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SPACES HAVING STAR-COUNTABLE k-NETWORKS

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INTRODUCTION

We investigate spaces with star-countable or locally countable k-networks.

Let us recall some basic definitions. Let X be a space, and let $\mathcal{P}$ be a cover (not necessarily open or closed) of X. Then $\mathcal{P}$ is a *k-network*, if whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathcal{P}^* \subset U$ for some finite $\mathcal{P}^* \subset \mathcal{P}$. If we replace "compact" by "single point", then such a cover is called a "network". A space is an *$\aleph$-space* [17] (resp. *$\aleph_0$-space* [12]) if it has a σ-locally finite k-network (resp. countable k-network). If we replace "k-network" by "network", then such a space is a *σ-space* (resp. *cosmic space* [13]). A *closed* (resp. *compact*; *Lindelöf*; *separable*) *k-network* is a k-network consisting of closed (resp. compact; Lindelöf; separable) subsets.

Let X be a space, and let $\mathcal{C}$ be a cover of X. Then X *is determined by* $\mathcal{C}$ [6] (= X has the weak topology with respect to $\mathcal{C}$ in the usual sense), if $F \subset X$ is closed in X if and only if $F \cap C$ is closed in C for every $C \in \mathcal{C}$. Here, we can replace "closed" by "open". A space X is a *k-space* (resp. *sequential space*) if it is determined by the cover of all compact (resp. compact metric) subsets of X. A space X is a *k_ω-space* [14] if it is determined by a countable cover of compact subsets of X. A space X is *Fréchet* if whenever $x \in \overline{A}$, then there exists a sequence in A converging to the point x. Any Fréchet space is sequential. Any sequential space, or k_ω-space is a k-space.

A collection in X is *star-countable* (resp. *point-countable*) if each member (resp. single point) meets only countable many members. Not every locally countable collection is star-countable, but any σ-locally countable collection of Lindelöf subsets is star-countable. Every star-countable collection is point-countable, but not necessarily locally countable.

We consider relationships among spaces with star-countable k-networks, spaces with locally countable k-networks, and spaces with point-countable k-networks, etc. We give characterizations of spaces with star-countable k-networks, and spaces with star-countable closed k-networks. Among k-spaces or Fréchet spaces, we give some characterizations for spaces to have star-countable closed k-networks. Also, as applications of star-countable or locally countable k-networks, we give charaterizations for certain quotient images of locally separable metric spaces, etc. Shou Lin [7] (resp. C. Liu and J. P. Song [11]) showed that every k-space with a locally countable k-network is characterized as the quotient, strong s-image (resp. locally Lindelöf image) of a metric space (resp. locally separable metric space), etc. The proofs of these characterizations are based on the classical construction of certain subsets of the 0-dimensional Baire space as the domains. We give another chracterization by a simpler proof in terms of weak topologies with respect to star-countable covers.

We assume that spaces are regular T_1, and maps are continuous and onto.

1. STAR-COUNTABLE, LOCALLY COUNTABLE, OR POINT-COUNTABLE k-NETWORKS

We consider the relationships among spaces with certain star-countable k-networks, and spaces with locally countable k-networks, and spaces with certain point-countable k-networks, etc.

The following lemma holds by the proof for (a) $\Rightarrow$ (b) of Theorem 1 in [22].

Lemma 1.1. *Let $\mathcal{C}$ be a star-countable cover of X. Then the following (1) and (2) hold.*

(1) X is a disjoint union of $\{X_\alpha;\ \alpha \in A\}$, where each X_α is a countable union of elements of $\mathcal{C}$.

(2) If X is determined by $\mathcal{C}$, then X is the topological sum of the collection $\{X_\alpha;\ \alpha \in A\}$ in (1), and the cover $\mathcal{C}$ is locally countable.

The following holds by Lemma 1.1(2).

Corollary 1.2. *Let X be a space determined by a star-countable cover $\mathcal{P}$. If $\mathcal{P}$ is a k-network (resp. compact k-network; network), then X is the topological sum of $\aleph_0$-spaces (resp. k_ω-and-$\aleph_0$-spaces; cosmic spaces).*

The following lemma is routinely shown, so we omit the proof.

Lemma 1.3. *(1) Let X be determined by $\{X_\alpha;\ \alpha \in A\}$, and each $X_\alpha \subset Y_\alpha$. Then X is determined by $\{Y_\alpha;\ \alpha \in A\}$.*

(2) Let X be determined by $\{X_\alpha;\ \alpha \in A\}$, and let each X_α be determined by $\{X_{\alpha\beta};\ \beta \in B\}$. Then X is determined by $\{X_{\alpha,\beta};\ \alpha \in A,\ \beta \in B\}$.

(3) Let $f : X \longrightarrow Y$ be a quotient map. If X is determined by $\{X_\alpha; \alpha \in A\}$, then Y is determined by $\{f(X_\alpha); \alpha \in A\}$.

E. Michael [15] characterized countably bi-k-spaces as the countably bi-quotient images of paracompact M-spaces. For the definition and properties of countably bi-k-spaces, see [15; 4E]. Any first countable space, or any locally compact space is countably-bi-k, Any countably bi-k-space is a k-space. We recall that a space is *meta-Lindelöf* if every open cover has a point-countable open refinement.

Theorem 1.4. *(1) For a space X, (a) or (b) $\Rightarrow$ (c) $\Rightarrow$ (d) holds.*

(a) X has a locally countable k-network,

(b) X has a σ-locally finite closed Lindelöf k-network,

(c) X has a star-countable closed k-network,

(d) X has a star-countable k-network.

(2) If X is a k-space, then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) holds.

(3) If X is a meta-Lindelöf space, then (a) $\Rightarrow$ (b) holds.

(4) If X is a countably bi-k-space, then (d) $\Rightarrow$ (a) and (b) holds, indeed (d) implies "X is locally separable metric".

Proof: (1) Obviously, (b) $\Rightarrow$ (c) holds, and clearly (c) $\Rightarrow$ (d) holds. To show that (a) $\Rightarrow$ (c) holds, let $\mathcal{P}$ be a locally countable k-network for X. Here, we can assume that each element of $\mathcal{P}$ is closed. For each $x \in X$, there exists a nbd V_x of x such that V_x meets only countably many elements of $\mathcal{P}$. Then each V_x is a Lindelöf space (indeed, $\aleph_0$-space). Let $\mathcal{P}^* = \{P \in \mathcal{P}; P$ is contained in V_x for some $x \in X\}$. Then it is easy to show that $\mathcal{P}^*$ is a locally countable closed Lindelöf k-network, hence $\mathcal{P}^*$ is a star-countable closed k-network for X.

(2) Let $\mathcal{P}$ be a closed k-network for X. Then each compact subset is contained in a finite union of elements of $\mathcal{P}$. Then, since X is a k-space, by Lemma 1.3(1), X is determined by the cover of all finite unions of elements of $\mathcal{P}$. But, any space is determined by any of its finite closed cover. Thus, by Lemma 1.3(2), X is determined by $\mathcal{P}$. Thus, (c) $\Rightarrow$ (a) and (b) holds by Lemma 1.1(2) and Corollary 1.2.

(3) Suppose (a) holds. Then each point of X has a nbd V_x of x such that V_x is an $\aleph_0$-space. Since X is meta-Lindelöf, there exists a point-countable open cover $\mathcal{C}$ of $\aleph_0$-spaces. But $\mathcal{C}$ is star-countable, and X is determined by $\mathcal{C}$, since $\mathcal{C}$ is an open cover of X. Then (b) holds by means of Lemma 1.1(2).

(4) Since X is a countable bi-k-space with a point-countable k-network, X has a point-countable base $\mathcal{B}$ in view of [6; Corollary 3.6]. Let $\mathcal{P}$ be a star-countable k-network for X. Since X is a first countable space, and $\mathcal{P}$ is a point-countable

k$-$*network* for X, by [6; Proposition 3.2], for each $x \in X$, $x \in \operatorname{int} \cup \mathcal{P}^*$ for some finite $\mathcal{P}^* \subset \mathcal{P}$. Since $\mathcal{P}$ is star-countable, each element of $\mathcal{P}$ is a cosmic space. Thus, X is locally separable. Then we can assume that each element of $\mathcal{B}$ is separable, hence that $\mathcal{B}$ is a star-countable base. Thus, by Lemma 1.1(2), X is the topological sum of separable metric spaces. Hence X is locally separable metric.

In the previous theorem, (a) $\Rightarrow$ (b) in (1), or (b) $\Rightarrow$ (a) in (3) doesn't hold in general. The k-ness of meta-Lindelöfness in (2) or (3) is essential. In (1) and (2), the closedness of the star-countable k-network is essential. In (4), it is impossible to replace "countably bi-k" by "countably compact". For counter examples, see Example 4.1 in Section 4.

Proposition 1.5. *Let X be a k-space (resp, meta-Lindelöf space). Then the following are equivalent.*

(a) X has a star-countable closed k-network (resp. locally countable k-network),

(b) X is the topological sum of $\aleph_0$-spaces,

(c) X is a paracompact, locally $\aleph_0$-space,

(d) X is a locally Lindelöf , $\aleph$-space,

(e) X is a locally separable, $\aleph$-space.

Proof: (a) $\Rightarrow$ (b) holds in view of the proof of Theorem 1.4(2) & (3). (b) $\Rightarrow$ (c) is obvious. To show (c) $\Rightarrow$(d), suppose (c) holds. Since X is a paracompact, locally $\aleph_0$-space, X has a locally finite closed cover of $\aleph_0$-spaces. But, each compact subset of X meets only finitely many elements of this cover. Thus, it is shown that X is an $\aleph$-space. Thus (d) holds. (d) $\Rightarrow$ (e) is easy. To show (d) $\Rightarrow$ (a), suppose (d) holds. Then, as in the proof of (a) $\Rightarrow$ (c) of Theorem 1.4, X has a σ-locally finite closed Lindelöf k-network. Hence X has a star-countable closed k-network. Thus (a) holds. To show (e) $\Rightarrow$ (d), suppose (e) holds. Since X is a σ-space, each compact subset of X is metric. But, X is a k-space. Then X is sequential. In [8], it is proved that every sequential space with a σ-locally countable

k-network is meta-Lindelöf . Then X is meta-Lindelöf. Hence every separable closed subset of X is Lindelöf. Thus (d) holds.

As is well-known, not every separable, first countable σ-space is Lindelöf. Also, not every separable space with a locally countable (hence star-countable), and σ-locally finite k-network is Lindelöf; see Example 4.1(5), and not every separable k-space with a point-countable compact k-network is Lindelöf ; cf. [6; Example 9.3]. And, obviously, not every connected metric space is Lindelöf. But the following holds.

Proposition 1.6. *(1) Let X be a separable space. Then each one of the following implies that X is Lindelöf. Indeed, X is an $\aleph_0$-space for (a), (b), or (c), and a cosmic space for (d).*

(a) X is a k-space with a star-countable closed k-network,

(b) X is a k-space with a σ-locally countable k-network,

(c) X is a Fréchet space with a point-countable k-network,

(d) X is determined by a star-countable network.

(2) Let X be a connected space. Then (a) or (d) implies that X is Lindelöf. Indeed, X is an $\aleph_0$-space for (a), and a cosmic space for (d).

Proof: We prove only (1). For (a), by Proposition 1.5, X is the topological sum of $\aleph_0$-spaces. Since X is separable, X is an $\aleph_0$-space. For (b), by the proof of (e) $\Rightarrow$ (d) in Proposition 1.5, X is Lindelöf. Then X is an $\aleph_0$-space. For (c), by [6; Theorem 5.2], X is an $\aleph_0$-space. For (d), by Corollary 1.2, X is the topological sum of cosmic spaces. Then X is a cosmic space

Not every Fréchet space with a point-countable separable k-network has a star-countable *closed*, or locally countable k-network ; see Example 4.1(6). But the following holds.

Proposition 1.7. *Let X be a Fréchet space. Then the following are equivalent.*

(a) X has a star-countable closed k-network,

(b) X has a point-countable separable closed k-network,

(c) X is a locally separable space with a point-countable k-network.

Proof: (a) $\Rightarrow$ (b) holds, for any element of a star-countable k-network is cosmic, hence separable. For (b) $\Rightarrow$ (c), let $\mathcal{P}$ be a point-countable separable closed k-network. Since X is Fréchet, for each $x \in X$, $x \in \text{ int } \bigcup\{P \in \mathcal{P};\ x \in P\}$ by [6; Lemma 5.1]. Then X is locally separable. For (c) $\Rightarrow$ (a), since X is a locally separable Fréchet space, X is the topological sum of $\aleph_0$-spaces by [6; Proposition 8.8]. Thus, X has a star-countable closed k-network.

2. Spaces with star-countable, or star-countable closed k-networks

We consider spaces with star-countable k-networks, and spaces with star-countable *closed* k-networks, and differences between these spaces.

Theorem 2.1. *Let $f : X \longrightarrow Y$ be a closed map. Then (1) and (2) below hold.*

(1) Let X have a star-countable k-network. Then each one of the following implies that Y has a star-countable k-network.

(a) X is a k-space,

(b) X is a paracompact space,

(c) Each point of X is a G_δ-set,

(d) Each $Bf^{-1}(y)$ (boundary of $f^{-1}(y)$) is Lindelöf.

(2) Let (a) or (b) below hold. Then Y has a star-countable k-network. When each $Bf^{-1}(y)$ is Lindelöf , Y has a locally countable k-network, hence, a star-countable closed k-network.

(a) X is determined by a star-countable k-network,

(b) X has a locally countable k-network,

Proof: (1) Let $\mathcal{P}$ be a star-countable k-network for X. For each $y \in Y$, take $x_y \in f^{-1}(y)$, and let $A = \bigcup\{x_y;\ y \in Y\}$. Let $\mathcal{P}^* = \{f(A \cap P);\ P \in \mathcal{P}\}$. Then $\mathcal{P}^*$ is a star-countable cover of Y. Also, the proof for Theorem 1.5 in [10] implies that one of the properties implies that $\mathcal{P}^*$ is a k-network for Y. Hence, Y has a star-countable k-network.

(2) (a) implies (b) by Corollary 1.2, and (b) implies that each point of X is a G_δ-set, and X has a star-countable k-network by Theorem 1.4. Thus Y has a star-countable k-network by (1). For the latter part, since f is closed, we can assume that each $f^{-1}(y)$ is Lindelöf (indeed, take $x_y \in f^{-1}(y)$ for each $y \in Y$, and let $C_y = Bf^{-1}(y)$ if $Bf^{-1}(y) \neq \emptyset$, and $C_y = \{x_y\}$ if $Bf^{-1}(y) = \emptyset$. Instead of X, consider a closed subset $C = \bigcup\{C_y;\ y \in Y\}$ of X with $f(C) = Y$). Let $\mathcal{P}$ be a locally countable k-network for X, and let $\mathcal{P}^* = \{f(P); P \in \mathcal{P}\}$. Then, since f is closed and each $f^{-1}(y)$ Lindelöf, it is easy to show that $\mathcal{P}^*$ is a locally countable network for Y. Then each point of Y is a G_δ-set. Then it is easy to see that each compact subset of Y is sequentially compact. Also, $\mathcal{P}^*$ is a point-countable cover of Y, f is closed, and each point of X is a G_δ-set. Then we see that $\mathcal{P}^*$ is a k-network for Y by means of Propostition 1.2(1) & Lemma 1.6 in [24]. Then $\mathcal{P}^*$ is a star-countable k-network for Y.

Not every closed image of a locally compact metric space has a star-countable *closed* k-network ; see Example 4.1(6). But the following holds.

Corollary 2.2. *Let $f : X \longrightarrow Y$ be a closed map. Let X be a k-space with a star-countable closed k-network. In particular, let X be a locally separable metric space. Then the following (1) and (2) hold.*

(1) Y has a star-countable k-network

(2) The following are equivalent.

(a) Every $Bf^{-1}(y)$ is Lindelöf ,

(b) Y has a point-countable closed k-network,

(c) Y has a star-countable closed k-network.

Proof: (1) follows from Theorem 2.1(1). For (2), (a) $\Rightarrow$ (c) holds by Theorem 2.1(2) and Theorem 1.4(2). (c) $\Rightarrow$ (b) is clear. Suppose (b) holds. While, X is a paracompact $\aleph$-space by Proposition 1.5. Then (a) holds by [6; Proposition 6.4]. Thus (b) $\Rightarrow$ (a) holds.

Let X be a space, and $\mathcal{C}$ be a closed cover of X. Then X *is dominated* by $\mathcal{C}$ [12] (= X has the weak topology with respect to $\mathcal{C}$ in the sense of [16]), if the union of any subcollection $\mathcal{C}^*$ of $\mathcal{C}$ is closed in X, and the union is determined by $\mathcal{C}^*$. Every space is dominated by a hereditarily closure preserving closed cover. Clearly, if X is dominated by $\mathcal{C}$, then X is determined by $\mathcal{C}$. When $\mathcal{C}$ is an increasing countable closed cover, then the converse holds, however, the converse doesn't hold in general. It is well known that every space dominated by paracompact spaces is paracompact; see [12], or [16].

Lemma 2.3. *(1) Let X be dominated by $\{X_\lambda;\ \lambda < \alpha\}$. For each $\lambda < \alpha$; let $Y_0 = X_0$, $Y_\lambda = X_\lambda - \bigcup\{X_\mu;\ \mu < \alpha\}$. If $x_\alpha \in Y_\lambda$ for each $\lambda < \alpha$, then $\{x_\alpha;\ \lambda < \alpha\}$ is closed and discrete in X.*

(2) Let $\mathcal{C}$ be a closed (resp. point-countable) cover. Let X be dominated (resp. determined) by $\mathcal{C}$. Let $\{K_n;\ n \in N\}$ be a decreasing sequence in X such that $K = \bigcap\{K_n;\ n \in N\}$ is compact, and any nbd of K contains some K_n. Then some K_m is contained in a finite union of elements of $\mathcal{C}$.

Proof: (1) is due to [25; Lemma 2.5]. For (2), note that if $x_n \in K_n$ for each $n \in N$, then $\{x_n;\ n \in N\}$ has an accumulation point in X. Then, by (1), some K_m is contained in a finite union of elements of the cover $\{Y_\lambda;\ \lambda < \alpha\}$ of X. Hence, (2) holds. The parenthetic part of (2) holds in view of the proof of Lemma 6 in [22].

Theorem 2.4. *(1) Let X be dominated by a closed cover of $\aleph_0$-spaces. Then the following (i) and (ii) hold.*

(i) X has a star-countable k-network.

(ii) In the following, (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) holds. If X is a k-space, then (e) $\Rightarrow$ (a), hence (a) $\sim$ (e) are equivalent.

(a) X is locally separable,

(b) X is locally Lindelöf,

(c) X is an $\aleph$-space,

(d) X has a σ-locally countable k-network,

(e) X has a star-countable closed k-network.

(2) Let X be a space determined by a point-countable cover of $\aleph_0$-spaces. Then (c) or (d) $\Rightarrow$ (e). If X is a k-space, then (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) holds. When X is a Fréchet space, (c), (d) and (e) hold.

Proof: (1) Let X be dominated by $\{X_\lambda;\ \lambda < \alpha\}$, and each X_λ be an $\aleph_0$-space. For each $\lambda < \alpha$, let $Y_0 = X_0$, $Y_\lambda = X_\lambda - \bigcup\{X_\mu;\ \mu < \alpha\}$, and let $\mathcal{P}_\lambda$ be a countable k-network for X_λ. To show (i) holds, let $\mathcal{P} = \bigcup\{\mathcal{P}_\lambda \cap Y_\lambda;\ \lambda < \alpha\}$. Then $\mathcal{P}$ is a star-countable cover of X. To see that $\mathcal{P}$ is a k-network for X, let $K \subset U$ with K compact, and U open in X. Then by Lemma 2.3(1) K meets only finitely many $Y_{\lambda_n}(n = 1, 2, \ldots, m)$. Since $K \cap X_{\lambda_n} \subset U$ for each λ_n, there exists a finite $\mathcal{P}_{\lambda_n} \subset \mathcal{P}_\lambda$ such that $K \cap X_{\lambda_n} \subset \cup\mathcal{P}_{\lambda_n} \subset U$ for each λ_n. Let $\mathcal{P}^* = \cup\{\mathcal{P}_{\lambda_n} \cap Y_{\lambda_n};\ n = 1, 2, \ldots, m\}$. Then $\mathcal{P}^* \subset \mathcal{P}$ is finite such that $K \subset \cup\mathcal{P}^* \subset U$. Then $\mathcal{P}$ is a k-network for X. Thus $\mathcal{P}$ is a star-countable k-network for X. For (ii), we note that X is paracompact, because X is dominated by paracompact spaces $X_\lambda(\lambda < \alpha)$. Thus (a) $\Rightarrow$ (b) holds. To see (b) $\Rightarrow$ (a) and (c), for $x \in X$, let V_x be a nbd of x whose closure is Lindelöf. Then, by Lemma 2.3(1), $\overline{V_x}$ is contained in a countable union of $X_{\lambda,n}$'s $(n \in N)$. Thus $\overline{V_x}$ is separable. Hence (a) holds. Besides, $\overline{V_x}$ is dominated by a closed cover $\mathcal{C} = \{X_{\lambda,n} \cap \overline{V_x}; n \in N\}$ of $\aleph_0$-spaces. Then each compact subset of $\overline{V_x}$ is contained in a finite union of elements of $\mathcal{C}$ by Lemma 2.3(2). Thus $\overline{V_x}$ is an $\aleph_0$-space. Hence X is a locally $\aleph_0$-space. But X is paracompact. Then X has a locally finite closed cover of $\aleph_0$-spaces. Thus, X is an $\aleph$-space. Hence (c) holds. (c) $\Rightarrow$ (d) is clear. To show (d) $\Rightarrow$ (e), let $\mathcal{P} = \cup\{\mathcal{P}_n;\ n \in N\}$ be a σ-locally countable closed k-network for X, here, assume that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$, and $\mathcal{P}_n$ is closed under finite intersections for each $n \in N$. Let K be a compact subset of X. Since K meets only countably many elements of $\mathcal{P}$, there exists a decreasing sequence $\{K_n;\ n \in N\}$ such that each K_n is a finite union of elements of $\mathcal{P}$, $K = \cap\{K_n;\ n \in N\}$, and any nbd of K contains some K_n. Thus, by Lemma 2.3(2), some K_m is contained in a finite union of X_λ's, hence the K_m is Lindelöf. This shows that $\mathcal{P}^* = \{P \in \mathcal{P};\ P \text{ is Lindelöf}\}$ is a k-network for X. Since $\mathcal{P}^*$ is a σ-locally countable closed Lindelöf k-network, $\mathcal{P}^*$ is a star-countable closed k-network for X. Thus (e) holds. If X is a k-space, then (e) $\Rightarrow$ (a) by Proposition 1.5, hence (a) $\sim$ (e) are equivalent.

(2) This holds by the proof of (d) $\Rightarrow$ (e) in the above, and Proposition 1.5. The latter part holds by [6; Corollary 8.9].

We note that, in the first half of Theorem 2.4(2), even if X is separable, and Lindelöf , (c) or (d) doesn't hold; see Example 4.1(7).

We give characterizations for k-spaces with a star-countable closed (or compact) network in terms of weak topologies.

Theorem 2.5. *The following are equivalent.*

(a) X is a k-space with a star-countable closed (resp. compact) k-network,

(b) X is the topological sum of k-and-$\aleph_0$-spaces (resp. k_ω-and-$\aleph_0$-spaces),

(c) X is determined by a countable closed cover of locally separable, k-and-$\aleph$-spaces (resp. locally compact, metric spaces),

(d) X is an $\aleph$-space determined by a point-countable closed cover of k-and-$\aleph_0$-spaces (resp. compact spaces),

(e) X is an $\aleph$-space dominated by k-and-$\aleph_0$-spaces (resp. compact spaces),

(f) X is a locally separable space dominated by k-and-$\aleph_0$-spaces (resp. compact metric spaces).

Proof: (a) $\Rightarrow$ (b) holds by Proposition 1.5. (b) $\Rightarrow$ (c) holds, because X is a locally separable, k-and-$\aleph$-space. (b) $\Rightarrow$ (d) and (e) is obvious. For (c) $\Rightarrow$ (d), let X be determined by a countable closed cover $\{X_n;\ n \in N\}$ of locally separable, k-and-$\aleph$-spaces. Then each compact subset of X is contained in a finite union of X_n's by Lemma 2.3(2), and each X_n is a closed subset which is an $\aleph$-space. Then X is an $\aleph$-space. While, each X_n is the topological sum of k-and-$\aleph_0$-spaces $X_{n,\alpha}(\alpha \in A)$ by Proposition 1.4. Hence X_n is determined by the point-countable cover $\{X_{n,\alpha};\ \alpha \in A\}$. Then, by Lemma 1.3(2), X is determined by a point-countable closed cover $\{X_{n,\alpha};\ n \in N,\ \alpha \in A\}$ of k-and-$\aleph_0$-spaces. Thus (c) $\Rightarrow$ (d) holds. For (d) or (e) $\Rightarrow$ (a), since X is determined by k-spaces, X is determined by a cover of compact spaces by Lemma 1.3(2), hence X is a k-space. Thus, (d) or (e) $\Rightarrow$ (a) holds by Theorem 2.4. (e) $\Leftrightarrow$ (f) holds in view of Theorem 2.4(1).

We prove the parenthetic part holds. For (a) $\Rightarrow$ (b), let $\mathcal{P}$ be a star-countable compact k-network. Since X is a k-space, it is determined by $\mathcal{P}$. Thus (a) $\Rightarrow$ (b) holds by Corollary 1.2. For (b) $\Rightarrow$ (c), (d), and (e), let X be the topological sum of $\{X_\lambda;\ \lambda < \alpha\}$ of k_ω-and -$\aleph_0$-spaces. Then X is an $\aleph$-space. Since each X_λ is a k_ω-and-$\aleph_0$-space, X_λ is dominated by an increasing countable cover of $\{X_{\lambda,n};\ n \in N\}$ of compact metric spaces. Then, X is dominated by a point-countable closed cover $\mathcal{C} = \{X_{\lambda,n};\ \lambda < \alpha,\ n \in N\}$ of compact metric spaces. Thus (d) and (e) hold. To show (c) holds, let $Y_n = \cup\{X_{\lambda,n};\ \lambda < \alpha\}$ for each $n \in N$. Then each Y_n is a closed subset of X which is locally compact metric space. But, X is determined by $\mathcal{C}$, and each $X_{\lambda,n} \subset Y_n$. Thus, by Lemma 1.3(1), X is determined by a countable closed cover $\{Y_n;\ n \in N\}$ of locally compact metric spaces. Thus (c) holds. Hence, (b) $\Rightarrow$ (c), (d), and (e) holds. The proof for (c) $\Rightarrow$ (d) is similar to the above one. for (d) or (e) $\Rightarrow$ (a), X is a k-space, and X has a star-countable compact k-network as in the proof of Theorem 2.4. Thus (d) or (e) $\Rightarrow$ (a) holds. (e) $\Leftrightarrow$ (f) holds by Theorem 2.4(1).

Corollary 2.6. *(1) Every space dominated by locally separable metric spaces has a star-countable k-network.*

(2) Every space determined by a star-countable closed cover of locally separable (resp. locally compact) metric spaces has a

star-countable closed (resp. compact) k-network.

(3) Every Fréchet space determined by a point-countable locally separable metric spaces has a star-countable closed k-network.

Proof: For (1), any locally separable metric space has a star-countable (closed) k-network. Thus, the proof for (i) of Theorem 2.4(1) suggests that X has a star-countable k-network. (2) follows form Lemma 1.1(2) and Theorem 2.5. For (3), X is determined by a point-countable cover of separable metric spaces by means of Lemma 1.3(2). Thus (3) holds by Theorem 2.4(2).

In (2) of the previous corollary, it is impossible to replace "star-countable " by "point-countable"; see Example 4.1(7).

Concerning CW-complexes, Proposition 2.8 below holds in terms of star-countable k-networks. A CW-complex is *countable* if it consists of countable cells. As for CW-complexes, see [28], for example. In the following lemma, (1) is well-known, and (2) is due to [26].

Lemma 2.7. *(1) Every CW-complex is a k-space dominated by a cover of compact metric spaces.*

(2) A CW-complex is an $\aleph$-space if and only if it is the topological sum of countable CW-complexes.

Proposition 2.8. *Let X be a CW-complex. Then the following hold.*

(1) X, as well as every closed image of X, has a star-countable k-network.

(2) X has a star-countable closed k-network if and only if X is the topological sum of countable CW-complexes.

(3) Let X have a star-countable closed k-network. If X is separable, or connected, then X is a countable CW-complex.

(4) X, as well as every closed image of X, is locally separable metric if it is a countable bi-k-space.

Proof: (1) follows from Theorems 2.1(1) & 2.4(1), and Lemma 2.7(1). (2) holds by Proposition 1.5 and Theorem 2.5, and Lemma 2.7. (3) holds by (2), and (4) holds by (1) and Theorem 1.4(4).

Note that not every CW-complex has a star-countable *closed* k-network. Indeed, not every Fréchet , connected CW-complex has a star-countable *closed* k-network ; see Example 4.1(6). Also, every connected CW-complex with a point-countable compact k-network is not a countable CW-complex; cf. [26], hence doesn't have a star-countable *closed* k-network.

3. Quotient strongly Lindelöf maps

We consider characterizations for certain quotient images of locally separable metric spaces by means of locally countable k-networks.

Let $f : X \longrightarrow Y$ be a map. Then f is *Lindelöf* if every $f^{-1}(y)$ is Lindelöf , and, f is an *s-map* if every $f^{-1}(y)$ is separable. Let us call f *strongly Lindelöf* if for any Lindelöf subset L of Y, $f^{-1}(L)$ is Lindelöf in X. Any strongly Lindelöf map is Lindelöf , and any closed Lindelöf map is strongly Lindelöf. Every closed Lindelöf image of a space with a locally countable k-network has a locally countable k-network ; see Theorem 2.1(2). Besides, we have the following.

Proposition 3.1. *Let $f : X \longrightarrow Y$ be a quotient strongly Lindelöf map, or open s-map. Let X be a k-space . If X has a locally countable (resp. locally countable compact) k-network, then so does Y (respectively).*

Proof: Let f be a quotient strongly Lindelöf map. In view of the proof of Theorem 1.4(1), X has a locally countable, closed Lindelöf k-network $\mathcal{P}$. Let $\mathcal{P}^* = \{f(P);\ P \in \mathcal{P}\}$. Since f is strongly Lindelöf and each element of $\mathcal{P}$ is Lindelöf , $\mathcal{P}^*$ is a star-countable network for Y. While, since X is a k-space, X is determined by the closed k-network $\mathcal{P}$. Since f is quotient, Y is determined by $\mathcal{P}^*$ from Lemma 1.3(3). Thus, by Lemma 1.1(2),

$\mathcal{P}^*$ is a locally countable network for Y. Thus each compact subset of Y is sequentially compact. While, since X has a locally countable network, each compact subset of X is metric, hence, a k-space X is sequential. Then, since $\mathcal{P}^*$ is a point-countable cover of Y, f is quotient, and X is sequential, $\mathcal{P}^*$ is a k-network for Y by means of Proposition 1.2(1) & Lemma 1.6 in [24]. Thus, Y has a star-countable k-network. Next, let f be an open s-map. To show f is strongly Lindelöf, let L be Lindelöf in Y. Since X is locally separable, there exist open separable subsets V_n in X such that $L \subset \cup\{f(V_n);\ n \in N\}$. Let $G = \cup\{f^{-1}(f(V_n));\ n \in N\}$. Then $f^{-1}(L) \subset G$. But, since each $f(V_n))$ is separable, and f is an open s-map, it is routinely shown that each $f^{-1}(f(V_n))$ is separable. Thus G is separable. Also, G is a k-space, for it is open in a k-space X. Thus, $f^{-1}(L)$ is Lindelöf by Proposition 1.6. Thus f is a quotient strongly Lindelöf map. Thus Y has also a locally countable k-network.

A map $f : X \longrightarrow Y$ is a *strong s-map* [7] if each point of Y has a nbd V in Y such that $f^{-1}(V)$ is separable. If $f : X \longrightarrow Y$ is a strong s-map, then X and Y are locally separable. We note that any open s-map defined on a locally separable space is a strong s-map in view of the proof of Proposition 3.1. A map $f : X \longrightarrow Y$ is *compact-covering* [13] if each compact subset of Y is the image of some compact subset of X.

Lemma 3.2. *Let $f : X \longrightarrow Y$. Then the following hold.*

(1) Let X be paracompact, and let X be a σ-space (resp. $\aleph$-space). If f is a strong s-map (resp. compact-covering, strong s-map), then X and Y have locally countable networks (resp. k-networks).

(2) Let X be a paracompact σ-space. Then f is a strong s-map if and only if it is a strongly Lindelöf map with Y locally Lindelöf.

(3) Let f be a quotient map, and X be a k-space with a locally countable k-network. Then f is a strong s-map if and only if it is strongly Lindelöf.

Proof: For (1) and (2), note that a subset of X is separable if and only if it is (hereditarily) Lindelöf, because X is a paracompact, σ-space. Thus (1) and (2) are routinely proved. For (3), X is a paracompact, σ-space by Proposition 1.5. Thus, the "only if" part holds by (2). The "if" part holds by (2), because Y is locally Lindelöf by Proposition 3.1.

Theorem 3.3. *The following are equivalent.*

(a) X is a k-space with a locally countable (resp. locally countable compact) k-network,

(b) X is the quotient strongly Lindelöf image of a locally separable (resp. locally compact) metric space,

(c) X is the quotient strong s-image of a locally separable (resp. locally compact) metric space.

Proof: To show (a) $\Rightarrow$ (b), suppose (a) holds. Then X has a star-countable closed (resp. compact) k-network by Theorem 1.4(1). Since X is a k-space, by Theorem 2.5, X is the topological sum of k-and-$\aleph_0$-spaces (resp. k_ω-and-$\aleph_0$-spaces) $X_\alpha (\alpha \in A)$. Let X be the topological sum of $\{X_\alpha;\ \alpha \in A\}$. Each X_α is the quotient image of a separable metric space M_α by [13; Corollary 11.5]. For the parenthetic part, let M_α be the topological sum of a countable compact k-network for X_α. Let M be the topological sum of $\{M_\alpha;\ \alpha \in A\}$. Let $f : M \longrightarrow X$ be the obvious map. Then M is locally separable (resp. locally compact) metric, and f is quotient strongly Lindelöf. (b) $\Rightarrow$ (a) follows from Proposition 3.1. (b) $\Leftrightarrow$ (c) holds by Lemma 3.2.(3).

Remark 3.4 We recall that a space is *hemicompact* if it has a countable cover $\mathcal{C}$ of compact subsets such that each compact subset is contained in a finite union of elements of $\mathcal{C}$. Here, we can replace "finite union of elements" by "some element". Any Lindelöf locally compact space is hemicompact. We note that a k_ω-space is precisely a hemicompact, k-space by Lemmas 1.1 & 3.2.

The following (1), (2), and (3) hold. Indeed, these are shown in view of Lemmas 1.1 & 3.2, and the proof of Theorem 3.3. Here, we recall that the fact that every cosmic space (resp. $\aleph_0$-space) is precisely the image (resp. compact-covering image) of a separable metric space [13].

(1) The following (a), (b), and (c) are equivalent.

(a) X is a space with a countable compact k-network (resp. k-space with a countable compact k-network),

(b) X is a hemicompact $\aleph_0$-space (resp. k_ω-and-$\aleph_0$-space).

(c) X is a hemicompact cosmic space (resp. k_ω and cosmic space),

(d) X is the compact-covering (resp. quotient) image of a separable locally compact metric space.

(2) X has a locally countable k-network (resp. locally countable compact k-network) $\Leftrightarrow$ X is the compact-covering, strong s-image of a locally separable (resp. locally compact) metric space.

(3) X has a locally countable network $\Leftrightarrow$ X is a disjoint and locally countable sum of cosmic spaces $\Leftrightarrow$ X is the strong s-image of a locally separable metric space.

Now, a map $f : X \longrightarrow Y$ is *pseudo-open* if for any $y \in Y$ and any open subset U of Y with $f^{-1}(y) \subset U$, $y \in \operatorname{int} f(U)$. Any open map or any closed map is pseudo-open, and every pseudo-open map is quotient. We recall that a quotient map from a metric space onto Y is pseudo-open if and only if Y is Fréchet [1].

In the following proposition, the equivalence among (a), (c), and (d) (resp. (c) and (d) for the parenthetic part) is shown in [8] (resp. [23]).

Proposition 3.5. *The following are equivalent.*

(a) X is a Fréchet space with a locally countable k-network (resp. locally countable compact k-network),

(b) X is the pseudo-open strongly Lindelöf image of a locally separable (resp. locally compact) metric space,

(c) X is the pseudo-open Lindelöf image of a locally separable (resp. locally compact) metric space,

(d) X is the closed Lindelöf image of a locally separable (resp. locally compact) metric space.

Proof: The equivalence between (a) and (b) holds by means of Theorem 3.3. To show (a) $\Rightarrow$ (d), let X have a locally countable closed (resp. compact) k-network. Since X is a k-space, by Theorem 1.4(2) & 2.5, X is the topological sum of Fréchet $\aleph_0$-spaces (resp. Fréchet , k_ω-and-$\aleph_0$-space). But, every Fréchet $\aleph_0$-space (resp. Fréchet , k_ω-and-$\aleph_0$-space) is the closed image of a separable (resp. locally compact separable) metric space by [4] (resp. [19]). Thus (d) holds. (b) $\Rightarrow$ (c) is clear. For (c) $\Rightarrow$ (a), note that X is the topological sum of $\aleph_0$-spaces X_α by mean of [6; Proposition 8.8], hence (a) holds. For the parenthetic part, let $f : M \longrightarrow X$ be a pseudo-open Lindelöf map with M a locally compact metric space. Then M is determined by a locally countable cover $\mathcal{C}$ of compact spaces. Since f is quotient and Lindelöf, by Lemma 1.3(3), X is determined by a point-countable cover $f(\mathcal{C})$ of compact spaces. But each X_α is closed in X. Then X_α is determined by a point-countable cover of compact spaces. But, each X_α is an $\aleph_0$-space. Then, by the same way as in the proof for (ii) of Theorem 2.4(1), each X_α has a countable compact k-network. Then X has a locally countable compact k-network. Thus (a) holds.

We note that it is impossible to replace "pseudo-open" by "quotient", and to omit " locally separable" in the previous proposition; see Example 4.2.

Reviewing the previous sections, as a summary, we have Theorems 3.6 and 3.7 below. Theorem 3.6 holds by Theorems 1.4, 2.5, and 3.3.

Theorem 3.6. *Let X be a k-space. Then the following are equivalent.*

(a) X has a star-countable closed (resp. compact) k-network,

(b) X has a locally countable (resp. locally countable compact) k-network,

(c) X is determined by a countable closed cover of $\aleph_0$-spaces (resp. locally compact metric spaces),

(d) X is the quotient strongly Lindelöf image of a locally separable (resp. locally compact) metric space.

Theorem 3.7. *Let X be a Fréchet space. Then the following are equivalent.*

(a) X has a star-countable closed (resp. compact) k-network,

(b) X has a locally countable (resp. locally countable compact) k-network,

(c) X has a point-countable separable closed (resp. point-countable compact) k-network,

(d) X is determined by a countable closed cover of $\aleph_0$-spaces (resp. locally compact metric spaces),

(e) X is determined by a point-countable cover of $\aleph_0$-spaces (resp. locally compact metric spaces),

(f) X is the pseudo-open (strongly) Lindelöf image of a locally separable (resp. locally compact) metric space,

(g) X is the closed Lindelöf image of a locally separable (resp. locally compact) metric space.

Proof: This holds by Propositions 1.7 & 3.5, and Theorem 2.4(2). For the parenthetic part, we show that (c) or (e) $\Rightarrow$ (a) holds. Let (c) hold, and let $\mathcal{P}$ be a point-countable compact k-network for X. But, every compact space with a point-countable k-network is metric by [2; Theorem 3.1]. Thus each element of $\mathcal{P}$ is separable. Then X has a locally countable closed k-network by the implication (c) $\Rightarrow$ (a). While, since X is a k-space, X is determined by $\mathcal{P}$. Then, as in the proof of Theorem 2.4(2). X has a star-countable compact k-network. Thus (a) holds. Let (e) hold. Then X is determined by a point-countable cover of compact metric spaces by Lemma 1.3(2). Then (a) holds by a similar argument.

4. Examples and questions

In Example 4.1 below, (1) shows that, in Theorem 1.4(1), (a) $\Rightarrow$ (b) doesn't hold in general, and that the k-ness or meta-Lindelöfness in Theorem 1.4(2) or (3) is essential. (2) shows that, in Theorem 1.4(3), (b) $\Rightarrow$ (a) doesn't hold in general. (3) shows that, in Theorem 1.4(2), the k-ness is essential, and that, in Theorem 1.4(4), it is impossible to replace "countably bi-k" by "countably compact". (4) or (5) shows that, in Proposition 1.5, the k-ness is essential. (5) shows that the k-ness in Proposition 1.6 is essential. (6) shows that the closedness of the star-countable k-network in Theorem 1.4 (1) & (2), and Propositions 1.5 & 1.7 is essential. (7) (resp. (8)) shows that, in Proposition 1.7, it is impossible to replace "Fréchet space" by "k-space " (resp. "separable" by "Lindelöf" in (b) or (c) of Proposition 1.7).

Example 4.1. (1) A σ-space X with a locally countable (hence, star-countable) compact k-network, but X is not an $\aleph$-space, and not meta-Lindelöf.

(2) A paracompact space X with a σ-locally finite compact k-network, but X has no locally countable network (indeed, X is not locally Lindelöf).

(3) A countably compact space X with a star-countable compact k-network, but X has no σ-locally countable k-networks (indeed, X is not a locally Lindelöf space, and it has a point which is not a G_δ-set).

(4) A Lindelöf space X with a star-countable compact k-network, but X is not locally separable.

(5) A separable space X with a σ-locally finite, and locally countable compact k-network (resp. σ-locally finite k-network), but X is not Lindelöf (resp. locally Lindelöf).

(6) A Fréchet , CW-complex X with a star-countable separable, Lindelöf k-network, but X has no star-countable closed k-networks (indeed, X has no point-countable closed k-networks and it is not locally separable).

(7) A separable, Lindelöf, and k-space X with a point-

countable compact k-network, but X has no star-countable k-networks and no σ-locally countable k-networks.

(8) A first countable, Lindelöf space X with a point-countable closed Lindelöf k-network, but X has no star-countable k-networks.

Proof: (1) There exists a σ-space with a locally countable k-network, but not an $\aleph$-space by [8; Example 1]. This space is not meta-Lindelöf by Theorem 1.4(3).

(2) is a modification of [9]. Let A be an uncountable set, and $X = \{p\} \cup (A \times \{n;\ n \in N\})$. Let any point of X except the point p be isolated and for the point p, let $\{p\} \cup (\cup\{B_n \times n;\ n \geq m\})$, where each $B_n \subset A$ with $A - B_n$ at most finite, be a basic nbd base of p in X. Then X is paracompact. Since any compact subset of X is finite, X has a σ-locally finite compact k-network $\{\{x\};\ x \in X\}$. But X is not locally Lindelöf.

(3) Let X be an infinite, countably compact space X whose compact subsets are finite; see [5]. Then X has a star-finite compact k-network. But X has no σ-locally countable k-networks, indeed, no σ-locally countable networks. Suppose that X has a σ-locally countable network. Then, each point of X is contained in a G_δ-set which is cosmic, hence each point of X is a G_δ-set. Thus, since X is countably compact, X is first countable. Hence X is discrete. This is a contradiction. Then X has no σ-locally countable networks. We note that X is not locally Lindelöf, because X is a countable compact space which is not locally compact.

(4) Let A be an uncountable set, and $X = A \cup \{p\}$. Let any point of X except the point p be isolated, and for the point p, let $\{p\} \cup B$, where $B \subset A$ with $A - B$ at most countable, be a basic nbd base of p in X. Then X is Lindelöf, but it is not locally separable. While, since each compact subset of X is finite, X has a star-finite compact k-network $\{\{x\};\ x \in X\}$.

(5) Let $X = P \cup (Q \times \{n :\ n \in N\})$, where P is the set of irrational numbers and Q is the set of rational numbers. Let each point of $X - P$ be isolated in X and for each $x \in P$, let

$\{x\} \cup (\cup\{((a_{n,x}, b_{x,n}) \cap Q) \times n;\ n \geq m\})$ be a basic nbd of x in X, where $x - 1/n < a_{n,x} < x < b_{x,n} < x + 1/n$, $n \in N$. Then X is a separable space, but it is not Lindelöf. Since any compact subset of X is finite, X has a locally countable, and σ-locally finite compact k-network $\{\{x\};\ x \in X\}$.

For the parenthetic part, let $X^* = \{p\} \cup (\cup\{X \times \{n\};\ n \in N\})$. Let $\{p\} \cup (\cup\{X \times \{n\};\ n \geq m\})$ be a basic nbd of p in X^* , and let each $X \times n$ be open in X^*. Then X^* has a σ-locally finite k-network $\{\{x\} \times n;\ x \in X,\ n \in N\} \cup \{\{p\} \cup (\cup\{X \times \{n\};\ n \geq m\});\ m \in N\})$, hence X^* is an $\aleph$-space. X^* is a separable space, but it is not locally Lindelöf.

(6) Let X be the quotient space obtained from the topological sum of ω_1 many closed unit intervals [0,1] by identifying all the zero points to a single point. Then X is a Fréchet CW-complex, and it has a star-countable separable, Lindelöf k-network. But X is not locally separable nor locally Lindelöf, and X has no point-countable closed k-networks in view of [21; Proposition 1].

(7) We show that the space X in [27; Example 1.6(2)] is the required space. Indeed, let $S = \{1/n;\ n \in N\} \cup \{0\}$, and let $I = [0, 1]$ be a subspace of the Euclidean space R. Let $\mathcal{C} = \{I \times 1/n;\ n \in N\} \cup \{I \times \{0\}\} \cup \{\{t\} \times S;\ t \in I\}$. Let $X = I \times S$ be the space determined by the point-finite cover $\mathcal{C}$ of compact metric subsets of R^2. Then X is a separable, Lindelöf space. Every compact subset of X is contained in a finite union of elements of $\mathcal{C}$ by Lemma 2.3(2). Then X has a point-countable compact k-network. To show that X has no star-countable k-networks, let $\mathcal{P}$ be a k-network for X. For each $\alpha \in I$, let $V_\alpha = \{(x, y) \in X;\ y > |x - \alpha|\} \cup \{I \times \{0\}\}$, and $C_\alpha = \{(\alpha, 1/n);\ n \in N\} \cup \{(\alpha, 0)\}$. Then each V_α is an open subset of X, and contains the compact set C_α. Then for each $\alpha \in [0, 1]$, there exists $P_\alpha \in \mathcal{P}$ such that $P_\alpha \subset V_\alpha$, and P_α contains a subsequence A_α of C_α. We note that $P_\alpha \neq P_\beta$ if $\alpha \neq \beta$. There exists $n \in N$ such that $L_n = \{(x, 1/n);\ x \in I\}$ meets ω_1 many A_α. But, since L_n is an open and compact subset of X, there exists a finitely many $P_i \in \mathcal{P}(i \leq m)$ such

that $L_n = \cup\{P_i;\ i \leq m\}$. Thus some P_i meets ω_1 many A_α, so P_i meets ω_1 many C_α. This shows that $\mathcal{P}$ is not star-countable. Thus X has no star-countable k-networks. Next, suppose that X has a σ-locally countable k-network. Since X is Lindelöf, X has a countable k-network, hence a star-countable k-network. This is a contradiction. Hence X has no σ-locally countable k-networks.

(8) Let X be the space in [3; Example 6.4]. As is seen there, X is a Lindelöf space with a point-countable base, but X has a closed subset which is not a G_δ-set. Obviously, X has a σ-disjoint (hence, point-countable) closed (hence, Lindelöf) k-network. Since X is a first countable space which is not metric, X has no star-countable k-networks by Theorem 1.3(4).

In Example 4.2 below, (1) shows that, in (d) of Theorem 3.3, it is impossible to replace " strongly Lindelöf image (resp. strong s-image)" by "Lindelöf image (resp. s-image)" and shows that, in Proposition 3.5, it is also impossible to replace "pseudo-open" by "quotient". (2) and (3) show that, in Proposition 3.5, the separability of the metric space is essential.

Example 4.2. (1) A separable, Lindelöf space X which is the quotient finite-to-one image of a locally compact metric space, but X is not the quotient strongly Lindelöf nor quotient strong s-image of a metric space, and X has no star-countable k-networks.

(2) A Lindelöf space X which is the open Lindelöf, s-image of a metric space, but X is not the closed, quotient strongly Lindelöf, nor quotient strong s-image of a metric space, and X has no star-countable k-networks.

(3) A space X which is the open finite-to-one, strongly Lindelöf image of a metric space, but X is not the closed image of a metric space.

Proof: (1) Let L be the topological sum of the cover $\mathcal{C}$ of the space X in Example 4.1(7), and $f : L \longrightarrow X$ be the obvious map. Then L is locally compact metric, and f is quotient finite-to-one. Suppose that X is the quotient strongly Lindelöf image of a metric space. Since X is Lindelöf, it is the quotient strongly Lindelöf image of a separable metric space. Thus X has a locally countable k-network by Proposition 3.1. This is a contradiction to Example 4.1(7). Then X is not the quotient strongly Lindelöf image of a metric space, hence not the quotient strong s-image of a metric space by Lemma 3.2(2).

(2) Let X be the Lindelöf space in Example 4.1(8). Since X has a point-countable base, X is the open Lindelöf, s-image of a metric space by [18]. Since X has a closed subset which is not a G_δ-set, X is not a closed image of a metric space. Also, since X does not have a star-countable k-network, X is not the quotient strongly Lindelöf or quotient strong s-image of a metric space as in the proof of (1).

(3) For each $r \in R$, let $X_r = \{(x, y);\ y = |x - r|\}$. Define a topology on X_r as follows:

Each point (x, y) with $y > 0$ is isolated, and for $(r, 0) \in X_r$, let $\{(x, y); y = |x - r| < 1/n\}$, where $n \in N$. be a basic nbd of $(r, 0)$ in X_r.

Let M be the topological sum of $\{X_r;\ r \in R\}$. Let X be an upper half plane and $f : M \longrightarrow X$ be the obvious function, and let X be the quotient space by f. Then M is metric, and f is open finite-to-one. Since the map f is finite-to-one and any Lindelöf subset of X is countable, f is strongly Lindelöf. But, X is not normal. Then X is not the closed image of a metric space.

We conclude this paper with Question 4.3 below. (1) is proposed in view of Theorem 1.4 and Example 4.1(1) & (3). (2), (3), and (4) are respectively proposed in view of Proposition 1.6, Proposition 1.7, and Theorem 2.1.

In view of results in this paper, (1) is affirmative if X is a k-space, meta-Lindelöf, or θ-refinable (here, every θ-refinable (= submetacompact) space with a locally countable network is a σ-space by [20]), (2) and (3) are affirmative if the k-network is closed, and (4) is affirmative if X is a k-space, paracompact, or each point of X is a G_δ-set, etc.

Question 4.3. (1) Is every space X with a locally countable k-network a σ-space, or space in which every closed subset is a G_δ-set?

(2) Is every separable k-space X with a star-countable k-network a Lindelöf space?

(3) Does every Fréchet space X with a point-countable separable k-network have a star-countable k-network ?

(4) Does every closed image of a space X with a star-countable k-network have a star-countable k-network, or a point-countable k-network?

Comment: Quite recently, Masami Sakai pointed out that (1) and (4) are negative if X is a *Hausdorff* space.

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