smaller than κ. On the other hand, from 1.1', we may think of collapse as a covering property. Its connection with almost κ-compactness is clear:

Lemma 2.1. *For any space X, $aic(X) > \kappa$ if and only if X is almost κ-compact.*

Viewing collapse as a covering property, we find a basic connection with pseudocompact and countably compact spaces. Recall that a space X is *feebly compact* if every countable open cover has a finite subset with dense union [PW]. This is equivalent to pseudocompactness for completely regular spaces and is certainly implied by countable compactness. Therefore we have $aic(X) > \omega$ if and only if X is feebly compact (for any space X, we have $aic(X) \geq \omega$), hence pseudocompact and countably compact spaces have uncountable collapse.

Neither pseudocompactness nor countable compactness are hereditary properties, nor are they preserved by products (to be compared with H-closed spaces [CF]). Similarly, we may have $aic(X) < aic(Y)$ for $X \subseteq Y$ and $aic(X\times Y) <\min\{aic(X), aic(Y)\}$ for spaces X, Y. To see this, consider open and closed subsets of the pseudocompact space ψ [GJ], and the example in [No] of subspaces X_1 and X_2 of $\beta\omega$ which are both countably compact, but for which $X_1 \times X_2$ is not pseudocompact. Pseudocompactness, however, is preserved in regular closed sets[2]

[2] A subset is *regular closed.* if it is the closure of an open set

[BCM], as is H-closedness [Ka]. Lemma 2.2 gives the analogous result for collapse.

Lemma 2.2. *Let Y be a regular closed non-compact subset of X. Then $aic(X) \leqslant aic(Y)$.*

Proof: Let $\mathcal{U}$ be a centered collection of open subsets of Y with $\cap\overline{\mathcal{U}}^Y = \emptyset$. Then $\mathcal{U}' = \{U \cap Y^0 : U \in \mathcal{U}\}$ is a centered collection of open sets in X with $\cap\overline{\mathcal{U}'}^X = \emptyset$. $\square$

With regard to products, it is easy to show that if $\{X_\lambda : \lambda \in \Lambda\}$ is a family of spaces,

$$aic(\Pi_{\lambda \in \Lambda} X_\lambda) \leqslant \min\{aic(X_\lambda) : \lambda \in \Lambda\}.$$

Although the product of pseudocompact spaces is not, in general, pseudocompact, the product of two first countable pseudocompact spaces is [BCM]. Moreover, the same holds true for countably compact spaces. Paul Gartside (Oxford) has shown that this also extends to collapse, namely that if X and Y are first countable spaces, then $aic(X \times Y) = \min\{aic(X), aic(Y)\}$. Partly prompted by this more stable behaviour, we examine $aic(X)$ for first countable spaces X in Section 3. Here, we present general results about collapse by using them as tools to calculate the collapse of concrete spaces. To begin with, we consider ordinal and ψ-like spaces.

Example 2.3. *Let $(\alpha, <)$ denote the limit ordinal α with the order topology. Then $aic(\alpha) = cf(\alpha)$.*

Proof: The obvious increasing cover by open intervals of size $cf(\alpha)$ has no member which is dense in α. Thus $aic(\alpha) \leqslant cf(\alpha)$. Conversely, any open cover with cardinality less than $cf(\alpha)$ contains an open set containing a final segment of α (since, in the cub filter, the intersection of less than $cf(\alpha)$ sets is non-empty). Initial closed segments are compact, so such an open cover has a finite subcover. Thus $aic(\alpha) \geqslant cf(\alpha)$. $\square$

The classic example of a pseudocompact space that is not countably compact is ψ [GJ 5.I], which is constructed using an infinite maximal almost disjoint family $\mathcal{A}$ on ω.

Proposition 2.4. *$aic(\psi) \geqslant \mathfrak{p}$.*

Proof: Let $\mathcal{U}$ be any centered collection of open subsets of ψ with $|\mathcal{U}| < \mathfrak{p}$. Without loss of generality, $\cup\mathcal{U} \subseteq \omega$. If $\cap\mathcal{U} \neq \emptyset$, we are done. So we may assume that $\cap\mathcal{U} = \emptyset$.

$\mathcal{U}$ has the strong finite intersection property: if there were $U_0, U_1, \ldots, U_n$ such that $\cap_{i=0}^{n} U_i = \{m_1, m_2, \ldots, m_l\}$, we could choose $U_{n+k} \in \mathcal{U}$ such that $m_k \notin U_{n+k}$. But then $\cap_{i=0}^{n+l} U_i = \emptyset$, contradicting $\mathcal{U}$ being centered. So $\mathcal{U}$ has the sfip and $|\mathcal{U}| < \mathfrak{p}$. Therefore $\mathcal{U}$ has an infinite pseudointersection, $P \subseteq \omega$. As $\mathcal{A}$ is maximal, P has a limit point $a \in \mathcal{A}$ and $a \in \overline{\mathcal{U}}$, as required. $\square$

Notice that in proving $aic(\psi) \geqslant \mathfrak{p}$, all we have used is that ψ has a countable dense set of points, every infinite subset of which has a limit point ('e-countably-compact' [Sc]). So the proof can be adapted to show that every separable, countably compact space has collapse $\geqslant \mathfrak{p}$ (an alternative proof would be to combine Definition 1.1' and Lemma 6.4 of [Va1]). We obtain an upper bound in Theorem 3.7, but present a coarser bound in Corollary 2.6.

Recall that the Lindelöf degree, $L(X)$, of a space X is defined as the smallest infinite cardinal κ such that every open cover of X has a subcover of cardinality $\leqslant \kappa$.

Lemma 2.5. *For any space $X, aic(X) \leqslant L(X)$.*

Proof: Let $\mathcal{U}$ be an open cover of X such that no finite subset has dense union and $|\mathcal{U}| = aic(X)$. There is a subcover $\mathcal{V}$ with $|\mathcal{V}| \leqslant L(X)$. But $\mathcal{V}$ covers X, has no finite subset with dense union and $\mathcal{U}$ was chosen with minimum cardinality for this to occur. Therefore $|\mathcal{V}| = |\mathcal{U}|$, so $aic(X) \leqslant L(X)$. $\square$

Corollary 2.6. *For any space $X, aic(X) \leqslant |X|$.*

It follows that $aic(\psi) \leqslant \mathfrak{c}$. As we saw in 2.3, the collapse of ordinal spaces is a regular cardinal (equal to their cofinality). Although this is not true in general (Example 2.13), the collapse of a countably compact space always has uncountable cofinality.

Proposition 2.7. *If X is countably compact, $cf(aic(X)) > \omega$.*

Proof: Let $\{U_\alpha : \alpha < \kappa\}$ be a centered collection of open sets and $\kappa = \sup\{\alpha_n : n \in \omega\}$, where each $\alpha_n < aic(X)$. We show that $\bigcap_{\alpha<\kappa}\overline{U_\alpha} \neq \emptyset$, hence $\kappa < aic(X)$. For each $n \in \omega$, pick $x_n \in \bigcap_{\alpha<\alpha_n}\overline{U_\alpha}$. If $\{x_n : n \in \omega\}$ is finite, then some x occurs infinitely often in this set; hence $x \in \bigcap_{n\in\omega}\bigcap_{\alpha<\alpha_n}\overline{U_\alpha} = \bigcap_{\alpha<\kappa}\overline{U_\alpha}$. If $\{x_n : n \in \omega\}$ is infinite, it has a limit point x. For any open V containing x, V contains infinitely many x_n's. It follows that $x \in \bigcap_{\alpha<\kappa}\overline{U_\alpha}$. □

Returning to Example 2.4, it is consistent with **ZFC** that $\mathfrak{p} > \omega_1$. However, if we generalize ψ's construction by taking mad families on uncountable sets, those spaces turn out to have collapse exactly ω_1 - which we demonstrate in Example 2.10.

Proposition 2.8. *Let X be a space with local cardinality κ. If $c(X) > \kappa$ then $aic(X) \leqslant \kappa^+$.*

Proof: Let $\mathcal{V} = \{V_\alpha : \alpha < \kappa^+\}$ be a collection of pairwise disjoint nonempty open sets. For $\alpha < \kappa^+$, define $U_\alpha = \bigcup_{\beta>\alpha}V_\beta$. Then $\mathcal{U} = \{U_\alpha : \alpha < \kappa^+\}$ is an open filter base on X. For any $x \in X$ there is an open W containing x with $|W| \leqslant \kappa$. Therefore there is some $\alpha < \kappa^+$ such that $W \cap V_\beta = \emptyset$ whenever $\beta > \alpha$. This implies that $x \notin \overline{U_\alpha}$ and hence $\bigcap\overline{\mathcal{U}} = \emptyset$. □

Corollary 2.9. *If X is compact then $c(X) \leqslant$ the local cardinality of X.*

Notice that (at least consistently) Proposition 2.8 cannot be improved by replacing cellularity by extent or spread. After all, it is consistent that $\mathfrak{p} = \mathfrak{c} = \omega_2$ [vD]. In such a model, $aic(\psi) = \omega_2$, but ψ is locally countable with spread and extent equal to ω_2.

Example 2.10. *For any uncountable cardinal κ, $aic(\psi_\kappa) = \omega_1$.*

Proof: ψ_κ is the generalization ψ, where $\psi_\kappa = \mathcal{A} \cup \kappa$ and $\mathcal{A}$ is a maximal almost disjoint family of ω-sequences of points from κ. Points of κ are isolated and a basic neighborhood containing $a \in \mathcal{A}$ is $\{a\} \cup (a - [a]^{<\omega})$. With the topology thus generated, ψ_κ is pseudocompact (because the family is maximal) and therefore $aic(\psi_\kappa) \geqslant \omega_1$. For the reverse inequality, note that ψ_κ is locally countable and has cellularity $\kappa > \omega$. By Proposition 2.8, $aic(\psi_\kappa) \leqslant \omega_1$. $\square$

Notice that, $L(\psi_\kappa) \geqslant \kappa$. So taking $\kappa > \omega_1$ shows that equality in 2.5 need not hold.

As mentioned above, the collapse of a space need not be a regular cardinal. Specifically, every cardinal value is achievable. To show this, we gather a collection of observations regarding character and π-character.

Lemma 2.11. (i) *If $X \subseteq Y$ and $\overline{X} - X \neq \emptyset$. Then*

$$aic(X) \leqslant min\{\chi(y, Y) : y \in \overline{X} - X\}.$$

(ii) *There is a proper T_2 extension $X' = X \cup \{\zeta\}$ of X such that $\chi(\zeta, X') = aic(X)$.*

(iii) *Suppose X is compact and x is any non-isolated point of X. Then $aic(X - \{x\}) \geqslant \pi\chi(x, X)$.*

Proof:

(i) Let $\mathcal{B}$ be a local base at $y \in \overline{X} - X$, with $|\mathcal{B}| = \chi(y, Y)$. Then $\mathcal{B}' = \{B \cap X : B \in \mathcal{B}\}$ is a centered collection of open subsets of X. Hence, $\cap\overline{\mathcal{B}'}^X = \emptyset$ (Y is Hausdorff) and $aic(X) \leqslant \chi(y, Y)$.

(ii) Let $\mathcal{U}$ be a centered collection of open sets in X satisfying $\cap\overline{\mathcal{U}} = \emptyset$ and $|\mathcal{U}| = aic(X)$. By defining $\mathcal{B} = \{\cap G : G \in [\mathcal{U}]^{<\omega}\}$, $\mathcal{B}$ also has these properties and is closed under finite intersections. If we let $X' = X \cup \{\zeta\}$, where $\zeta \notin X$, and $\{\{\zeta\} \cup B : B \in \mathcal{B}\}$ be a local base at ζ then X' is a proper Hausdorff extension of X with $\chi(\zeta, X') = |\mathcal{B}| = aic(X)$.

(iii) Let Y denote $X - \{x\}$. It is straightforward to verify that if $\mathcal{V}$ is a collection of open sets of a compact space Z and $\cap\overline{\mathcal{V}} = \{\zeta\}$ then $\{\cap F : F \in [\mathcal{V}]^{<\omega}\}$ is a local π-base at ζ.
As x is not isolated in X, Y is not H-closed, so $aic(Y)$ is well defined. If $\mathcal{U}$ is any open filter base in Y with $\cap\overline{\mathcal{U}}^Y = \emptyset$, then $\cap\overline{\mathcal{U}}^X = \{x\}$. By the remark above, it follows that $|\mathcal{U}| = |[\mathcal{U}]^{<\omega}| \geqslant \pi\chi(x, X)$. □

Corollary 2.12. *There is no Hausdorff extension of ψ with character smaller than $\mathfrak{p}$ and no first countable Hausdorff extension of ω_1.*

Proof: Lemma 2.11(i)

Example 2.13. *For every infinite cardinal κ, there is a space X with $aic(X) = \kappa$.*

Proof: Every point $p \in 2^\kappa$ has character κ. Moreover, each such p has π-character κ. Given a family of fewer than κ basic open sets $\mathcal{P}$, there is a coordinate α not restricted by any member of $\mathcal{P}$. Therefore, one of $\pi_\alpha^{-1}(\{0\})$ or $\pi_\alpha^{-1}(\{1\})$ is a neighborhood of p that contains no member of $\mathcal{P}$. Lemma 2.11(i,iii) implies that $2^\kappa - \{p\}$ has collapse κ. □

The fact that every cardinal value of collapse can be witnessed should be compared with the following:

Lemma [Bu 9.1] *Suppose X is a non-compact space and m is the cardinal number minimal with respect to the condition that there exists an open cover $\mathcal{U}$, $|\mathcal{U}| = m$, but $\mathcal{U}$ has no finite subcover. Then m is regular.*

As a final example, we apply Lemma 2.11 to the Stone-Čech compactification of the integers, $\beta\omega$. For any point $x \in \beta\omega - \omega$, $\beta\omega - \{x\}$ is countably compact but not compact [En]. Therefore $aic(\beta\omega - \{x\})$ is well defined and, by Lemma 2.11(i), $aic(\beta\omega - \{x\}) \leqslant \chi(x, \beta\omega)$. In fact, equality holds.

Example 2.14. *For any $x \in \beta\omega - \omega$, $aic(\beta\omega - \{x\}) = \chi(x, \beta\omega)$ and hence $\mathfrak{u} \leqslant aic(\beta\omega - \{x\}) \leqslant \mathfrak{c}$.*

Proof: For convenience, let $Y = \beta\omega - \{x\}$. Let $\mathcal{U}$ be a centered collection of open subsets of Y with $\cap\overline{\mathcal{U}}^Y = \emptyset$. We may assume that $\cup\mathcal{U} \subseteq \omega$, since ω is a dense open subset of Y. As $\beta\omega$ is compact, $\cap\overline{\mathcal{U}}^{\beta\omega} \neq \emptyset$ and hence $\cap\overline{\mathcal{U}}^{\beta\omega} = \{x\} \ldots (*)$.

Every point $z \in \beta\omega - \omega$ can be thought of as a free ultrafilter on ω (see [Wa]) and one can show that the collection of sets

$$\{a \cup \{y \in \beta\omega - \omega : a \in y\} : a \in z\}$$

forms a neighborhood base at z. $\mathcal{U}$ is a centered collection of subsets of ω with empty intersection, so generates a free ultrafilter on ω. To satisfy $(*)$, $\mathcal{U}$ must generate a unique free ultrafilter, namely x. Consequently, $\mathcal{U}$ generates a neighborhood base at x and therefore $|\mathcal{U}| \geqslant \chi(x, \beta\omega)$. This completes the proof that $aic(Y) = \chi(x, \beta\omega)$.

Notice that, by definition, $|\mathcal{U}| \geqslant \mathfrak{u}$. Moreover $\mathcal{U}$ is a collection of subsets of ω, so $|\mathcal{U}| \leqslant \mathfrak{c}$. □

To close this section, we mention how the use of collapse can generalize results on compactness that do not use its full strength - our motivating observation. A space is *κ-Baire* if the intersection of fewer than κ many open dense sets is dense. Recall that feebly compact spaces are ω_1-Baire [En] - i.e. the intersection of countably many open dense subsets is again dense. As detailed in [Ku], Martin's Axiom implies that all CCC compact T_2 spaces are κ-Baire for all $\kappa < \mathfrak{c}$.

Proposition 2.15 ($MA(\kappa)$) *If X is regular, CCC and $aic(X) \geqslant \kappa$, then X is λ-Baire for all $\lambda < \kappa$.*

Proof: Mimic the proof of Lemma 2.2.22 in [Ku]. □

3. Conditions imposing an upper bound on the collapse

In Example 2.3 and 2.13, we saw that there is no overall bound on collapse, but that, in 2.6, the collapse of a space is bounded by the cardinality of the underlying set. In [Ar], Arhangel'skiĭ proved that every first countable, compact Hausdorff space has cardinality at most $\mathfrak{c}$. There are first countable, pseudocompact and first countable, countably compact spaces with arbitrarily large cardinality: Example 2.10 and ordinal spaces with points of uncountable cofinality removed, respectively. We consider whether this extends to collapse. To be precise,

Question 3.1 *Given a cardinal κ, is there a regular, first countable space X with $aic(X) \geqslant \kappa$?*

So we restrict our attention to regular first countable spaces. We show that, with local compactness, the value of collapse is at most $\mathfrak{c}$ and, more curiously, that any first countable space that is not countably compact has collapse at most $\mathfrak{d}$. In doing so, we appeal to two facts about first countable spaces:

(1) if $\{C_\alpha : \alpha < \omega_1\}$ is a collection of closed sets satisfying $C_\alpha \subseteq C_\beta$ whenever $\alpha < \beta$, then $\cup_{\alpha<\omega_1} C_\alpha$ is closed;
(2) for any $A \subseteq X$, $|\overline{A}| \leqslant |A|^\omega$.

It is shown in [JNW] that, assuming $V = L$, there is a locally countable, locally compact (hence first countable) countably compact space with cardinality κ if and only if $\kappa = \omega$ or $\kappa = \kappa^\omega$. At least consistently, then, there is no bound on the cardinality of locally compact, first countable, countably compact spaces. However, as we see from Corollary 3.5, the collapse of such spaces is bounded.

Theorem 3.2. *Let κ be an infinite cardinal such that $\kappa^\omega = \kappa$. If X is a regular first countable space that has cardinality $> \kappa$ but local cardinality $\leqslant \kappa$, then X has a non-compact clopen subset Y with $|Y| \leqslant \kappa$ and aic(X)$\leqslant \omega_1$.*

Proof: Suppose we have inductively defined open U_β for all $\beta < \alpha < \omega_1$ such that $|U_\beta| \leqslant \kappa$ and $\overline{U_\gamma} \subsetneq U_\beta$ whenever $\gamma < \beta$. Then $U = \bigcup_{\beta<\alpha} U_\beta$ has cardinality at most $\kappa.\omega = \kappa$ and, by first countability, $|\overline{U}| \leqslant \kappa^\omega = \kappa$. As $|X| > \kappa$, there is some $x_\alpha \notin$

U and for each $x \in Y \cup \{x_\alpha\}$, there is an open V_x containing x with $|V_x| \leqslant \kappa$. Defining U_α to be the union of these V_x's, we have $\overline{U} \subsetneq U_\alpha$ and $|U_\alpha| \leqslant \kappa$. This completes the inductive stage at step α.

We now consider $Y = \overline{\bigcup_{\beta<\omega_1} U_\beta}$, a regular closed subset of X. Observe that $|Y| \leqslant \kappa^\omega = \kappa$. Using first countability again and the fact that the U_β's are nested, $Y = \bigcup_{\beta<\omega_1} \overline{U_\beta} = \bigcup_{\beta<\omega_1} U_\beta$. Hence, Y is open. Notice that the collection $\{U_\beta : \beta < \omega_1\}$ is an open cover of Y, no finite subsets of which can have dense union: if $A \subseteq \omega_1$ is finite and $\alpha = \max A$, then $\overline{\bigcup_{\beta\in A} U_\beta} = \overline{U_\alpha} \subsetneq U_{\alpha+1}$. So, by Lemma 2.1, we have $aic(X) \leqslant \omega_1$. □

Incidentally, Theorem 3.2 shows that if X is connected, first countable and has local cardinality $\leqslant \kappa$ then $|X| \leqslant \kappa^\omega$. More generally, our method of proof shows that if X is first countable and every point has a neighborhood with cardinality at most κ, then $aic(X) \leqslant \kappa^\omega$. The crucial step in the proof is showing that the space has as strongly increasing chain (equivalently, a strongly decreasing chain):

Definition 3.3. *A sequence of open sets $\{V_\alpha : \alpha < \kappa\}$ is called a* strongly increasing chain *if $\overline{V_\alpha} \subsetneq V_\beta$ whenever $\alpha < \beta < \kappa$.*

This notion appears in [Ju] as part of the definition of the cardinal function *depth.* Arguing as in 3.2, if X is a first countable space with an uncountable strongly increasing chain, then $aic(X) \leqslant \omega_1$.

Corollary 3.4. *Every locally compact, first countable, non-compact space X has a clopen non-compact subspace Y with $|Y| \leqslant \mathfrak{c}$.*

Proof: If $|X| \leqslant \mathfrak{c}$, the result is trivial, so we may assume $|X| > \mathfrak{c}$. By [Ar], every first countable compact space has cardinality at most $\mathfrak{c}$. Theorem 3.2 now applies.

Corollary 3.5. *For a first countable, locally compact non-compact space X, $aic(X) \leqslant \mathfrak{c}$.*

This implies that all the spaces of [JNW] have collapse ω_1

Question 3.6. *Is there a locally compact first countable space X in ZFC with $aic(X) = \mathfrak{c}$?*

Our final result not only gives an upper bound for the collapse of all first countable spaces that are not countably compact, but also implies there is no **ZFC** example of such a space with collapse exactly $\mathfrak{c}$. So any space answering 3.6 must be countably compact. Of course, there are consistent examples: ψ and ω_1, assuming **CH**, and those in [JNW].

Theorem 3.7. *If X is a regular, first countable space that is not countably compact, then $aic(X) \leqslant \mathfrak{d}$.*

Proof: Let $A = \{x_n : n \in \omega\}$ be an infinite closed discrete set. For each n, let $\{B(n,m) : m \in \omega\}$ be a neighborhood base at x_n with the additional properties that $\overline{B(n,m+1)} \subseteq B(n,m)$ for all n,m and $\overline{B(n,0)} \cap \overline{B(n',0)} = \emptyset$ whenever $n \neq n'$. Let $D \subseteq {}^{\omega}\omega$ be a dominating family with cardinality $\mathfrak{d}$. For $f \in D$, define $U_f = \bigcup\{B(n, f(n)) : n \in \omega\}$. The collection

$$\mathcal{U} = \{U_f - \bigcup_{n \in F} \overline{B(n,0)} : f \in D,\ F \in [\omega]^{<\omega}\}$$

is a centered collection of open sets and $|\mathcal{U}| \leqslant |D| = \mathfrak{d}$. We show that $\bigcap \overline{\mathcal{U}} = \emptyset$.

Suppose, for a contradiction, that there were some $\zeta \in \bigcap \overline{\mathcal{U}}$. Notice that, for any finite $F \subseteq \omega, \bigcup_{n \in F} B(n,0) \cap \bigcap \overline{\mathcal{U}} = \emptyset$. Therefore $A \cap \bigcap \overline{\mathcal{U}} = \emptyset$ and, for any open V containing ζ, V must meet infinitely many $B(n,0)$'s. Because $\zeta \notin A$, there is an open V such that $\zeta \in V$ and $\overline{V} \cap A = \emptyset$. It follows that, for every $n \in \omega$, there is an m_n such that $B(n, m_n) \cap V = \emptyset$. Define $f \in {}^{\omega}\omega$ by $f(n) = m_n$. As D is dominating, there is some $g \in D$ such that $f <^* g$. For this g, there is a finite $G \subseteq \omega$ such that $n \in G$ if and only if $B(n, g(n)) \cap V \neq \emptyset$. Then $W = U_g - \bigcup_{n \in G} \overline{B(n,0)}$ is in $\mathcal{U}$ and $V \cap W = \emptyset$. Thus $\zeta \notin \overline{W}$, contradicting $\zeta \in \bigcap \overline{\mathcal{U}}$. $\square$

Returning to Example 2.4, we now know that $\mathfrak{p} \leqslant aic(\psi) \leqslant \min\{|\psi|, \mathfrak{d}\}$. We can consistently arrange a mad family so that the corresponding ψ has collapse $< \mathfrak{d}$: it is consistent with **ZFC** that $\mathfrak{a} < \mathfrak{d}$ [vD]. Constructing a mad family so that $aic(\psi) > \mathfrak{p}$ seems considerably harder.

Question 3.8. *Is it consistent with ZFC that there is a mad family on ω such that $aic(\psi) > \mathfrak{p}$?*

In light of Theorem 3.7, the search for first countable spaces with large collapse (i.e. $> \mathfrak{c}$) forces us to look for countably compact spaces with strong completeness properties. To this end, it may be possible to use an Ostaszewski-type argument [Os], modified in such a way that the induction continues beyond stage ω_1.

As a concluding remark, we note that any first countable space with large collapse cannot be either *CCC* or perfect:

(a) [Ho] If X is *CCC* and first countable then $|X| \leqslant \mathfrak{c}$, and hence $aic(X) \leqslant \mathfrak{c}$ by 2.6. So when looking for spaces with large collapse, we are not hindered by the issue of [Ny]. Note also that Proposition 2.8 gives some local cardinality information.

(b) In [PW], it is shown that perfect, feebly compact spaces are *CCC*.

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