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A METRIC CHARACTERIZATION OF A SUBSPACE OF THE REAL LINE

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Dedicated to Professor Ryôsuke Nakagawa on his 60th

ABSTRACT. We prove: A separable metrizable space is homeomorphic to a subspace of the real line if and only if it admits a metric which induces the original topology and satisfies (a) the cardinality of any subset consisting of points which are equidistant from two distinct points is at most 1; and (b) the cardinality of any subset consisting of points which are equidistant from a point is at most 2. A separable metric space satisfying (a) or (b) alone need not be homeomorphic to a subspace of the real line.

1. INTRODUCTION

Every subspace of the real line $\mathbb{R}$ with the Euclidean metric d has the following two properties: (a) There exist no four distinct points x, y, p_1 and p_2 such that $d(x,p_i) = d(y,p_i)$ for each $i = 1, 2$; (b) There exist no four distinct points x, p_1, p_2 and p_3 such that $d(x,p_1) = d(x,p_2) = d(x,p_3)$. The first purpose of this note is to prove conversely the following theorem:

Theorem 1. *A separable metric space (X, d) having the properties* (a) *and* (b) *is homeomorphic to a subspace of* $\mathbb{R}$.

The second purpose is to show that a separable metric space (X, d) having (a) alone need not be homeomorphic to a subspace of $\mathbb{R}$. Since the circle with a metric inherited from the Euclidean metric d on the plane has (b), a metric space (X, d)

having (b) alone need not be homeomorphic to a subspace of $\mathbb{R}$.

In the remaining part of this section, we discuss the relationship between the property (a) and other metric properties. Recall from [1] that a metric space (X, d) has the *unique midset property* (abbreviated UMP) if for every pair of distinct points x and y, there exists one and only one point p such that $d(x,p) = d(y,p)$. A metric space (X, d) having UMP has (a) and the converse holds if (X, d) is connected (cf. [7], Lemma 2.1). Berard [1] showed that a connected metric space consisting of more than one point and having UMP is homeomorphic to an interval in $\mathbb{R}$. Nadler [7] showed that a locally compact and separable metric space having UMP is homeomorphic to a subspace of $\mathbb{R}$. However, UMP is too restrictive to describe a subspace of $\mathbb{R}$, because not all subspaces X of $\mathbb{R}$ have a compatible metric d for which (X, d) has UMP (see Remarks 2). The following theorem can be proved by the same way as the proof of Nadler's theorem stated above.

Theorem 2. *A locally compact and separable metric space* (X, d) *having the property* (a) *is homeomorphic to a subspace of* $\mathbb{R}$.

An alternative proof of Theorem 2 will be given later. Two subsets A and B of a metric space (X, d) are said to be *congruent* if there exists a bijection $f : A \to B$ such that $d(x, y) = d(f(x), f(y))$ whenever x and y in A. We say that a metric space has the *unique triangle property* (abbreviated UTP) if no distinct subsets of cardinality 3 are congruent. Janos [4] proved that the dimension of a locally compact and separable metric space having UTP is at most 1. A metric space (X, d) having UTP has the property (a), because if $d(x, p_i) = d(y, p_i)$ for each $i = 1, 2$, then the sets $\{x, p_1, p_2\}$ and $\{y, p_1, p_2\}$ are congruent. Hence, Theorem 2 sharpens Janos's theorem. For further background, the reader is referred to [3] and [5].

For a point p of a metric space (X, d) and $\varepsilon > 0$, we write $B(p, \varepsilon) = \{x \in X : d(p, x) \leq \varepsilon\}$ and $S(p, \varepsilon) = \{x \in X : d(p, x) = \varepsilon\}$. Moreover, we often write simply X to denote a metric space (X, d) and abbreviate 'closed-and-open' to 'clopen'. The letter $\mathbb{N}$ denotes the set of positive integers. Other terms and notation follow [2].

2. Proof of the theorems

To prove Theorems 1 and 2, we use the following theorem due to Purisch [8] and some lemmas. A space is called *suborderable* if it is homeomorphic to a subspace of a linearly ordered space.

Theorem 3. (Purisch) *A metrizable space X is suborderable if and only if X satisfies the following four conditions:*

(i) *Each component of X is orderable.*

(ii) *The set of cut points of each component is open.*

(iii) *Each component of X has a clopen neighborhood base (i.e., every open set including a component C includes a clopen set including C).*

(iv) $\operatorname{Ind} U = 0$ *for every hereditarily disconnected, closed subspace U of X.*

In [8, Theorem 3.3], the set U in the condition (iv) is not assumed to be closed; however, we can add 'closed' by the remark after the proof of [8, Theorem 3.3]. Nadler [7] essentially proved that a connected subspace, consisting of more than one point, of a metric space having the property (a) has UMP, and hence it is homeomorphic to an interval by Berard's theorem stated in the introduction. Hence, the following lemma is a direct consequence of [7, Lemma 2.3]; the statement is included for reader's convenience. Recall that a point p of a connected set C is a *cut point* of C if $C - \{p\}$ is disconnected.

Lemma 1. (Nadler) *Let (X, d) be a metric space having the property* (a) *and C a connected closed subset of X. Then every point in the boundary of C in X is not a cut point of C.*

Recall from [2] that a *quasi-component* of a point p of a space X is the intersection of all clopen sets of X containing p.

Lemma 2. *Let* (X, d) *be a metric space having the properties* (a) *and* (b). *Then every quasi-component of a point of* X *is connected.*

Proof: Let Q be a quasi-component of a point of X. It suffices to show that every pair of distinct points in Q is included in a connected subset of Q. To show this, let q_1 and q_2 be distinct points in Q. Put $X_1 = \{x \in X : d(x, q_1) \leq d(x, q_2)\}, X_2 = \{x \in X : d(x, q_1) \geq d(x, q_2)\}$ and $B = X_1 \cap X_2$. Then $|B| \leq 1$ by the property (a). If $B = \emptyset$, then X_1 is clopen in X and contains q_1 but not q_2. This contradicts the fact that Q is a quasi-component. Thus, we put $B = \{b\}$. Suppose that b is not in Q and G is a clopen set of X such that $Q \subset G \subset X - \{b\}$. Then $G \cap X_1$ is clopen in X and contains q_1 but not q_2, which is a contradiction. Hence, $b \in Q$. Put $\gamma = d(b, q_1) = d(b, q_2)$ and $Q_i = Q \cap B(b, \gamma) \cap X_i$ for $i = 1, 2$. It is clear that $Q_1 \cap Q_2 = \{b\}$ and $q_i \in Q_i$ for $i = 1, 2$. It remains to show that each Q_i is connected. First, we show that $|Q_i \cap S(b, \varepsilon)| = 1$ for each positive number $\varepsilon \leq \gamma$ and each $i = 1, 2$. If $\varepsilon = \gamma$, then $Q_i \cap S(b, \varepsilon) = \{q_i\}$, because $|S(b, \varepsilon)| \leq 2$ by the property (b). If $\varepsilon < \gamma$ and $Q_i \cap S(b, \varepsilon) = \emptyset$, choose a clopen set H in X such that $Q \subset H$ and $H \cap (S(b, \varepsilon) - Q) = \emptyset$. This can be done because $|S(b, \varepsilon)| \leq 2$. Then $(H - B(b, \varepsilon)) \cap X_i$ is clopen in X and contains q_i but not q_{3-i}, which is a contradiction. Hence, $Q_i \cap S(b, \varepsilon) \neq \emptyset$ for each $i = 1, 2$. Since $|S(b, \varepsilon)| \leq 2$, this implies that $|Q_i \cap S(b, \varepsilon)| = 1$ for each $i = 1, 2$. Hence, by the property (b), $Q_1 \cup Q_2 = B(b, \gamma)$. To complete the proof, suppose that Q_i is disconnected. Then there is a proper clopen subset V in Q_i with $b \in V$. Pick a point p of $Q_i - V$ and let $\delta = d(b, p)$. Since $Q_1 \cup Q_2 = B(b, \gamma)$, $(V \cap B(b, \delta)) \cup X_{3-i}$ is clopen in X and contains b but not p. This contradiction completes the proof. $\square$

Proof of Theorem 1: By [6, Corollary 5.6], a separable suborderable metric space X can be embedded in a separable linearly ordered metric space X^*, and X^* can be embedded in $\mathbb{R}$ by [2, 6.3.2(c) p.373]. Hence, it suffices to show that a separable metric space X having the properties (a) and (b) is suborderable. We show that X satisfies the conditions (i)-(iv) in Theorem 3. To prove (i)-(iii), let C be a component of X. Since C is a connected metric space having the property (a), C has UMP. Thus, C is a singleton or homeomorphic to an interval by Berard's theorem stated in the introduction. Hence, (i) is satisfied. The number of non-cut points of C is at most two, and by Lemma 1, each cut point of C is in the interior of C. Hence, (ii) is satisfied. Moreover, it follows from Lemma 2 that C is an intersection of clopen sets in X. In case $|C| = 1$, say $C = \{x\}$, it is easily checked that C has a clopen neighborhood base, because $|S(x, \varepsilon)| \leq 2$ for each $\varepsilon > 0$ by the property (b). In case C has no boundary point, C itself is clopen in X. Thus, it remains to settle the case where C is homeomorphic to an interval and has a boundary point. By Lemma 1, the boundary of C consists of at most two points. We put $Y = X - \operatorname{Int}_X C$. Then Y has the properties (a) and (b) and $\{y\}$ is a component in Y for each boundary point y of C. Hence, similarly to the case where $|C| = 1$, we can find a clopen neighborhood base $\mathcal{V}(y)$ of y in Y. If the boundary of C consists of only one point y, put $\mathcal{V} = \{C \cup V : V \in \mathcal{V}(y)\}$. If the boundary of C has two points y_1 and y_2, put $\mathcal{V} = \{C \cup V_1 \cup V_2 : V_i \in \mathcal{V}(y_i), i = 1, 2\}$. In each case, $\mathcal{V}$ is a clopen neighborhood base of C in X, thus proving (iii). Finally, to prove (iv), let U be a hereditarily disconnected, closed subspace of X. Since U has the properties (a) and (b), by the same way as above, we can find a clopen neighborhood base of x in U for each $x \in U$. Hence, $\operatorname{ind} U = 0$. Since U is a separable metric space, it follows from [2, Theorem 7.1.11] that $\operatorname{Ind} U = 0$. This completes the proof. $\square$

As stated in the preceding proof, a separable suborderable metric space is homeomorphic to a subspace of $\mathbb{R}$. Hence, Theorem 2 follows from the next theorem.

Theorem 4. *A locally compact, metric space* (X,d) *having the property* (a) *is suborderable.*

Proof: We show that X satisfies the conditions (i)-(iv) in Theorem 3. Similarly to the proof of Theorem 1, (i) and (ii) are proved. Since X is locally compact, a component C of X with $|C| = 1$ has a clopen neighborhood base by the proof of [2, Theorem 6.2.9]. Hence, by the same argument as in the proof of Theorem 1, (iii) can be proved. Finally, (iv) follows from [2, Theorem 7.1.12]. The proof is complete. □

Remarks 1. The example in the next section shows that local compactness cannot be removed from Theorems 2 and 4. By [7, Example 3.4], separability cannot be removed from Theorems 1 and 2. A space is called *peripherally compact* if each point has a neighborhood base consisting of open sets with compact boundary. All locally compact spaces and all metric spaces having the property (b) are peripherally compact. It is open whether a peripherally compact and separable metric space (X,d) having the property (a) is homeomorphic to a subspace of $\mathbb{R}$.

3. An example and remarks

The following example shows that a separable metric space having the property (a) alone need not be homeomorphic to a subspace of $\mathbb{R}$.

Example. There exists a separable metric space (X,d) having the property (a) which is not suborderable.

Proof: Let ∞ be an upper bound of $\mathbb{N}$ and put $\mathbb{N}^{\#} = \mathbb{N} \cup \{\infty\}$. Define $2^{-\infty} = 0$. Let X be the subspace of the Euclidean plane defined by $X = \bigcup\{I_n : n \in \mathbb{N}^{\#}\}$, where $I_n = \{(x, 2^{-n}) : 2^{-n} \leq x \leq 1\}$ for each $n \in \mathbb{N}$ and $I_\infty = \{(0,0)\}$. Since the component I_∞ has no clopen neighborhood base, it follows from Theorem 3 that X is not suborderable. We define a compatible metric

d on X for which (X,d) has the property (a). To do this, put $b_n = 1 + 2^{-n}$ for each $n \in \mathbb{N}$ and define a sequence $\{a_{n,k} : k \in \mathbb{N}^{\#}\}$ for each $n \in \mathbb{N}^{\#}$ by the following rules: In case $n = \infty$, $a_{\infty,k} = 0$ for each $k \in \mathbb{N}^{\#}$. In case $n \neq \infty$, $a_{n,\infty} = 1 - 2^{-2(n+1)}$ and

$$a_{n,k} = \begin{cases} 1 - 2^{-2(n+k+1)} & \text{for } k = 1, 2, ..., n-1 \\ a_{n,\infty} + 2^{-2(k+1)} & \text{for } k = n, n+1, n+2, \dots . \end{cases}$$

Then

$$1 - 2^{-2(n+1)} = a_{n,\infty} < \cdots < a_{n,n+2} < a_{n,n+1} < 1 - 2^{-2(n+2)},$$

and

$$1 - 2^{-2(n+2)} = a_{n,1} < a_{n,2} < \cdots < a_{n,n} = 1.$$

For every pair of points $p = (x, 2^{-m})$ and $q = (y, 2^{-n})$, with $m \leq n$, in X, define

$$d(p,q) = d(q,p) = \begin{cases} |x-y|, & \text{if } m = n \\ a_{m,n}x + b_m(2^{-m} - 2^{-n}) + a_{n,m}y, & \text{if } m < n. \end{cases}$$

Note that n may be ∞ and $2^{-\infty} = 0$ by our convention. We must show the triangle inequality $d(p,q) \leq d(p,r) + d(r,q)$ for $p = (x, 2^{-m})$, $q = (y, 2^{-n})$ and $r = (z,\ 2^{-k})$. Now, we prove it only in the cases where $m < k < n$ and $m < n < k$; the proof for other cases is left to the reader.

Case 1. $m < k < n$. Since $z \geq 2^{-k}$,

$$(b_m - b_k)(2^{-k} - 2^{-n}) \leq 2^{-k} = 2^{-1} \cdot 2^{-k} + 2^{-1} \cdot 2^{-k} \leq a_{k,m}z + a_{k,n}z,$$

which implies that

$$b_m(2^{-k} - 2^{-n}) \leq a_{k,m}z + a_{k,n}z + b_k(2^{-k} - 2^{-n}).$$

Since $a_{n,m} < a_{n,k}$ and $a_{m,n} < a_{m,k}$, it follows that

$$\begin{aligned} d(p,q) &= a_{m,n}x + b_m(2^{-m} - 2^{-n}) + a_{n,m}y \\ &\leq a_{m,k}x + b_m(2^{-m} - 2^{-k}) + b_m(2^{-k} - 2^{-n}) + a_{n,k}y \\ &\leq a_{m,k}x + b_m(2^{-m} - 2^{-k}) + a_{k,m}z + a_{k,n}z \\ &\quad + b_k(2^{-k} - 2^{-n}) + a_{n,k}y \\ &= d(p,r) + d(r,q). \end{aligned}$$

Case 2. $m < n < k$. Since

$$a_{m,n}x - a_{m,k}x \leq a_{m,n} - a_{m,k} < 2^{-2(n+1)} < 2^{-n} - 2^{-k} < b_m(2^{-n} - 2^{-k})$$

and

$$a_{n,m}y - a_{n,k}y \leq a_{n,m} - a_{n,k} < 2^{-2(n+1)} < 2^{-n} - 2^{-k} < b_n(2^{-n} - 2^{-k}),$$

we have

$$a_{m,n}x < a_{m,k}x + b_m(2^{-n} - 2^{-k})$$

and

$$a_{n,m}y < a_{n,k}y + b_n(2^{-n} - 2^{-k}),$$

respectively. Hence,

$$\begin{aligned} d(p,q) &= a_{m,n}x + b_m(2^{-m} - 2^{-n}) + a_{n,m}y \\ &< a_{m,k}x + b_m(2^{-n} - 2^{-k}) + b_m(2^{-m} - 2^{-n}) \\ &\quad + a_{n,k}y + b_n(2^{-n} - 2^{-k}) \\ &= a_{m,k}x + b_m(2^{-m} - 2^{-k}) + a_{n,k}y + b_n(2^{-n} - 2^{-k}) \\ &\leq d(p,r) + d(r,q). \end{aligned}$$

Thus, d is a metric on X and it is easily checked that d is compatible with the topology of X. To show that (X, d) has the property (a), let $p = (x, 2^{-m})$ and $q = (y, 2^{-n})$ be distinct points in X. Put $B = \{r \in X : d(p,r) = d(q,r)\}$. All we have to show is that $|B| \leq 1$. If $m = n$, then $|B| = 1$ by the definition of d. Thus, we suppose that $m < n$. Then $|B \cap I_k| \leq 1$ for each $k \in \mathbb{N}^{\#}$ by the definition of d. Hence, it suffices to show that $B \cap I_k \neq \emptyset$ for at most one k. For this end, define a sequence $\{\Delta_k : k \in \mathbb{N}^{\#}\}$ by the following rules: For $k \leq m$,

$$\begin{aligned} \Delta_k &= (a_{n,k}y + b_k(2^{-k} - 2^{-n})) - (a_{m,k}x + b_k(2^{-k} - 2^{-m})) \\ &= a_{n,k}y - a_{m,k}x + b_k(2^{-m} - 2^{-n}); \end{aligned}$$

for $m \leq k \leq n$,

$$\Delta_k = (a_{n,k}y + b_k(2^{-k} - 2^{-n})) - (a_{m,k}x + b_m(2^{-m} - 2^{-k}));$$

for $k \geq n$,

$$\Delta_k = (a_{n,k}y + b_n(2^{-n} - 2^{-k})) - (a_{m,k}x + b_m(2^{-m} - 2^{-k})).$$

If $(z, 2^{-k}) \in B \cap I_k$ for $k < m$, then $\Delta_k = \delta_k z$, where $\delta_k = a_{k,m} - a_{k,n}$. Since $2^{-k} \leq z \leq 1$, this implies that $2^{-k}\delta_k \leq \Delta_k \leq \delta_k$. Conversely, if the last inequalities hold, then $(z, 2^{-k}) \in B \cap I_k$, where $z = \delta_k^{-1}\Delta_k$. Thus, we have:

(1) For $k < m$, $2^{-k}\delta_k \leq \Delta_k \leq \delta_k$ if and only if $B \cap I_k \neq \emptyset$.

Similarly, putting $\delta_k = a_{k,m} - a_{k,n}$ for each k, we have the following:

(2) $-(1 + a_{m,n}) \leq \Delta_m \leq -2^{-m}(1 + a_{m,n})$ if and only if $B \cap I_m \neq \emptyset$;

(3) for $m < k < n$, $2^{-k}\delta_k \leq \Delta_k \leq \delta_k$ if and only if $B \cap I_k \neq \emptyset$;

(4) $2^{-n}(1 + a_{n,m}) \leq \Delta_n \leq 1 + a_{n,m}$ if and only if $B \cap I_n \neq \emptyset$;

(5) for $n < k < \infty$, $\delta_k \leq \Delta_k \leq 2^{-k}\delta_k$ if and only $B \cap I_k \neq \emptyset$;

(6) $\Delta_\infty = 0$ if and only if $B \cap I_\infty \neq \emptyset$.

By the definition of the $a_{i,j}$'s, (1)-(5) above imply the following (1')-(5'), respectively:

(1') for $k < m$, $0 < \Delta_k < 2^{-2(m+1)}$ if $B \cap I_k \neq \emptyset$;

(2') $\Delta_m < -2^{-m}$ if $B \cap I_m \neq \emptyset$;

(3') for $m < k < n$, $0 < \Delta_k < 2^{-2(k+1)}$ if $B \cap I_k \neq \emptyset$;

(4') $\Delta_n > 2^{-n}$ if $B \cap I_n \neq \emptyset$;

(5') for $n < k < \infty$, $-2^{-2(k+2)} < \Delta_k < 0$ if $B \cap I_k \neq \emptyset$.

Hence, if the following (7)-(9) were proved, then it would follow from (1')-(5') and (6) that $B \cap I_k \neq \emptyset$ for at most one k.

(7) If $k < \ell \leq m$, then $2^{-2(m+1)} < \Delta_k - \Delta_\ell < 2^{-m}$.

(8) If $m \leq k < n$, then $\Delta_k - \Delta_{k+1} > 2^{-(k+1)}$.

(9) If $n \leq k < \ell \leq \infty$, then $2^{-2(k+1)} < \Delta_k - \Delta_\ell < 2^{-k}$.

It remains to prove (7)-(9).

Proof of (7). If $k < \ell \leq m$, then

$$\Delta_k - \Delta_\ell = (a_{n,k} - a_{n,\ell})y - (a_{m,k} - a_{m,\ell})x + (b_k - b_\ell)(2^{-m} - 2^{-n}).$$

Since

$$-2^{-2(m+2)} < -2^{-2(n+2)} < a_{n,k} - a_{n,\ell} \leq (a_{n,k} - a_{n,\ell})y < 0,$$

$$-2^{-2(m+2)} \leq a_{m,k} - a_{m,\ell} \leq (a_{m,k} - a_{m,\ell})x < 0$$

and

$$\begin{aligned} 2^{-(2m+1)} &\leq 2^{-(k+1)} \cdot 2^{-(m+1)} \leq (b_k - b_\ell)(2^{-m} - 2^{-n}) \\ &< 2^{-1} \cdot 2^{-m} = 2^{-(m+1)}, \end{aligned}$$

we have

$$\Delta_k - \Delta_\ell > 2^{-(2m+1)} - 2^{-2(m+2)} > 2^{-2(m+1)}$$

and

$$\Delta_k - \Delta_\ell < 2^{-(m+1)} + 2^{-2(m+2)} < 2^{-m}. \quad \square$$

Proof of (8). If $m \leq k < n$, then

$$\begin{aligned} (10)\ \Delta_k - \Delta_{k+1} &= (a_{n,k} - a_{n,k+1})y - (a_{m,k} - a_{m,k+1})x \\ &\quad + \Big[b_k(2^{-k} - 2^{-n}) - b_m(2^{-m} - 2^{-k}) \\ &\quad - b_{k+1}(2^{-(k+1)} - 2^{-n}) + b_m(2^{-m} - 2^{-(k+1)})\Big]. \end{aligned}$$

Since

$$(a_{n,k} - a_{n,k+1})y \geq a_{n,k} - a_{n,k+1} \geq -2^{-2(n+2)} > -2^{-2(k+2)}$$

and

$$(a_{m,k} - a_{m,k+1})x \leq a_{m,k} - a_{m,k+1} = 2^{-2(k+1)} - 2^{-2(k+2)} < 2^{-2(k+1)},$$

we have

$$\begin{aligned} (a_{n,k} - a_{n,k+1})y - (a_{m,k} - a_{m,k+1})x &> -(2^{-2(k+2)} + 2^{-2(k+1)}) \\ &> -2^{-(2k+1)}. \end{aligned}$$

On the other hand, the last term $[\cdots]$ of (10) is:

$$\begin{aligned}[\cdots] &= (b_k - b_{k+1})(2^{-(k+1)} - 2^{-n}) + b_k(2^{-k} - 2^{-(k+1)}) \\ &\quad + b_m(2^{-k} - 2^{-(k+1)}) \\ &\geq (b_k + b_m) \cdot 2^{-(k+1)} \\ &> 2 \cdot 2^{-(k+1)} \\ &= 2^{-k}.\end{aligned}$$

Hence, $\Delta_k - \Delta_{k+1} > 2^{-k} - 2^{-(2k+1)} > 2^{-(k+1)}$. □

Proof of (9). If $n \leq k < \ell \leq \infty$, then

$$\begin{aligned}\Delta_k - \Delta_\ell &= (a_{n,k} - a_{n,\ell})y - (a_{m,k} - a_{m,\ell})x + b_n(2^{-n} - 2^{-k}) \\ &\quad - b_m(2^{-m} - 2^{-k}) - b_n(2^{-n} - 2^{-\ell}) + b_m(2^{-m} - 2^{-\ell}) \\ &= (a_{n,k} - a_{n,\ell})y - (a_{m,k} - a_{m,\ell})x + (b_m - b_n)(2^{-k} - 2^{-\ell}).\end{aligned}$$

Since

$$0 < (a_{n,k} - a_{n,\ell})y \leq a_{n,k} - a_{n,\ell} = 2^{-2(k+1)} - 2^{-2(\ell+1)} < 2^{-2(k+1)},$$
$$0 < (a_{m,k} - a_{m,\ell})x \leq a_{m,k} - a_{m,\ell} = 2^{-2(k+1)} - 2^{-2(\ell+1)} < 2^{-2(k+1)}$$

and

$$2^{-(2k+1)} \leq 2^{-(m+1)} \cdot 2^{-(k+1)} \leq (b_m - b_n)(2^{-k} - 2^{-\ell}) \leq 2^{-m} \cdot 2^{-k},$$

we have

$$\Delta_k - \Delta_\ell > 2^{-(2k+1)} - 2^{-2(k+1)} = 2^{-2(k+1)}$$

and

$$\Delta_k - \Delta_\ell < 2^{-(m+k)} + 2^{-2(k+1)} < 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.$$

Hence, the proof is complete. □

Remarks 2. The authors do not know whether the example can be strengthened by replacing the property (a) by UMP or UTP. We now show that a metric space need not have UMP even if it is homeomorphic to a subspace of $\mathbb{R}$. The simplest example is a metric space consisting of two points. A less trivial one is a convergent sequence. To see this, let $\mathbb{N}^\# = \mathbb{N} \cup \{\infty\}$ be a convergent sequence, where $n \to \infty$, and suppose that $\mathbb{N}^\#$ has a compatible metric d for which $(\mathbb{N}^\#, d)$ has UMP. Then for every pair of distinct points m and n in $\mathbb{N}^\#$, there is a unique $p = p(m,n) \in \mathbb{N}^\#$ such that $d(m,p) = d(n,p)$. Choose $k \in \mathbb{N}$ such that $d(m,n) < d(1,\infty)/3$ for each $m, n \geq k$. Then, $p(1,n) < k$ for each $n \geq k$. For, if $p = p(1,n) \geq k$ for some $n \geq k$, then $d(1,\infty) \leq d(1,p) + d(p,\infty) = d(n,p) + d(p,\infty) < d(1,\infty)$, which is a contradiction. Hence, there exist $p_1 < k$ and an infinite $N_1 \subset \mathbb{N}$ such that $p_1 = p(1,n)$ for each $n \in N_1$. Similarly, using p_1 instead of 1, we can find $p_2 \in \mathbb{N}$ and an infinite $N_2 \subset N_1$ such that $p_2 = p(p_1, n)$ for each $n \in N_2$. Choose distinct m and n in N_2. Then, $d(m,p_i) = d(n,p_i)$ for each $i = 1, 2$, which contradicts UMP. It is interesting to know what kind of subspace X of $\mathbb{R}$ has a compatible metric d for which (X,d) has UMP. For examples of such a subspace, see [1] and [7].

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