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Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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COMPLETE NORMALITY AND COUNTABLE COMPACTNESS

PETER J. NYIKOS, BORIS SHAPIROVSKIĬ, ZOLTÁN SZENTMIKLÓSSY AND BOBAN VELIČKOVIĆ

One of the classical separation axioms of topology is complete normality. A topological space X is *completely normal* if for every pair of subsets A and B of X which are separated (i.e. $\bar{A} \cap B = \emptyset = A \cap \bar{B}$) there are disjoint open sets containing A and B respectively. A standard exercise is to show that this is equivalent to hereditary normality. We will refer to completely normal Hausdorff spaces as T_5 spaces.

Until now it has been somewhat of a mystery how well-behaved countably compact T_5 spaces can be. Under Gödel's axiom of constructibility ($V = L$) they can be quite pathological: in [1] there is a $V = L$ construction of a compact T_5 space X of cardinality 2^c in which every subspace is separable, yet the space has no infinite closed 0-dimensional subspaces; in particular, X has no nontrivial convergent sequences.

Our main results show that these spaces are much better behaved under the Proper Forcing Axiom (PFA), introduced in [2]. It implies that every countably compact T_5 space is sequentially compact. [The consistency of this is new even for the compact case.] Much more strongly, it implies that every countable subset of a countably compact T_5 space has compact, Fréchet-Urysohn closure. [A space is called *Fréchet-Urysohn* if whenever a point x is in the closure of a subset A, then there is a sequence from A converging to x.] Hence, in particular, a separable subspace can have cardinality at most c.

A striking corollary of this and the Tychonoff theorem is that, under PFA, the product of any number of countably compact T_5 spaces is countably compact, although the T_5 property may be lost. This establishes the set-theoretic independence of the productivity of countable compactness when the factors are T_5; earlier, J. Vaughan [3] showed the consistency of the existence of a family of sequentially compact perfectly normal (T_6) spaces whose product is not countably compact.

We have also shown that the PFA implies every compact T_5 space is pseudo-radial of chain-net order ≤ 2. [A space is called *pseudo-radial* if closures can be obtained by iterating the operation of taking limits of well-ordered nets.] In fact, the closure of any subset A can be taken by taking the set $\hat{A}$ of all limits of convergent sequences in A, and adding to $\hat{A}$ the set of all points which are the limit of a well-ordered net in $\hat{A}$.

The proofs of these results use the following "reduction theorem". For the definitions of "countably tight" and "free sequence" see [4], [5], or [6]. "Separable" is synonymous with having a countable dense subset.

Theorem 1 [4] *The following are equivalent.*

- *(a) Every separable, T_5, compact space is countably tight.*
- *(b) Every free sequence in a separable, T_5, countably compact space is countable.*
- *(c) A separable, T_5, countably compact space cannot contain ω_1.*
- *(d) No version of γN is T_5.*

Here γN is the generic symbol for a locally compact Hausdorff space X with a countable dense set of isolated points, identified with the set N of positive integers, such that $X \backslash N$ is homeomorphic to ω_1. We will also identify $X \backslash N$ with ω_1 using a definition of N that makes it disjoint from ω_1. Baumgartner and the first author have independently observed that using only the usual axioms of set theory one can construct versions of γN in which the union of N with the successor ordinals is not normal. On the other hand, it is also consistent that there are other versions which are T_5 ([4, 12]). One of our main results is that this cannot happen under the following version of the Open Coloring Axiom (OCA):

If X is a separable metric space and

$$[X]^2 = K_0 \cup K_1$$

is a partition with K_0 open in the product topology then either there exists an uncountable 0-homogeneous subset of X, of else X can be covered by countably many 1-homogeneous sets.

As usual, $[A]^2$ stands for the collection of two-element subsets of A. A subset H of X is called *i-homogeneous* if $[H]^2 \subseteq K_i$. In saying K_0 is open in the product topology, what we really mean is that $\{\langle x, y\rangle : \{x, y\} \in K_0\}$ is open.

The OCA was introduced and proved relatively consistent with ZFC + MA + $2^{\aleph_0} = \aleph_2$ by Todorčević ([15]), who extended and refined the previous work of Abraham, Rubin, and Shelah ([0]). (For many other applications of OCA see [13, 14].) Answering a question of the first author, the fourth author showed:

Theorem 2. [12] *under* OCA *no version of γN can be completely normal.*

Outline of Proof: For each $\alpha < \omega_1$ let $\alpha_\alpha \subseteq \omega$ be such that $\alpha_\alpha \cup [0, \alpha]$ is a compact neighborhood of $[0, \alpha]$. It is easily seen that $\alpha_\alpha \subset_* a_\beta$ and $a_\beta \backslash a_\alpha \subset_* U$, for every neighborhood U of $(\alpha, \beta]$ whenever $\alpha < \beta$. As usual, identify $\{\alpha_\alpha : \alpha < \omega_1\}$ with a subset of the Cantor set,

Let S be the set of all $\langle \alpha_\xi, \alpha_\eta, \alpha_\mu\rangle$ such that $\xi < \eta < \mu$ and define the partition

$$[S]^2 = K_0 \cup K_1$$

by $\{\langle a, b, c, \rangle\ \langle \bar{a}, \bar{b}, \bar{c}\rangle\} \in K_0$ iff

$$a \neq \bar{a} \text{ and } [(a\backslash b) \cap (\bar{c}\backslash \bar{b}) \neq \emptyset \text{ or } (c\backslash b) \cap (\bar{b}\backslash \bar{a}) \neq \emptyset].$$

Then K_0 is open in the product topology.

It is not hard to show in ZFC that S can not be the union of a sequence $\{S_n : n < \omega\}$ of 1-homogeneous sets. So, by OCA, there is an uncountable 0-homogeneous subset H of S. By cutting H down if necessary we may assume $\mu < \bar{\xi}$ whenever $\langle \alpha_\xi, \alpha_\eta, \alpha_\mu \rangle$ and $\langle \alpha_{\bar{\xi}}, \alpha_{\bar{\eta}}, \alpha_{\bar{\mu}} \rangle$ are two distinct members of H such that $\xi < \bar{\xi}$. Then

$$A = \bigcup \{(\xi, \eta] : \langle \alpha_\xi, \alpha_\eta, \alpha_\mu \rangle \in H\}$$

and

$$B = \bigcup \{(\eta, \mu] : \langle \alpha_\xi, \alpha_\eta, \alpha_\mu \rangle \in H\}$$

are separated in γN. If there were an open subset U of γN such that $A \subset U$ and $\mathrm{cl} U \cap B = \emptyset$, we could let $c = U \cap N$ and have $a_\eta \backslash a_\xi$ almost contained in c and $a_\mu \backslash a_\eta$ almost disjoint from c whenever $\langle \alpha_\xi, \alpha_\eta, \alpha_\mu \rangle \in H$. Now, for every ξ there are at most one η and μ such that $\langle \alpha_\xi, \alpha_\eta, \alpha_\mu \rangle \in H$. If this happens choose $n(\xi) \in N$ such that

$$[(a_\eta \backslash a_\xi) \backslash c] \cup [(a_\mu \backslash a_\eta) \cap c] \subseteq [0, n(\xi)].$$

Then there is an uncountable subset I of $H, n \in N$, and $a \subseteq [0, n]$ such that whenever $\langle \alpha_\xi, \alpha_\eta, \alpha_\mu \rangle \in I$ then $n(\xi) = n$ and $a_\eta \cap [0, n] = a$. But then any pair of distinct elements of I is in K_1, a contradiction.

In the proof of Theorem 2, if is possible to strengthen the conditions for membership of $\{\langle \alpha_\xi, \alpha_\eta, \alpha_\mu \rangle \langle \alpha_{\bar{\xi}}, \alpha_{\bar{\eta}}, \alpha_{\bar{\mu}} \rangle\}$ in K_0 by requiring that $a_\eta \backslash a_\xi$ meet $a_{\bar{\mu}} \backslash a_{\bar{\eta}}$ whenever $\xi < \bar{\xi}$, and still show that PFA gives an uncountable 0-homogeneous set. Of course, the resulting K_0 will not be open, but we can use a different consequence of PFA formulated by the third author and of independent interest. If X is a set and $\boldsymbol{H}$ is a proper σ-ideal on X, call a graph $\mathcal{G}$ on X $\boldsymbol{H}$*-sparse* if for all $Z \notin \boldsymbol{H}$ there exists $H \in \boldsymbol{H}$ and a countable $Q \subset Z$ such that for all $b \in [X \backslash H]^{<\omega}$ there exists $z \in Q$ such that $\{z, y\} \notin \mathcal{G}$ for all $y \in b$.

The Sparse Graph Axiom. *Let X be a set and let $\boldsymbol{H}$ be a proper σ-ideal on X. Let $\{H_\alpha : \alpha < \omega_1\} \subset \boldsymbol{H}$. Every $\boldsymbol{H}$-sparse graph on X has an independent subset $I \not\subset H_\alpha$ for all $\alpha < \omega_1$.*

The proof that PFA implies the Sparse Graph Axiom uses a standard technique of interpolating countable elementary submodels of large enough fragments of set theory, made explicit in the works of Todorčević [14], [15].

Besides the following theorem, the Sparse Graph Axiom also implies the axiom $\omega_1 \to (\omega_1, (\omega_1;\ \text{fin}\ \omega_1))^2$ of Todorčević [16]. This axiom, which implies that there are no S-spaces [16], is easily seen equivalent to the Sparse Graph Axiom applied to $X = \omega_1$, $\boldsymbol{H} = [\omega_1]^{\leq\omega}$, and $H_\alpha = \alpha$. It can be phrased as follows: if $\mathcal{G}$ is a graph on ω_1, then either $\mathcal{G}$ has an uncountable independent subset, or else there is a pair $S, \mathcal{B}$ such that S is an uncountable subset of ω_1 and $\mathcal{B}$ is an uncountable disjoint family of finite subsets of ω_1 such that whenever $s \in S$ and $b \in \mathcal{B}$ satisfy $s < b$, there is an edge in $\mathcal{G}$ from s to b; more formally, $[\{s\}, b] \cap \mathcal{G} \neq \emptyset$.

Theorem 3. *The Sparse Graph Axiom implies no γN can be T_5.*

The proof of Theorem 3 applies the Sparse Graph Axiom to $X = [\omega_1]^3$, $\boldsymbol{H} = \{H \subset X :$ there is a club $F \subset \omega_1$ such that $[F]^3 \cap H = \emptyset\}$, and $H_\alpha = [\omega_1]^3 \setminus [\omega_1 \backslash \alpha]^3$. The graph involved is the set of all pairs from X not in K_0, so that "independent" is synonymous with "0-homogeneous."

Given either Theorem 2 or Theorem 3, we can combine Theorem 1(a) and (b) with the results from [5] to show that the closure of every countable set is compact [5, Theorem 1 and following sentence] and sequential [5, Theorem 2], [6]. This takes us to the threshold of:

Theorem 4. [PFA] *In a countably compact T_5 space, every countable subset has compact, Fréchet-Urysohn closure.*

The final three steps are taken in ZFC. A compact sequential space has the property that every countably compact subset is compact [7]. Every pseudocompact subset of a T_5 space is countably compact [8, 3.10.21]. And if a countably compact space has the property that every pseudocompact subset is compact, then the space is Fréchet-Urysohn [9].

Corollary 1. [PFA] *Every countably compact T_5 space is sequentially compact.*

Corollary 2. [PFA] *If X is a product of countably compact T_5 spaces, then X is countably compact.*

Indeed, one need only take a countably infinite set S in the product, project it to each factor space, take the respective compact closures, and find an accumulation point of S in the compact product of these subspaces.

Corollary 2 show that an affirmative answer to the Scarborough-Stone problem is consistent in the T_5 case. Scarborough and Stone showed [10] that the product of $\aleph_1$ sequentially compact spaces is countably compact and asked whether this continued to hold for any number of factors. The first author has solved this problem [11] by producing a family of T_5 sequentially compact spaces whose product is not countably compact, but it is still important to know what happens if higher separation axioms are imposed on the factor spaces. Corollary 2, coupled with the results of [3], give an independence result for the Scarborough-Stone problem for T_5 and T_6 spaces. For the case of T_3, Tychonoff, and T_4 spaces we only know that the negative answer is consistent.

Another sweeping corollary of Theorem 4 comes by way of the following curious concept.

Definition 1. A dense subset S of a space X is *super-dense* if S meets every closed infinite subset A of X.

Lemma 1. *Every dense, countably compact subspace of a T_5 space is super-dense.*

Outline of Proof: Let A be closed infinite, and let D be an infinite discrete subspace of A. Let $B = \mathrm{cl}(D) \setminus D$. By T_5, we can expand D to a discrete-in-$(X \setminus D)$-family of open sets, and if we pick points of S one apiece from these open sets, the only accumulation points will be in B, and at least one of these is in S.

Definition 2. A space X is *c-radial* if, whenever a point x of X is in the closure of a countably compact subset S, then there is a well-ordered net from S converging to x.

A standard transfinite induction, using regularity of X and super-density of S, gives:

Theorem 5. *Every compact T_5 space is c-radial.*

Corollary 3. [PFA] *Every compact T_5 space is pseudo-radial, of chain-net order ≤ 2.*

Proof: Let $A \subset X$ and let $\hat{A}$ be the set of all limits of sequences from A. Now $\hat{A}$ is countably compact. Indeed, if D is any countably infinite subset of $\hat{A}$, then there is a countable subset B of A such that D is a subset of $\mathrm{cl}(B)$; then, by Theorem 4, $\mathrm{cl}(B)$ is Fréchet-Urysohn, hence a subset of $\hat{A}$; of course, $\mathrm{cl}(B)$ is (countably) compact, so any accumulation point of D is actually in $\hat{A}$. Now apply Theorem 5.

Finally, here is a curious corollary which does not mention countable compactness.

Corollary 4. [PFA] *Every locally compact, T_5, separable, first countable space of cardinality $\aleph_1$ is a normal Moore space.*

In [4], it is shown that the statement of Corollary 4 is equivalent to any of the (equivalent) statements in Theorem 1 if one assumes MA + $\neg$CH.

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University of South Carolina
Columbia, SC 29208

University of California, Berkeley
Berkeley, CA 94720