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NORMALITY OF PRODUCTS OF GO-SPACES AND CARDINALS

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Dedicated to Professor Akihiro Okuyama on his 60th birthday

ABSTRACT. In this paper, we characterize the normality of $X \times \kappa$, where X is a GO-space and κ a regular uncountable cardinal.

1. INTRODUCTION

Gruenhage, Nogura and Purish [GNP] proved that if X is a GO-space of countable tightness, then $X \times \omega_1$ is normal. As was written in the introduction of their paper, the author conjectured that $X \times \kappa^+$ is normal whenever X is a GO-space of tightness $\leq \kappa$. But this was false, because consider $\omega_1 \times \omega_2$. Since it contains the non-normal closed subspace $\omega_1 \times (\omega_1 + 1)$, it is not normal. But ω_2 is of tightness $\leq \omega_1$. In this paper, we characterize normality of a product space of a GO-space and a regular uncountable cardinal using the minimal linearly ordered compactification of a GO-space.

2. LINEARLY ORDERED COMPACTIFICATIONS

Let $<$ be a linear order on a set X. $\lambda(X)$ denotes the topology on X which is generated by $\{(\leftarrow, a) : a \in X\} \cup \{(b, \rightarrow)$

$: b \in X\}$ as a subbase, where $(\leftarrow, a) = \{x \in X : x < a\}$ and $(b, \rightarrow)$ similarly. $(a, b]$ denotes the usual half open interval $\{x \in X : a < x \leq b\}$ if a and b are in X. Analogously we define $(a, b), [a, b]$, etc.. If necessary, we write $<_X$ and $(a, b)_X$ instead of $<$ and (a, b). A "*LOTS X*" means the triple $< X, <, \lambda(X) >$. A "*GO − space X*" is a triple $< X <, \tau >$, where $<$ is a linear order on X and τ is a topology on X which has a base consisting of convex open sets such that $\lambda(X) \subset \tau$. For a given GO-space $< X, <, \tau >$, $\tilde{X}$ denotes the LOTS with the lexicographic order, where $\tilde{X} = \{< x, -1 >: x \in X$ and $[x, \rightarrow) \in \tau - \lambda(X)\} \cup X \times \{0\} \cup \{< x, 1 >: x \in X$ and $(\leftarrow, x] \in \tau - \lambda(X)\}$. By identifying $X \times \{0\}$ with X, the LOTS $\tilde{X}$ contains the GO-space X as a dense subspace, and the restricted order of $<_{\tilde{X}}$ on X coincides with $<_X$ (in this situation, we say "$\tilde{X}$ contains X densely and linearly"). It is known that any LOTS L, which contains a GO-space X densely and linearly, also contains the LOTS $\tilde{X}$ densely and linearly ([MK]). So the LOTS $\tilde{X}$ is considered as the smallest LOTS which contains X densely and linearly. A linearly ordered compactification (abbreviated as *LOC*) of a GO-space X is a compact LOTS which contains X densely and linearly. Note that a compact GO-space is a LOTS. Kaufman constructed in [Ka] the minimal LOC X' of a LOTS X, i.e., for each LOC L of X, there is a countinuous map $f : L \rightarrow X'$ such that the restriction $f|X$ of f on X is the identity map 1_X on X. For a GO-space X, ℓX denotes the compact LOTS $(\tilde{X})'$. Then, by these consideration, ℓX is the minimal LOC(abbreviated as *MLOC*) of X. It is not difficult to show that if there is a continuous map f on a LOC cX of X to a LOC $c'X$ with $f|X = 1_X$, then f must be order preserving in the sense $f(a) \leq f(b)$ if $a \leq b$. The following property $P(L, X)$ for a linearly ordered set $< L, <>$ and a subset X of L, is useful in our discussion.

$$P(L, X) : (a, b) \neq 0 \text{ for any } a \text{ and } b \text{ in } L - X \text{ with } a < b.$$

Lemma 2.1. *Let cX be a LOC of a GO-space X. Then cX*

is minimal if and only if $P(cX, X)$ *holds.*

Proof: First assume $P(cX, X)$ does not hold. Then there are a and b in $cX - X$ with $a < b$ such that $(a,b)_{cX} = 0$. Then it is easy to show that the quotient space obtained by identifying $a = b$ is a LOC of X. So cX is not minimal.

Next assume cX is not minimal. By the minimality of ℓX, there is a continuous map $f : cX \to \ell X$ such that $f|X = 1_X$. Then it is easy to show that f is not 1-1. So there are a and b in cX with $a < b$ such that $f(a) = f(b)$. By [En, 3.7.14]. we have a, b and $f(a)$ are not in X. If $(a,b)_{cX} \neq 0$, take a point x in $(a,b) \cap X$ by the density of X. Since f is order preserving, $f(a) \leq f(x) = x \leq f(b)$. So we have $f(a) = x \in X$. This is a contradiction.

3. Cofinality and Normal Sequences

Let L be a compact LOTS and $x \in L$. Note that each subset F of a compact LOTS L has a least upper bound (abbreviated as sup$F$ or $\sup_L F$ if necessary), and has a greatest lower bound inf$F$, see [En, 3.12.3(a)]. A subset A of $(\leftarrow, X)$ is said to be 0-*unbounded* for x (in L) if, for each $y < x$, there is $a \in A$ such that $y \leq a$. Otherwise, A is said to be 0-*bounded* for x. Analogously, 1-unboundedness of $A \subset (x, \to)$ for x is defined. Note that 0-unboundedness and 1-unboundedness are dual notions, so we only define and prove "0- ... ". Of course, we can do for "1-... ". 0-*cofinality* for x in L is defined as follows:

$$0 - cfx = \min\{|A| : A \text{ is } 0 - \text{unbounded for } x.\}.$$

If necessary we write 0-$\mathrm{cf}_L x$ instead of 0-cf$x$. Observe that 0-cf$x$ = 0 if x is the first element of L, 0-cf$x$ = 1 if x has the immediate predecessor in L, and 0-cf$x$ is a regular infinite cardinal otherwise. For a fixed cardinal κ, a strict increasing sequence $< x(\alpha) : \alpha < \kappa >$ in L is said to be a 0-*unbounded sequence* for x if $\{x(\alpha) : \alpha < \kappa\}$ is 0-unbounded for x. Furthermore a 0-unbounded sequence is said to be a 0-*normal sequence* for x if $x(\alpha) = \sup\ \{x(\beta) : \beta < \alpha\}$ for each limit ordinal $\alpha < \kappa$.

Since each subset of a compact LOTS has a least upper bound, we can easily show the following lemma by induction:

Lemma 3.1. *Let x be a point in a compact LOTS L, then there always exists a 0-normal sequence for x of length 0-cf$x$.*

Remark 3.2. Note that if $< x(\alpha) : \alpha < \kappa >$ is a 0-normal sequence for x, then $\{x(\alpha) : \alpha < \kappa\}$ is a closed subspace of the subspace $(\leftarrow, x)$ and it is homeomorphic to the ordinal space 0-cf$x$. The proof of 1) in the next lemma is a routine and 2) follows form 1), so we left it to the reader.

Lemma 3.3. *Let x be a point in a compact LOTS L with $0-cfx \geq \omega_1$. If $< x(\alpha) : \alpha < 0-cfx >$ and $< x'(\alpha) : \alpha < 0-cfx >$ are 0-normal sequences for x, then we have:*

(1) *$\{\alpha < 0-cfx : x(\alpha) = x'(\alpha)\}$ is closed unbounded in $0-cfx$.*
(2) *if $X \subset L$, then $\{\alpha < 0-cfx : x(\alpha) \in X\}$ is stationary in $0-cfx$ iff so is $\{\alpha < 0-cfx : x'(\alpha) \in X\}$.*

From now on, we shall apply the above arguments for $L = \ell X$, where ℓX is the MLOC of a GO-space X. In our argument, for each $x \in \ell X$, we always fix a 0-normal sequence $< x(\alpha) : \alpha < 0\text{-cf}_{\ell X} x >$ and put $0(X\Delta x) = \{\alpha < 0\text{-cf}x : x(\alpha) \in X\}$. Of course, we define $1(X\Delta x)$ analogously. By the above lemma, observe that if 0-cf$x \geq \omega_1$, then the "stationarity" of $0(X\Delta x)$ does not depend on choices of 0-normal sequences for x. The following lemma is easy to prove.

Lemma 3.4. *Let x be a point in the MLOC ℓX of a GO-space X. Then the following holds.*

(1) *If $x \in \ell X - X$, then $\{x(\alpha) : \alpha \in 0(X\Delta x)\}$ is a closed subset of X and homeomorphic to $0(X\Delta x)$.*
(2) *If $x \in X$, then $\{x(\alpha) : \alpha \in 0(X\Delta x)\} \cup \{x\}$ is a closed subset of X and homeomorphic to $0(X\Delta x) \cup \{0-cfx\}$.*
(3) *If $0-cfx \geq \omega_1$ and $0(X\Delta x)$ is not stationary in $0-cfx$, then $(\leftarrow, x)_{\ell X} \cap X$ is the free union of $0-cfx$ many 0-bounded for x, closed and open subsets of X.*

(4) *If* $0{-}cfx = \omega$, *then* $(\leftarrow, x)_{\ell X} \cap X$ *is the locally finite union of countable many 0-bounded for* x, *closed subsets of* X.

Lemma 3.5. *Let* x *be a point in the MLOC* ℓX *of a GO-space* X. *If any one of the following holds, then there is a 0-unbounded sequence* $< x(\alpha) : \alpha < 0\text{-}cfx >$ *for* x *such that* $x(\alpha) \in X$ *for each* $\alpha <$ *0-cfx.*

(1) $0{-}cfx = 0$.

(2) $x \in \ell X - X$ *and* $0{-}cfx = 1$.

(3) $0{-}cfx \geq \omega$.

Proof: 1): Since the empty sequence is 0-unbounded if $0{-}\mathrm{cf}x = 0$, this is evident.

2): Assume $x \in \ell X - X$ and $0{-}\mathrm{cf}x = 1$. Let $< x(\alpha) : \alpha < 1 >$ be a (or the fixed) 0-normal sequence for x in ℓX. Since $x(0)$ is the last element of $(\leftarrow, x)_{\ell X}$, it suffices to show $x(0) \in X$. Assume, on the contrary, that $x(0) \in \ell X - X$. Then we have $(x(0), x)_{\ell X} \neq 0$ by Lemma 2.1. This is a contradiction.

3): Assume 0-cf$x \geq \omega$. Let $< x(\alpha) : \alpha < 0{-}\mathrm{cf}x >$ be a 0-normal sequence for x in ℓX.

Case 1. $0(X\Delta x)$ is unbounded in 0-cf$x$.

In this case, enumerate $(X\Delta x) = \{\alpha(\beta) : \beta < 0{-}\mathrm{cf}x.\}$ in the increasing order. Put $y(\beta) = x(\alpha(\beta))$ for each $\beta <$ 0-cf$x$. Then $< y(\beta) : \beta <$0-cf$x >$ is the desired sequence.

Case 2. $(X\Delta x)$ is bounded in 0-cf$x$.

In this case, take $\alpha_0 < 0{-}\mathrm{cf}x$ with $0(X\Delta x) \subset \alpha_0$. Since $x(\alpha_0+\alpha) \in \ell X - X$ for each $\alpha < 0{-}\mathrm{cf}x$. By Lemma 2.1 and the density of X in ℓX, pick $y(\alpha)$ in $(x(\alpha_0+\alpha), x(\alpha_0+\alpha+1))_{\ell X} \cap X$. Then $< y(\alpha) : \alpha < 0{-}\mathrm{cf}x >$ is the desired sequence.

Remark 3.6. Let X be the GO-space $[0,1) \cup [2,3]$ with the subspace topology of the reals. By Lemma 2.1, $\ell X = [0,1] \cup$ [2.3]. Observe that $2 \in X$ and $0{-}\mathrm{cf}2 = 1$, but there does not exist such a 0-unbounded sequence for 2 in the above lemma.

For a point x in the MLOC ℓX of a GO-space X, put cf$x$ =max$\{0-$cf$x, 1-$cf$x\}$. It is easy to show that the character and the tightness at x in X is equal to cf$x$, and cf$x \geq \omega$ holds for $x \in \ell X - X$.

4. Normality of $X \times \kappa$.

A space is $< \kappa$-paracompact if it is λ-paracompact for each cardinal $\lambda < \kappa$. The proof of the following lemma is analogous to that of [Ke, Theorem 4], so we left it to the reader.

Lemma 4.1. *Let X be a GO-space and κ an uncountable cardinal. Then X is $< \kappa$-paracompact if and only if, for each $x \in \ell X - X$ and $i \in 2, i(X \Delta x)$ is not stationary in i-cfx whenever $\omega < i-$cfx $< \kappa$.*

Remark 4.2. Note that all GO-spaces are normal and countably paracompact and that countable paracompactness is inversely preserved by quasi-perfect maps. So $X \times \kappa$ is countably paracompact if X is a GO-space and κ is a regular uncountable cardinal. Next we prove our main theorem.

Theorem 4.3. *Let X be a GO-space and κ a regular uncountable cardinal. Then $X \times \kappa$ is normal if and only if the following two conditions hold:*

a) $i-$cfx $\neq \kappa$ for each $x \in X$ and $i \in 2$,

b) X is $< \kappa$-paracompact.

Proof: "only if" part: Assume $X \times \kappa$ is normal. For each cardinal $\lambda < \kappa$, $X \times \kappa$ contains $X \times (\lambda + 1)$ as a closed subspace. Therefore, by Kunen's Theorem [Pr, Corollary 3.7], X is $< \kappa$-paracompact. To show a), assume that there are $x \in X$ and $i \in 2$ such that $i-$cf$x = \kappa$. Without loss of generality, we may assume $i = 0$. By Lemma 3.5, take a 0-unbounded sequence $< x(\alpha) : \alpha < \kappa >$ for x such that $x(\alpha) \in X$ for each $\alpha < \kappa$. Put $H(0) =$cl$\{< x(\alpha), \alpha >: \alpha < \kappa\}$ and $H(1) = \{x\} \times \kappa$, here cl denotes the closure. Then it is not difficult to show that $H(0)$ and $H(1)$ are disjoint closed sets. So, by the normality of $X \times \kappa$, there are disjoint open sets $U(0)$ and $U(1)$ such that $H(j) \subset U(j)$ for each $j \in 2$. Since $U(0)$ is open and $< x(\alpha), \alpha > \in U(0)$ for each $\alpha < \kappa$, there is $f(\alpha) < \alpha$ such that $\{x(\alpha)\} \times (f(\alpha), \alpha] \subset U(0)$. Then, by the Pressing Down Lemma (abbreviated as PDL), there are a stationary set $S \subset \kappa$ and $\alpha_0 < \kappa$ such that $f(\alpha) < \alpha_0 < \alpha$ for each $\alpha \in S$. Since $\{x(\alpha) : \alpha \in S\}$ is 0-unbounded for x, we have $< x, \alpha_0 > \in \mathrm{cl}\{< x(\alpha), \alpha_0 >: \alpha \in S\} \subset \mathrm{cl}\ U(0)$. This is a contradiction because of $< x, \alpha_0 > \in H(1) \subset U(1)$.

"if" part: Assume a) and b) hold, but $X \times \kappa$ is not normal. We shall obtain a contradiction. Then there are disjoint closed sets $H(0)$ and $H(1)$ which can not be separated by disjoint open sets in $X \times \kappa$. Put $Y(x, y) = [x, y]_X \times \kappa$ for x and y in X with $x \leq y$. For x and y in X define $x \simeq y$ by one of the following clauses.

1) $x = y$,

2) $x < y$, moreover $H(0) \cap Y(x, y)$ and $H(1) \cap Y(x, y)$ are separated by disjoint open sets in $Y(x, y)$,

3) $y < x$, moreover $H(0) \cap Y(y, x)$ and $H(1) \cap Y(y, x)$ are separated by disjoint open sets in $Y(y, x)$.

Then it is easy to show that $\simeq$ is an equivalence relation on X.

Let $X/\simeq$ be the set of all equivalence classes. Observe that each equivalence class is convex in X. We shall show that each equivalence class E in $X/\simeq$ is open in X. To show this, let x be a point in E. First we show:

Claim 1. *If* $0\text{-}cfx \neq 0$, *then there is* $a \in (\leftarrow, x)_{\ell X}$ *such that* $(a, x]_{\ell X} \cap X \subset E$.

Proof: Put $\lambda = 0\text{-cf}x$. When $\lambda = 1$, it is almost clear. So assume $\omega \leq \lambda$. Note that $\lambda \neq \kappa$ by a). Let $< x(\alpha) : \alpha < \lambda >$ be the 0-normal sequence for x. Since $H(0)$ and $H(1)$ are disjoint closed sets, there are $f(\beta) < \lambda$, $g(\beta) < \beta$ and $j(\beta) \in 2$ such that $((x(f(\beta)), x]_{\ell X} \cap X) \times (g(\beta), \beta] \cap H(j(\beta)) = 0$ for each $\beta < \kappa$. Then, by the PDL, there are a stationary set $S \subset \kappa, \beta_0 < \kappa$ and $j_0 \in 2$ such that $g(\beta) = \beta_0$ and $j(\beta) = j_0$ for each $\beta \in S$.

Subclaim. *There is* α_0 *such that* $((x(\alpha_0), x]_{\ell X} \cap X) \times (\beta_0, \kappa) \cap H(j_0) = 0$.

Proof: There are two cases.

Case 1. $\kappa < \lambda$.

In this case, put $\alpha_0 = \sup\{f(\beta) : \beta \in S\}$. Then it is not difficult to show that this α_0 works.

Case 2. $\omega \leq \lambda < \kappa$.

In this case, again applying the PDL to S and f, there are a stationary set $S' \subset S$ and $\alpha_0 < \lambda$ such that $f(\beta) = \alpha_0$ for each $\beta \in S'$. Then this α_0 is the desired one.

This completes the proof of subclaim.

Since X is $< \kappa$-paracompact and $\beta_0 < \kappa$, $X \times [0, \beta_0]$ is normal by Kunen's Theorem. So there are disjoint open sets $V(0)$ and $V(1)$ in $X \times [0, \beta_0]$ such that $H(j) \cap X \times [0, \beta_0] \subset V(j)$ for each $j \in 2$. To complete the proof of claim 1, pick a point y in $(x(\alpha_0), x]_{\ell X} \cap X$. Put $U(j_0) = V(j_0) \cap Y(y, x)$ and $U(1 - j_0) = [V(1-j_0) \cup ((x(\alpha_0), x]_{\ell X} \cap X) \times (\beta_0, \kappa)] \cap Y(y, x)$. Then $U(0)$ and $U(1)$ separates $H(0) \cap Y(y, x)$ and $H(1) \cap Y(y, x)$. Therefore we have $y \simeq x$, so $y \in E$. This completes the proof of claim 1.

Claim 2. *If 1-cfx* $\neq 0$, *then there is* $b \in (x, \rightarrow)_{\ell X}$ *such that* $[x, b)_{\ell X} \cap X \subset E$.

Furthermore observe that if $0-\text{cf}x = 0(1-\text{cf}x = 0)$, then $(\leftarrow, x]_{\ell X} \cap X = \{x\} \subset E([x, \rightarrow)_{\ell X} \cap X = \{x\} \subset E$, respectively). So E is open in X by this observation and claim 1,2. Therefore X is the free union of $X/\simeq$, so there is $E \in X/\simeq$ such that $H(0) \cap E \times \kappa$ and $H(1) \cap E \times \kappa$ can not be separated by disjoint open sets in $E \times \kappa$. Pick a point $y \in E$ and put $F = \{x \in E : y \leq x\}$ and $F' = \{x \in E : x \leq y\}$. Since $E \times \kappa$ is the union of the closed subspaces $F \times \kappa$ and $F' \times \kappa$, we may assume that

(*) $\quad H(0) \cap F \times \kappa$ and $H(1) \cap F \times \kappa$ can not be separated by disjoint open sets in $F \times \kappa$.

Put $x = \sup_{\ell X} F$. If $x \in F$, then we have $y,\ x \in E$ and $F = [y, x]_X$. This contradicts (*). Se we have $x \notin F$ and $F = [y, x)_{\ell X} \cap X$. Since X is the free union of $X \backslash \simeq$, we also have $x \in \ell X - X$. Put $\lambda = 0-$ cf$x$, then clearly we have $\omega \leq \lambda$. Assume $\lambda = \omega$. Then we easily have a contradiction to $(*)$, using 4) of Lemma 3.4. So we have $\lambda \geq \omega_1$. Let $< x(\alpha) : \alpha < \lambda >$ be a 0-normal sequence for x. We may assume that $y < x(0)$ by 2) of Lemma 3.3. Assume $0(X\Delta x)$ is not stationary in λ. Then similarly we also have a contradiction to $(*)$ using 3) of Lemma 3.4. So $0(X\Delta x)$ is stationary in λ. Therefore we have $\kappa \leq \lambda$ by $<\kappa$-paracompactness and Lemma 4.1. For each $\alpha \in 0(X\Delta x) \cap$ Lim(λ) and $\beta < \kappa$, fix $j(\alpha, \beta) \in 2$ with $< x(\alpha), \beta > \notin H(j(\alpha, \beta))$. Here Lim$(\lambda)$ denotes all limit ordinals less than λ. Furthermore fix $f(\alpha, \beta) < \alpha$ and $g(\alpha, \beta) < \beta$ such that $([x(f(\alpha, \beta)), x(\alpha)]_{\ell X} \cap X) \times (g(\alpha, \beta), \beta] \cap H(j(\alpha, \beta)) = 0$. There are two cases.

Case 1. $\lambda = \kappa$.

In this case, applying the PDL to $j(\alpha, \alpha)$, $f(\alpha, \alpha)$ and $g(\alpha, \alpha)$, we have a stationary set $S \subset 0(X\Delta x) \cap$ Lim(λ), $\alpha_0 < \lambda = \kappa$, $\beta_0 < \kappa$ and $j_0 \in 2$ such that $f(\alpha, \alpha) = \alpha_0$, $g(\alpha, \alpha) = \beta_0$ and $j(\alpha, \alpha) = j_0$ for each $\alpha \in S$. So $Z_0 = ([x(\alpha_0), x)_{\ell X} \cap X) \times (\beta_0, \kappa)$ is disjoint from $H(j_0)$. Since X is $<\kappa$-paracompact and F is closed in X, $Z_1 = F \times [0, \beta_0]$ is normal by Kunen's Theorem. Next put $Z_2 = ([y, x(\alpha_0)]_{\ell X} \cap X) \times \kappa$. Take $z \in F$ with $x(\alpha_0) < z$. Since Z_2 is a closed subspace of $[y, z]_X \times \kappa$ and $y \simeq z$, $H(0) \cap Z_2$ and $H(1) \cap Z_2$ can be separated by disjoint open sets. Since $F \times \kappa$ is the union of the closed subspaces Z_0, Z_1 and Z_2, $H(0) \cap F \times \kappa$ and $H(1) \cap F \times \kappa$ can be separated by disjoint open sets in $F \times \kappa$. This contradicts $(*)$.

Case 2. $\kappa < \lambda$.

In this case, for each $\beta < \kappa$, applying the PDL we have a stationary set $S(\beta) \subset 0(X\Delta x) \cap$ Lim(λ), $f(\beta) < \lambda$, $g(\beta) < \beta$ and $j(\beta) \in 2$ such that $f(\alpha, \beta) = f(\beta)$, $g(\alpha, \beta) = g(\beta)$ and $j(\alpha, \beta) = j(\beta)$ for each $\alpha \in S(\beta)$. Put $\alpha_0 = \sup\{f(\beta) : \beta < \kappa\}$. Then we have $\alpha_0 < \lambda$ by $\kappa < \lambda$. Again, applying the PDL to g and j, we have a stationary set $T \subset \kappa$, $\beta_0 < \kappa$ and $j_0 \in 2$ such that $g(\beta) = \beta_0$ and $j(\beta) = j_0$ for each $\beta \in T$. Then $Z_0 = ([x(\alpha_0), x)_{\ell X} \cap X) \times (\beta_0, \kappa)$ is disjoint from $H(j_0)$. Then, as in the above case, $H(0) \cap F \times \kappa$ and $H(1) \cap F \times \kappa$ can be separated by disjoint open sets in $F \times \kappa$. So this also contradicts $(*)$.

This completes the proof of the theorem.

Corollary 4.4. *Let X be a GO-space. Then $X \times \omega_1$ is normal if and only if $i - cfx \neq \omega_1$ for each $x \in X$ and $i \in 2$.*

Corollary 4.5. ([GNP]) *If X is a GO-space of countable tightness, then $X \times \omega_1$ is normal.*

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