Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
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E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Vol. 18, 1993

QUASI-UNIFORM SPACES — ELEVEN YEARS LATER

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ABSTRACT. We discuss results obtained in the last ten years that are related to problems posed in the book "[17] P. Fletcher and W.F. Lindgren, Quasi-uniform Spaces, Dekker, 1982". [1] The selection of the material has mainly been influenced by the interests of the author. In the appendix a recent question of G.C.L. Brümmer about functorial quasi-uniformities is answered.

In this paper no attempt is made to survey the work done in the area of quasi-uniform spaces during the past decade. We refer the reader to [2,24,26,33,38,40,41,51] for some further contributions to problems contained in "Quasi-uniform Spaces" which are not mentioned in this article because of lack of space.

1. PRELIMINARIES

In order to make this article accessible to the nonspecialist we begin by recalling some basic concepts of the theory of quasi-uniform spaces.

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[1] The present survey article is partially based on talks given by the author at the "Workshop di Topologia 1992", held at Vietri sul Mare, Salerno, Italy. Some further results on quasi-uniformities that we have presented at this meeting are dealt with in [42].

Partly this paper was written while the author was supported by the Swiss National Science Foundation under grant 21-30585.91.

Definition: Let X be a (nonempty) set. A filter $\mathcal{U}$ on $X \times X$ is called a *quasi-uniformity on* X provided that:
(i) Each member of $\mathcal{U}$ is a reflexive relation.
(ii) For each $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that $U^2 \subseteq V$.

(Here $U^2 = \{(x,y) \in X \times X$: There exists $z \in X$ such that $(x,z) \in U$ and $(z,y) \in U\}$.)

A quasi-uniformity $\mathcal{U}$ is called a *uniformity* if
(iii) $U \in \mathcal{U}$ implies $U^{-1} \in \mathcal{U}$, where $U^{-1} = \{(x,y) \in X \times X : (y,x) \in U\}$.

If $\mathcal{U}$ and $\mathcal{V}$ are quasi-uniformities on a set X and $\mathcal{U} \subseteq \mathcal{V}$, then $\mathcal{U}$ is called *coarser* than $\mathcal{V}$. A map f from a quasi-uniform space $(X,\mathcal{U})$ to a quasi-uniform space $(Y,\mathcal{V})$ is called *quasi-uniformly continuous* if $(f \times f)^{-1}(V) \in \mathcal{U}$ whenever $V \in \mathcal{V}$.

The *topology* $\mathcal{T}(\mathcal{U})$ *induced* by a quasi-uniformity $\mathcal{U}$ on X is $\{G \subseteq X$: For each $x \in G$ there exists $V \in \mathcal{U}$ such that $V(x) \subseteq G\}$. (Here $V(x) = \{y \in X : (x,y) \in V\}$ whenever $V \in \mathcal{U}$ and $x \in X$.)

Observations: The filter $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is a quasi-uniformity on X provided that $\mathcal{U}$ is a quasi-uniformity on X. Each collection of quasi-uniformities on a set X has a supremum and, thus, an infimum.

Given a quasi-uniformity $\mathcal{U}$ on X, $\mathcal{U}^*$ will denote the coarsest uniformity on X which is finer than $\mathcal{U}$. (Obviously, $\mathcal{U}^*$ is the filter on $X \times X$ generated by the filterbase $\{U \cap U^{-1} : U \in \mathcal{U}\}$.)

The topology induced by the supremum of a family of quasi-uniformities on a set X is equal to the supremum of the topologies induced by the members of the family on X.

Each quasi-uniformly continuous map between quasi-uniform spaces is continuous with respect to the induced topologies.

Definition: Let X be a (nonempty) set. A function d from $X \times X$ into the nonnegative reals is called a *quasi-pseudo-metric* if
(i) $d(x,x) = 0$ whenever $x \in X$, and
(ii) $d(x,y) \leq d(x,z) + d(z,y)$ whenever $x,y,z \in X$.

The filter generated by $\{U_\epsilon : \epsilon > 0\}$ on $X \times X$ is called the *quasi-pseudo-metric quasi-uniformity* $\mathcal{U}_d$ on X. (Here $U_\epsilon = \{(x,y) \in X \times X : d(x,y) < \epsilon\}$ whenever $\epsilon > 0$.)

The topology $\mathcal{T}(\mathcal{U}_d)$ is called the *topology* induced by d on X.

A quasi-pseudo-metric d is called a (T_1)-*quasi-metric* provided that for each $x, y \in X$, $d(x,y) = 0$ implies $x = y$.

A quasi-pseudo-metric d is called *non-archimedean* if it satisfies $d(x,y) \leq \max\{d(x,z), d(z,y)\}$ whenever $x, y, z \in X$.

Remarks: Let d be a quasi-pseudo-metric on a set X. Then d^{-1} (defined by $d^{-1}(x,y) = d(y,x)$ whenever $x, y \in X$) is also a quasi-pseudo-metric on X. Little is known about connections between $\mathcal{T}(\mathcal{U}_d)$ and $\mathcal{T}(\mathcal{U}_{d^{-1}})$. It is shown in [25] that both topologies have the same weight and that the height of $\mathcal{T}(\mathcal{U}_d)$ is equal to the width of $\mathcal{T}(\mathcal{U}_{d^{-1}})$.[2]

Example 1. *(a) On the set $X = \boldsymbol{R}$ of reals let $d(x,y) = 1$ if $y < x$, and $d(x,y) = y - x$ if $y \geq x$. Then (X,d) is a quasi-metric space. It is called the* Sorgenfrey line.

(b) Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be the collection of all reflexive transitive relations V on X for which $V(x) \in \mathcal{T}$ for all $x \in X$. Then $\mathcal{B}$ is a (filter)base for the so-called fine transitive quasi-uniformity $\mathcal{U}$ *of X. (This quasi-uniformity is* compatible *with the topology of X, i.e. $\mathcal{T}(\mathcal{U}) = \mathcal{T}$.)*

(c) Let X be a topological space. The filter generated by $\{[G \times G] \cup [(X \setminus G) \times X] : G$ is open in $X\}$ on $X \times X$ is the so-called Pervin quasi-uniformity *of X. It is also compatible with the topology of X.*

(d) Let $\mathcal{L}$ be the quasi-uniformity on the set $\boldsymbol{R}$ of real numbers generated by the base $\{Q_\epsilon : \epsilon > 0\}$ where $Q_\epsilon = \{(x,y) \in \boldsymbol{R} \times \boldsymbol{R} : x - y < \epsilon\}$ whenever $\epsilon > 0$. By definition, the

[2]The *height* of a topological space $(X, \mathcal{T})$ is equal to $\sup\{L(\mathcal{T}|A) : A \subseteq X\}$ and the *width* is equal to $\sup\{d(\mathcal{T}|A) : A \subseteq X\}$. Here $L(\mathcal{T}|A)$ ($d(\mathcal{T}|A)$, respectively) denotes the Lindelöf degree (the density) of the subspace A of X.

semi-continuous quasi-uniformity $\mathcal{SC}$ *of a topological space* X *is the coarsest quasi-uniformity on* X *for which each continuous function* $f : X \to (\boldsymbol{R}, \mathcal{L})$ *is quasi-uniformly continuous. It is established in [17] that* $\mathcal{SC}$ *is finer than the Pervin quasi-uniformity, but coarser than the fine transitive quasi-uniformity of* X.

The constructions (b), (c) and (d) show that each topological space is quasi-uniformizable, *i.e. it admits a compatible quasi-uniformity.*

It seems hard to find an elegant (topological) characterization of those topologies that admit a compatible quasi-pseudometric. It is known that these topologies are exactly those that can be induced by a local quasi-uniformity[3] $\mathcal{U}$ with a countable base having the property that $\mathcal{U}^{-1}$ is a local quasi-uniformity, too (see e.g. [17,25]). For a recent study of that problem we refer the interested reader to [24].

2. Basic results

The results mentioned in this section answer some natural questions that have been left open in [17]. By $\boldsymbol{N}$ we shall denote the set of positive integers.

Proposition 1. [31,35] (positive solution to Problem B (b) in [17]): *A topological space* X *admits a unique quasi-uniformity if and only if (1) it does not have any strictly increasing sequence* $(G_n)_{n\in\boldsymbol{N}}$ *of open sets (i.e. it is hereditarily compact) and (2) there does not exist any strictly decreasing sequence* $(G_n)_{n\in\boldsymbol{N}}$ *of open sets such that* $\bigcap_{n\in\boldsymbol{N}} G_n$ *is open. (Equivalently, such a space* X *is characterized by the property that each open collection* $\mathcal{B}$ *of* X *such that* $\bigcap\mathcal{B}'$ *is open for any subcollection* $\mathcal{B}'$ *of* $\mathcal{B}$ *is finite.)*

[3] A filter $\mathcal{U}$ of reflexive binary relations on a (nonempty) set X is called a *local quasi-uniformity* provided that for any $x \in X$ and $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2(x) \subseteq U(x)$.

In [35] it is shown that this property of topological spaces is finitely productive. Typical examples of topological spaces admitting a unique quasi-uniformity are the cofinite topologies on uncountable sets and the hereditarily compact quasi-sober (in particular, the finite) topologies.[4]

Definition: [17] A quasi-uniformity $\mathcal{U}$ on a set X is called ***totally bounded*** provided that for each $U \in \mathcal{U}$ there exists a finite cover $\mathcal{A}$ of X such that $A \times A \subseteq U$ whenever $A \in \mathcal{A}$ (equivalently, if the (quasi)-uniformity $\mathcal{U}^*$ is *precompact*, i.e. for each $U \in \mathcal{U}^*$ there is a finite subset F of X such that $U(F) = X$).

Observations: The Pervin quasi-uniformity of any topological space X is totally bounded. In fact [17,p.28], it is the finest compatible totally bounded quasi-uniformity on X.

A quasi-uniformity $\mathcal{U}$ is totally bounded if and only if both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are hereditarily precompact [37, Lemma 1.1].

Examples: [43,46] (a) Let $X = \boldsymbol{N}$ and for $x, y \in X$, set $d(x,y) = 0$ if $x = y$, $d(x,y) = \frac{1}{x}$ if x is even and $x < y$, $d(x,y) = \frac{1}{y}$ if y is odd and $x > y$, and $d(x,y) = 1$ otherwise. Then both the quasi-metric quasi-uniformities $\mathcal{U}_d$ and $\mathcal{U}_{d^{-1}}$ are precompact, but $\max\{d, d^{-1}\}$ is the discrete metric on X.

(b) Let $X = \boldsymbol{N} \cup \{\infty\}$ equipped with its usual order. For $x, y \in X$, set $d(x,y) = 0$ if $x = y$, $d(x,y) = \frac{1}{n}$ if $x = n, n \in \boldsymbol{N}$ and $x < y$, and $d(x,y) = 1$ otherwise. The metric $\max\{d, d^{-1}\}$ is discrete, although $\mathcal{U}_d$ is hereditarily precompact.

Hereditarily precompact quasi-uniformities have been studied in [43]. Among other things it is shown that a (nonempty) product of quasi-uniform spaces is hereditarily precompact if and only if each factor space is hereditarily precompact and

[4] A nonempty topological space X is called ***irreducible*** if each pair of nonempty open sets of X has a nonempty intersection. A topological space is called ***quasi-sober*** if the only irreducible closed subspaces are the closures of singletons. A quasi-sober T_0-space is called ***sober***.

that each regular hereditarily precompact quasi-pseudo-metric space is second countable.

Proposition 2. [27] (negative solution to Problem B (a) in [17]): *A topological space X admits a unique totally bounded quasi-uniformity if and only if its topology $\mathcal{T}$ is the unique base of open sets for $\mathcal{T}$ that is closed under finite unions and finite intersections. (Here we use the convention that $\bigcap\emptyset = X$.)*

Typical examples of topological spaces admitting a unique totally bounded quasi-uniformity are the hereditarily compact spaces and the set ω_0 equipped with the lower topology $\{[0,n] : n \in \omega_0\} \cup \{\emptyset, \omega_0\}$.

The space with carrier set $\omega_0 + 2$ and topology $\{[0,n] : n \in \omega_0\} \cup \{(\omega_0+2)\setminus\{\omega_0+1\}, \omega_0+2, (\omega_0+2)\setminus\{\omega_0\}, \omega_0, \emptyset\}$ admits a unique totally bounded quasi-uniformity, while this is not true for its subspace $(\omega_0 + 2) \setminus \{\omega_0\}$.

It is known that a topological space is hereditarily compact if and only if it admits a unique totally bounded quasi-uniformity and each of its ultrafilters has an irreducible limit set [5, Proposition 4]. The following elementary problem however remains open:

Problem 1 [35] *Is each topological T_1-space X whose topology is the unique base of open sets that is closed under finite unions and finite intersections hereditarily compact?*

It is shown in [35, Proposition 2.4] that Problem 1 has a positive answer provided that each point of X is a G_δ-set.

A basic result ([17, Theorem 1.33]): Each compatible quasi-uniformity $\mathcal{U}$ on a topological space contains a coarser compatible totally bounded quasi-uniformity. In fact there is a finest quasi-uniformity of this kind, namely the supremum of the family of totally bounded quasi-uniformities coarser than $\mathcal{U}$. (The supremum of a family of totally bounded quasi-uniformities is totally bounded.)

Question [17, p. 19]: Which topological spaces admit a coarsest compatible quasi-uniformity? The following characterization of these spaces has been obtained.

Definition 1. [32] *Let $(X,\mathcal{T})$ be a topological space and let $G_1, G_2 \in \mathcal{T}$. We write $G_1 < G_2$ if for each ultrafilter $\mathcal{G}$ on X containing G_1 there exists a finite collection $\mathcal{M}$ of open sets of X such that each element of $\mathcal{M}$ contains a limit point of $\mathcal{G}$ and $\cap\mathcal{M} \subseteq G_2$. (Conventions: $\bigcap\emptyset = X$. If $G_1 < G_2$, then we say that G_1 is* handy *in G_2 (with respect to X).)*

Remark: [32] A topological space admits a unique totally bounded quasi-uniformity if and only if each of its open sets is handy in itself.

Proposition 3. [29,32] *A topological space X admits a coarsest quasi-uniformity if and only if its handy-relation is* approximating *(i.e. for each open set G of X we have $G = \bigcup\{G' : G' < G$ and G' is open in $X\})$.*

Remark: [32] It is known that this property is closed-hereditary, but not open-hereditary and that each core-compact space[5] satisfies the condition. If a quasi-uniformity is minimal among the compatible quasi-uniformities on a topological space X, then it is necessarily the coarsest compatible quasi-uniformity on X.

A sober T_1-space that is not core-compact, but has an open base consisting of sets that are handy in themselves (and thus admits a coarsest quasi-uniformity) is constructed in [32].

A topological space that admits a coarsest quasi-uniformity and in which each convergent ultrafilter has an irreducible limit set is core-compact (compare [32, Corollary 3]).

In particular, each locally compact space[6] X admits a coars-

[5] A topological space is called *core-compact* if each open set is the union of open sets that are relatively compact in it. An open set G is called *relatively compact* in an open set B if each open cover of B contains a finite subcollection covering G.

[6] A topological space is called *locally compact* provided that each point has a neighborhood base consisting of compact sets.

est quasi-uniformity $\mathcal{U}$. It is generated by the subbase $\{[(X \setminus K) \times X] \cup [X \times G] : K \subseteq G, K$ is compact and G is open in $X\}$. The topology $\mathcal{T}(\mathcal{U}^{-1})$ is generated by the sets $X \setminus K$ where K is compact and saturated[7] in X [29]. The coarsest quasi-uniformity of a locally compact Hausdorff space X is a uniformity if and only if X is compact [17, Proposition 1.47].

Some well-known results on locally compact spaces can be generalized to the class of topological spaces admitting a coarsest quasi-uniformity. For instance it is shown in [32] that a (nonempty) product of topological spaces admits a coarsest quasi-uniformity if and only if each factor space admits a coarsest quasi-uniformity and all but finitely many factor spaces are compact.

3. Transitivity

Definition: [17] A quasi-uniformity is called *transitive* if it has a base consisting of transitive relations. A topological space X is called *transitive* if the finest compatible quasi-uniformity on X is transitive (i.e. the fine quasi-uniformity of X is the fine transitive quasi-uniformity of X).

It is known that the semi-continuous quasi-uniformity of any topological space is transitive [17, Corollary 2.13].

Example 2. *(The Kofner plane) [17] Let $X = \boldsymbol{R}^2$. For each $x \in X$ and $\epsilon > 0$ let $C(x,\epsilon)$ be the (closed) disk of radius ϵ lying above the horizontal line through x and tangent to this line at x. For $x, y \in X$ define $d(x,y)$ as follows: Set $d(x,y) = 1$ if $y \notin C(x,1)$, set $d(x,y) = r$ if $r \leq 1$, $y \in C(x,r)$ and $y \notin C(x,s)$ for all $s < r$, and set $d(x,y) = 0$ if $x = y$.*

An application of the Baire Category Theorem shows that the quasi-metric space (X,d) is not non-archimedeanly quasi-metrizable. Hence it cannot be transitive, since this would imply that it admits a quasi-uniformity with a countable base of transitive relations.

[7] A set in a topological space is called *saturated* if it is equal to the intersection of its open neighborhoods.

Essentially, Kofner's approach is the only method known to construct nontransitive spaces [35].

Problem 2. *(a) Characterize the nontransitive subspaces of the Kofner plane.*

(b) Is there in ZFC a nontransitive T_1-space of cardinality $\aleph_1$. (It is known that a countable union of closed transitive spaces is transitive [17, Theorem 6.15].)

Example 3. [28] (partial solution to Problem M in [17]): *There exists a transitive space that is the union of two subspaces homeomorphic to the (nontransitive) Kofner plane. Let* $X = (\boldsymbol{R}^2 \times \{1\}) \cup (\boldsymbol{R}^2 \times \{-1\})$. *For each* $r \in \boldsymbol{R}^2$ *and each* $n \in \boldsymbol{N}$ *let* $S_n(r)$ *be the open disk of radius* 2^{-n} *lying above the horizontal line through* r *and tangent to this line at* r. *Similarly,* $S_n^{-1}(r)$ *will denote the open disk of radius* 2^{-n} *lying below the horizontal line through* r *and tangent to this line at* r. *Construct a base for a topology on* X *by defining for each* $r \in \boldsymbol{R}^2, n \in \boldsymbol{N}$ *and* $i \in \{1, -1\}$ *basic open neighborhoods* $K_n(r, i)$ *at the point* (r, i), *where*

$$K_n(r, 1) = [(S_n(r) \cup \{r\}) \times \{1\}] \cup [S_n(r) \times \{-1\}]$$

and

$$K_n(r, -1) = [S_n^{-1}(r) \times \{1\}] \cup [(S_n^{-1}(r) \cup \{r\}) \times \{-1\}].$$

Obviously, the subspaces $\boldsymbol{R}^2 \times \{1\}$ *and* $\boldsymbol{R}^2 \times \{-1\}$ *of* X *are homeomorphic to the Kofner plane. However* X *can be shown to be transitive.*

Results [17]: (a) Every subspace of a transitive space that is the intersection of an open set and a closed set is transitive.

(b) (Junnila) Orthocompact[8] semi-stratifiable[9] spaces are transitive. (In particular each metric space is transitive.)

(c) (Kofner) Each generalized ordered space[10] is transitive.

(d) (Kofner) Each T_1-space with an ortho-base[11] is transitive.

Example 4. [28] (solution to Problem N in [17]): *Define a topology* $\mathcal{K}'$ *on* $\boldsymbol{R}^2$ *with basic neighborhoods* $K'_n(r) = S_n^{-1}(r) \cup \{r\} \cup S_n(r)$ $(n \in \boldsymbol{N})$ *at the point* $r \in \boldsymbol{R}^2$. *Then the space* $(\boldsymbol{R}^2, \mathcal{K}')$ *is a semi-stratifiable orthocompact space, and hence transitive. Let* $\mathcal{S}$ *denote the Sorgenfrey topology on* $\boldsymbol{R}$. *Since generalized ordered spaces are transitive,* $(\boldsymbol{R}, \mathcal{S})$ *is transitive. The space* $(\boldsymbol{R}^3, \mathcal{S} \times \mathcal{K}')$ *is not transitive: The plane* $P = \{(x,y,z) \in \boldsymbol{R}^3 : x = z\}$ *is closed in* $(\boldsymbol{R}^3, \mathcal{S} \times \mathcal{K}')$. *However as a subspace of* $(\boldsymbol{R}^3, \mathcal{S} \times \mathcal{K}')$, P *is homeomorphic to the Kofner plane. Since transitivity is a closed-hereditary property, we conclude that* $(\boldsymbol{R}^3, \mathcal{S} \times \mathcal{K}')$ *is not transitive.*

Many questions about transitive spaces that are easy to state remain open.

Problem 3. [17, Problem M] *Are compact Hausdorff spaces transitive?*

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[8]Let X be a topological space. A relation $V \subseteq X \times X$ is called a *neighbornet* on X if for each $x \in X$ the set $V(x) = \{y \in X : (x,y) \in V\}$ is a neighborhood at x. A neighbornet is called *transitive* if it is a transitive relation. A topological space X is called *(countably) orthocompact* provided that each (countable) open cover of X has a refinement $\{T(x) : x \in X\}$ where T is a transitive neighbornet of X.

[9]A topological space X is called *semi-stratifiable* provided that to each closed set E of X one can assign a sequence $(E_n)_{n \in \boldsymbol{N}}$ of open subsets of X so that $\bigcap_{n \in \boldsymbol{N}} E_n = E$ and for each $n \in \boldsymbol{N}$, $E_n \subseteq F_n$ whenever $E \subseteq F$.

[10]A *generalized ordered space* is a triple $(X, \mathcal{T}, \leq)$ in which $\leq$ is a linear order on X and $\mathcal{T}$ is a topology on X such that the open-interval topology of $\leq$ is coarser than $\mathcal{T}$ and for each $x \in X$ the $\mathcal{T}$-neighborhood filter of x has a base consisting of (possibly degenerate) intervals.

[11]A base $\mathcal{B}$ of a topological space is called an *ortho-base* if whenever $\mathcal{B}'$ is a subset of $\mathcal{B}$, $x \in \bigcap \mathcal{B}'$ and $x \notin \operatorname{int} \bigcap \mathcal{B}'$, then $\mathcal{B}'$ is a neighborhood base at x.

Remark: [28] (van Douwen) There exists a counterexample under $b = c$, in particular under CH.

[35] Hereditarily compact spaces are transitive.

Problem 4. [17, Problem P] *Are non-archimedeanly quasi-(pseudo)-metrizable spaces transitive?*

Remark: It seems not to be known whether each topological space with a σ-disjoint base is transitive. However, it is shown in [35] that each non-archimedeanly quasi-pseudo-metrizable space with a σ-locally finite network[12] is transitive. Moreover, topological spaces with a countable network are transitive [35].

Problem 5. *(a)* [17, Problem R] *Are quasi-metrizable Moore*[13] *spaces non-archimedeanly quasi-metrizable?*

(b) [17, Problem R] *Are Moore spaces transitive?*

(c) Are monotonically normal[14] *spaces transitive?*

(d) [23] *Is the compact open (continuous) image of any transitive space transitive?*

Problem 6. [31] *Is each locally compact* γ-space[15] *(non-archimedeanly) quasi-metrizable?*

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[12]A collection $\mathcal{B}$ of subsets of a topological space X is a *network* if for each open set G of X and each point $x \in G$ there is $B \in \mathcal{B}$ with $x \in B \subseteq G$.

[13]A T_1-space X is called *developable* if there is a sequence $(\mathcal{G}_n)_{n \in N}$ of open covers of X such that if $x \in X$ and G is an open set containing x then there is an $n \in N$ such that $\mathrm{st}(x, \mathcal{G}_n) \subseteq G$. A regular developable space is called a *Moore space*.

[14]A T_1-space X is said to be *monotonically normal* if to each pair (H, K) of disjoint closed subsets of X, one can assign an open set $D(H, K)$ such that

(i) $H \subseteq D(H, K) \subseteq \overline{D(H, K)} \subseteq X \setminus K$;

(ii) if $H \subseteq H'$ and $K \supseteq K'$, then $D(H, K) \subseteq D(H', K')$.

[15]A topological T_1-space is called a *γ-space* if it possesses a compatible local quasi-uniformity with a countable base. Each quasi-metrizable space is a γ-space. The converse does not hold (see e.g. [18]).

Results: [31] Each regular submetalindelöf[16] locally compact γ-space is a non-archimedeanly quasi-metrizable Moore space.

[31] Each locally compact zero-dimensional γ-space is non-archimedeanly quasi-metrizable.

Problems about (non-archimedeanly) quasi-metrizable spaces often have more or less obvious generalizations to higher cardinals. Sometimes those questions also lead to important problems in the theory of bispaces. We are going to give a few examples next.

Definition 2. [6] *For a quasi-uniformity $\mathcal{U}$ call* $w(\mathcal{U}) = \min\{card(\mathcal{B}) : \mathcal{B}$ *is a filterbase of* $\mathcal{U}\} + \omega_0$ *the* weight *of* $\mathcal{U}$. *Given an arbitrary topological space* X *define the* transitivity degree *of* X *as* $tq(X) = \min\{w(\mathcal{U}) : \mathcal{U}$ *is a compatible transitive quasi-uniformity on* $X\}$ *and the* quasi-uniform weight *of* X *as* $q(X) = \min\{w(\mathcal{U}) : \mathcal{U}$ *is a compatible quasi-uniformity on* $X\}$.

Remarks: [6] For any compact T_1-space X, the weight of X is equal to $q(X)$. A T_1-space X is quasi-metrizable if and only if $q(X) = \omega_0$. A T_1-space X is non-archimedeanly quasi-metrizable if and only if $tq(X) = \omega_0$. Each transitive space X satisfies $q(X) = tq(X)$.

Proposition 4. [6] *For each infinite cardinal* β *the exists a quasi-metrizable space* X_β *such that* $tq(X_\beta) > \beta$.

Remark: [6,19] We note that it is an open problem whether for any infinite cardinal m each completely regular topological space that admits a local uniformity with a base of cardinality m also admits a uniformity with a base of cardinality m. In [6] a corresponding problem for (local) quasi-uniformities has been dealt with.

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[16]A topological space X is called *submetalindelöf* provided that each open cover of X has a sequence $(\mathcal{G}_n)_{n\in N}$ of open refinements such that for each $x \in X$ there is an $n \in N$ with $\mathrm{ord}(x, \mathcal{G}_n) \leq \omega_0$.

Definition: (e.g.[39]) A *bispace* $(X, \mathcal{T}, \mathcal{S})$ consists of a (non-empty) set X equipped with two topologies $\mathcal{T}$ and $\mathcal{S}$. It is called *completely regular* if there exists a quasi-uniformity $\mathcal{U}$ on X such that $\mathcal{T}(\mathcal{U}) = \mathcal{T}$ and $\mathcal{T}(\mathcal{U}^{-1}) = \mathcal{S}$. (One says that $\mathcal{U}$ is *compatible* with the topologies of X.) A completely regular bispace is called *strongly zero-dimensional* if its finest compatible totally bounded quasi-uniformity is transitive. It is observed in [39, Proposition 4] that the finest compatible quasi-uniformity on a non-archimedeanly quasi-pseudo-metrizable bispace is transitive.

Problem 7. [39] *(a) Characterize those strongly zero-dimensional bispaces whose finest compatible quasi-uniformity is transitive.*

(b) Let X be a strongly zero-dimensional quasi-pseudo-metrizable bispace such that both its topologies are non-archimedeanly quasi-pseudo-metrizable. Is X non-archimedeanly quasi-pseudo-metrizable?

By [39, Proposition 3], the finest compatible quasi-uniformity on a strongly zero-dimensional bispace $(X, \mathcal{P}, \mathcal{Q})$ is transitive if each locally finite collection of open sets in the topological space $(X, \mathcal{P} \vee \mathcal{Q})$ is countable. In the light of the next example, part (a) of Problem 7 is not expected to have a simple solution.

Example 5. (e.g. [39]) *Let X be a topological space and let $\mathcal{P}$ be the Pervin quasi-uniformity of X. Then the finest compatible quasi-uniformity on the strongly zero-dimensional bispace $(X, \mathcal{T}(\mathcal{P}), \mathcal{T}(\mathcal{P}^{-1}))$ is transitive if and only if X is transitive.*

The answer to part (b) of Problem 7 is known to be positive provided that both topologies of X have a σ-point-finite base [39, Proposition 5].

Some open questions on transitive quasi-uniformities are related to basic problems in the theory of open coverings of topological spaces. The following problem is well known.

Problem 8. [17, Problem I, p. 102] *Is each countably orthocompact σ-orthocompact*[17] *topological space orthocompact?*

Some partial results: [17, Propositions 5.9 and 5.13] Each countably metacompact σ-orthocompact space is orthocompact. Hence every separable countably orthocompact σ-orthocompact T_1-space is orthocompact.

[34] Each regular weakly Lindelöf [18] countably point-star preorthocompact[19] space is countably metacompact.

[34] The subspace X' of the non-isolated points of an almost preorthocompact[20] space X with a G_δ-diagonal is countably metacompact.

[34] A normal almost preorthocompact space with a G_δ-diagonal is countably paracompact.

Each σ-orthocompact P-space[21] is orthocompact.

4. Completeness

At present there does not exist a generally accepted notion of completeness for quasi-uniform spaces.

Maybe the most appealing [1,22,49] definition of the concept of a Cauchy sequence in a quasi-pseudo-metric space is the following: A sequence $(x_n)_{n \in N}$ in a quasi-pseudo-metric space (X, d) is called *left K-Cauchy* if for each $\epsilon > 0$ there is $k \in N$

[17]A topological space X is called *σ-orthocompact* provided that for any open cover $\mathcal{C}$ of X there exists a sequence $(T_n)_{n \in N}$ of transitive neighbornets of X such that for each $x \in X$ there are an $n \in N$ and a $C \in \mathcal{C}$ with $T_n(x) \subseteq C$.

[18]A topological space X is called *weakly Lindelöf* provided that each open cover $\mathcal{C}$ of X has a countable subfamily whose union is dense in X.

[19]A topological space X is called *countably point-star preorthocompact* provided that for each countable open cover $\mathcal{C}$ of X there is an open neighbornet V of X such that $V^2(x) \subseteq \mathrm{st}(x, \mathcal{C})$ whenever $x \in X$.

[20]A topological space X is called *almost preorthocompact (= almost 2-fully preorthocompact* [30]) if for each open cover $\mathcal{C}$ of X there is a neighbornet V of X such that if $x \in X$, $a \in V^2(x)$ and $b \in V(x)$, then $\{a, b\} \subseteq C$ for some $C \in \mathcal{C}$.

[21]A topological space is called a *P-space* provided that the intersection of any countable collection of open sets is open.

such that $d(x_m, x_n) < \epsilon$ for all $m, n \in \boldsymbol{N}$ with $k \leq m \leq n$. (The corresponding concept for nets has been studied, too (see e.g. [54]; for filters also compare [49,42]).)

Unfortunately there are convergent sequences in quasi-pseudo-metric spaces that do not satisfy the stated condition. In fact, a regular quasi-metric space in which each convergent sequence has a left K-Cauchy subsequence is metrizable [43, Proposition 4]. Nevertheless with the help of that notion of a Cauchy sequence some classical theorems about complete metric spaces generalize satisfactorily to the quasi-metric setting. For instance, in [14] a Baire category theorem is derived and in [43, Theorem 2] it is shown that a quasi-pseudo-metric space is compact if and only if it is precompact and each left K-Cauchy sequence converges. Note also that by [43, Theorem 3] a quasi-pseudo-metric space is hereditarily precompact if and only if each of its sequences has a left K-Cauchy subsequence.

We restrict our attention in the following to some completeness properties considered in [17]. But we would like to stress that several very interesting completion theories for quasi-uniform spaces have been developed in the last years [7,8,9,10,11,52,53] none of which is discussed in this paper.

Definition: [17] A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called a *($\mathcal{U}$-)Cauchy filter* provided that for each $U \in \mathcal{U}$ there exists $x \in X$ such that $U(x) \in \mathcal{F}$. A quasi-uniform space $(X,\mathcal{U})$ is called *complete* if each Cauchy filter has a cluster point in X.

A sequence $(x_n)_{n \in \boldsymbol{N}}$ in a quasi-pseudo-metric space (X,p) is said to be a *(left p-)Cauchy sequence* if its elementary filter is a $\mathcal{U}_p$-Cauchy filter. A quasi-pseudo-metric space (X,p) is called *sequentially complete* if every Cauchy sequence has a cluster point in X. It is said to be *complete* if the quasi-uniformity $\mathcal{U}_p$ is complete.

Remarks: These definitions are compatible with the usual ones in uniform and metric spaces.

By [38, Proposition 4 and Lemma 5] each Tychonoff sequentially complete quasi-metric space is Čech complete and has a base of countable order. [22]

There are (zero-dimensional) sequentially complete quasi-metric spaces that are not complete [38, Example 4], but such examples cannot have property wD [23] [38, Proposition 2]. On the other hand, by [38, Proposition 1], each Cauchy filter converges in a Hausdorff quasi-metric space all Cauchy sequences of which are convergent.

In [16] it is observed that the topological property of admitting a complete quasi-uniformity is closed-hereditary, productive and preserved under perfect continuous surjections. The fine (transitive) quasi-uniformity of each (weakly) orthocompact and each regular almost realcompact[24] space is known to be complete [15,16].

Question [17, Problem C]: Is the finest compatible quasi-uniformity on a topological space always complete? The answer to this question has turned out to be negative.

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[22] A T_1-space X has a *base of countable order* if and only if there is a sequence $(\mathcal{B}_n)_{n\in N}$ of bases for X that satisfies the following condition: Whenever $x \in X$ and $(b_n)_{n\in N}$ is a decreasing sequence of subsets of X such that $x \in b_n \in \mathcal{B}_n$ for each $n \in N$, then $\{b_n : n \in N\}$ is a neighborhood base at x.

A Tychonoff space X is *Čech complete* if there exists a countable family $\{\mathcal{G}_n : n \in N\}$ of open covers of X such that whenever $\mathcal{F}$ is a family of closed subsets of X that has the finite intersection property and that contains for each $n \in N$ a member F_n with $F_n \subseteq G_n$ for some $G_n \in \mathcal{G}_n$, then $\mathcal{F}$ has a nonempty intersection.

[23] A subset B of a topological space X is called a *discrete* set in X if the family $\{\{b\} : b \in B\}$ is discrete. A topological space X is said to *have property wD* if for each countably infinite discrete set B in X there are a countably infinite subset C of B and a discrete collection $\{G_c : c \in C\}$ of open subsets of X such that $G_c \cap C = \{c\}$ for each $c \in C$.

[24] A topological space is called *almost realcompact (closed complete)* if each open (closed) maximal filter $\mathcal{F}$ without cluster point has a countable subcollection $\mathcal{F}'$ such that $\bigcap\{\overline{F} : F \in \mathcal{F}'\} = \emptyset$.

Proposition 5. [16, Corollary 4.1] *An ω-bounded*[25] *weakly Lindelöf regular space that admits a complete quasi-uniformity is compact.*

Example 6. [16] *Let $D = \{0,1\}$ equipped with the discrete topology. Set $P = D^{\mathbf{R}}$ and $\Sigma_0 = \{(x_i)_{i\in\mathbf{R}} \in P : x_i \neq 0$ for at most countably many $i \in \mathbf{R}\}$. It is known that Σ_0 is weakly Lindelöf and ω-bounded. Since $\beta\Sigma_0 = P$, Σ_0 is not compact. Hence Σ_0 does not admit a complete quasi-uniformity.*

Another useful criterion to show that certain spaces do not admit complete quasi-uniformities is established in [34, Proposition 3.2]. The following problem however remains still open.

Problem 9. [13] *Is the fine quasi-uniformity of each (regular) quasi-pseudo-metrizable space complete?*

Definition: [17] A quasi-uniform space $(X,\mathcal{U})$ is called *bicomplete* provided that the uniformity $\mathcal{U}^*$ is complete.

Example: [13, Example 4] Let κ be an uncountable regular cardinal (equipped with its order topology) and let $\mathcal{U}$ be the supremum quasi-uniformity of all quasi-pseudo-metric quasi-uniformities $\mathcal{U}_p$ on κ such that $\mathcal{T}(\mathcal{U}_p)$ is coarser than the topology of κ and such that the pseudo-metric $\max\{p, p^{-1}\}$ generates on κ a topology of density strictly smaller than κ. Then $\mathcal{U}$ is a compatible quasi-uniformity on κ that is not bicomplete.

Proposition 6. [13] *A topological space admits a bicomplete quasi-uniformity if and only if its fine quasi-uniformity is bicomplete.*

[5,20] *The Pervin quasi-uniformity of a topological space X is bicomplete if and only if X is hereditarily compact and quasi-sober.*

[13] *The fine transitive quasi-uniformity of any quasi-sober, any countable or any first-countable T_1-space is bicomplete.*

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[25] A topological space X is called *ω-bounded* provided that each countable subset of X has a compact closure in X.

The fine quasi-uniformity of any quasi-pseudo-metrizable space is bicomplete.

[44] *A quasi-pseudo-metric space that admits only bicomplete quasi-pseudo-metrics is hereditarily compact and quasi-sober (and thus admits a unique quasi-uniformity and has a countable topology).*

Remarks: [13, Proposition 7] The fine quasi-uniformity of the cofinite topology on an uncountable set is not bicomplete.

Those topological spaces that have a bicomplete semicontinuous quasi-uniformity are characterized in [13, Proposition 8]. Among other things it is proved in [13] that the semicontinuous quasi-uniformity of a quasi-sober hereditarily countably metacompact space X is bicomplete if and only if X is hereditarily closed-complete. Furthermore the semicontinuous quasi-uniformity of any hereditarily realcompact completely regular space is shown to be bicomplete.

Problem 10. *(a)* [13,16] *Is the fine quasi-uniformity of a topological space complete (bicomplete) if and only if its fine transitive quasi-uniformity is complete (bicomplete)?*

(b) [50] *Which quasi-pseudo-metrizable spaces admit bicomplete quasi-pseudo-metrics?*

Example: [44] The Euclidean topology on the rationals $\boldsymbol{Q}$ admits a bicomplete quasi-pseudo-metric, while this is not true for the lower topology on $\boldsymbol{Q}$.

The concept of a bicomplete quasi-uniform space is quite important, because the corresponding well-developed completion theory extends the known theory for uniform spaces in a straightforward way. Furthermore the generalized theory has applications in various areas of mathematics. It also plays a decisive role in the completion theory of the so-called topological quasi-uniform spaces introduced recently by M.B. Smyth and Ph. Sünderhauf [53,54].

5. BICOMPLETION AND TOPOLOGICALLY ORDERED SPACES

All topological and quasi-uniform spaces considered in this section are T_0-spaces.

A basic result: [17] Each quasi-uniform space $(X,\mathcal{U})$ has an (up to quasi-uniform isomorphism) unique bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$ in the sense that the space $(\widetilde{X},\widetilde{\mathcal{U}})$ is a bicomplete extension of $(X,\mathcal{U})$ in which $(X,\mathcal{U})$ is $\mathcal{T}(\widetilde{\mathcal{U}}^*)$-dense. (The uniformities $(\widetilde{\mathcal{U}})^*$ and $\widetilde{\mathcal{U}^*}$ coincide; see below.)

If D is a $\mathcal{T}(\mathcal{U}^*)$-dense subspace of a quasi-uniform space $(X,\mathcal{U})$ and $f : (D,\mathcal{U}|D) \to (Y,\mathcal{V})$ is quasi-uniformly continuous where $(Y,\mathcal{V})$ is a bicomplete quasi-uniform space, then there exists a (unique) quasi-uniformly continuous extension of f to X.

Construction of the bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$ *of a quasi-uniform space* $(X,\mathcal{U})$: [17] Let $(X,\mathcal{U})$ be a quasi-uniform space. By $\widetilde{X}$ we denote the set of all minimal $\mathcal{U}^*$-Cauchy filters on X. Moreover let $\widetilde{\mathcal{U}}$ be the quasi-uniformity on $\widetilde{X}$ that is generated by all sets $\widetilde{U}$ where U belongs to $\mathcal{U}$. Here $\widetilde{U} = \{(\mathcal{F},\mathcal{G}) \in \widetilde{X} \times \widetilde{X}$: there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U\}$. Often a $\mathcal{T}(\mathcal{U}^*)$-convergent minimal $\mathcal{U}^*$-Cauchy filter $\eta^*(x) \in \widetilde{X}$ is identified with its limit point in $(X,\mathcal{U}^*)$ and using this identification $(X,\mathcal{U})$ is considered a subspace of $(\widetilde{X},\widetilde{\mathcal{U}})$.

Remark: (e.g. [5]) Suppose that $\mathcal{U}$ is totally bounded. Then the topological space $(\widetilde{X},\mathcal{T}(\widetilde{\mathcal{U}}))$ is a locally compact space in which the limit set of any ultrafilter is the closure of some (unique) singleton. Furthermore $\widetilde{\mathcal{U}}$ is the coarsest quasi-uniformity that the topological space $(\widetilde{X},\mathcal{T}(\widetilde{\mathcal{U}}))$ admits and the uniform topology $\mathcal{T}(\widetilde{\mathcal{U}}^*)$ is compact.

Next we construct both the sobrification of a topological and the Fell-compactification of a locally compact space with the help of the bicompletion.

Definition: [20] The *Skula topology* or *b-topology* of a topological space X is the topology $\mathcal{T}(\mathcal{P}^*)$ where $\mathcal{P}$ is the Pervin quasi-uniformity of X. Let ${}^sX$ be the set of all closed irreducible subsets of X. Define a topology ${}^s\mathcal{T}$ on ${}^sX$ as follows: ${}^s\mathcal{T} = \{[G] : G \text{ is open in } X\}$ where $[G] = \{F \in {}^sX : F \cap G \neq \emptyset\}$ whenever G is open in X. The sober space $({}^sX, {}^s\mathcal{T})$ is called the *sobrification* of X. If X is a T_0-space, then $i : X \to {}^sX$ defined by $i(x) = \mathrm{cl}\{x\}$ whenever $x \in X$ is a topological embedding of X onto a b-dense subspace of ${}^sX$.

[21] A family of subsets $\mathcal{H}$ of a topological space X is called *well-monotone* provided that the partial order $\subseteq$ of set inclusion is a well-order on $\mathcal{H}$. The compatible quasi-uniformity $\mathcal{M}$ on a topological space X which has as a subbase the set of all binary relations of the form $T = \bigcup\{\{x\} \times (\bigcap\{G : x \in G \in \mathcal{H}\}) : x \in X\}$ where $\mathcal{H}$ is a well-monotone open cover of X is called the *well-monotone open covering quasi-uniformity of* X. It coincides with the Pervin quasi-uniformity of X if and only if X is hereditarily compact [13, Remark 1].

Proposition 7. [13] *Let X be a topological space and let $\mathcal{M}_X$ be the well-monotone open covering quasi-uniformity of X. Then $(\widetilde{X}, \mathcal{T}(\widetilde{\mathcal{M}_X}))$ is the sobrification of X and $\widetilde{\mathcal{M}_X}$ is the well-monotone open covering quasi-uniformity of $(\widetilde{X}, \mathcal{T}(\widetilde{\mathcal{M}_X}))$. (Here we identify each minimal $\mathcal{M}_X^*$-Cauchy filter with its $\mathcal{T}(\mathcal{M})$-limit set.) In particular, $\mathcal{M}_X$ is bicomplete if and only if X is quasi-sober.*

Definition: [12] Let X be a locally compact topological space and let $\mathcal{L}$ be the set of limit sets of ultrafilters on X. Consider the topology $\mathcal{T}$ on $\mathcal{L}$ which is generated by the subbase $\{< X, G >: G \text{ is open in } X\} \cup \{< X \setminus K >: K \text{ is compact in } X\}$ where $< X, G >= \{F \in \mathcal{L} : G \cap F \neq \emptyset\}$ and $< X \setminus K >= \{F \in \mathcal{L} : K \cap F = \emptyset\}$. It is known that $(\mathcal{L}, \mathcal{T}, \subseteq)$ is a compact (partially) ordered Hausdorff space, the so-called *Fell-compactification* of X.

Proposition 8. [32,29] *Let $\mathcal{U}$ be the coarsest quasi-uniformity on a locally compact space X. Then $(\widetilde{X}, \mathcal{T}(\widetilde{\mathcal{U}}^*), \bigcap\widetilde{\mathcal{U}})$ is the Fell* *compactification of* X. *(Again, minimal* $\mathcal{U}^*$*-Cauchy filters are identified with their* $\mathcal{T}(\mathcal{U})$*-limit sets.) The quasi-uniformity* $\mathcal{U}$ *is bicomplete if and only if the limit set of each ultrafilter on* X *is the closure of some (unique) singleton. If* X *is a locally compact noncompact Hausdorff space, then* $(\widetilde{X}, \mathcal{T}(\widetilde{\mathcal{U}}^*))$ *is the one-point-compactification of* X *with* $\emptyset$ *added as the point-at-infinity. (If* X *is a compact Hausdorff space, then* $\mathcal{U} = \mathcal{U}^*$ *and* $\mathcal{U}$ *is complete.)*

In the light of the last proposition we are going to finish this section with some remarks on topological ordered spaces.

A *topological ordered space* $(X, \mathcal{T}, \leq)$ is a topological space $(X, \mathcal{T})$ equipped with a $\mathcal{T} \times \mathcal{T}$-closed partial order $\leq$. For any quasi-uniform T_0-space $(X, \mathcal{U})$, the triple $(X, \mathcal{T}(\mathcal{U}^*), \cap\mathcal{U})$ is a topological ordered space. The class of topological ordered spaces determined by this construction is exactly the class of completely regular ordered spaces (introduced by L. Nachbin; see [48]).[26]

Any compact topological ordered space $(X, \mathcal{T}, \leq)$ is determined by a unique quasi-uniformity $\mathcal{U}$ [17, Theorem 1.20]. It consists of the set of all $\mathcal{T} \times \mathcal{T}$-neighborhoods of $\leq$ in the product space $X \times X$. Of course, $\mathcal{U}$ is totally bounded and bicomplete. Furthermore $\mathcal{T}(\mathcal{U})$ is the upper and $\mathcal{T}(\mathcal{U}^{-1})$ the lower topology [27] of X.

Remarks: In [40] those topological ordered spaces $(X, \mathcal{T}, \leq)$ for which the set $\mathcal{N}$ of all neighborhoods of $\leq$ forms a quasi-

[26] A topological ordered space $(X, \mathcal{T}, \leq)$ is called *completely regular ordered* provided that the following two conditions are satisfied:

(i) If $a, b \in X$ such that $f(a) \leq f(b)$ whenever $f : X \to [0,1]$ is continuous and increasing, then $a \leq b$.

(ii) For any point $a \in X$ and neighborhood V of a there are two continuous maps $f, g : X \to [0,1]$ such that f is continuous and increasing, g is continuous and decreasing, $f(a) = g(a) = 1$ and $\min\{f(x), g(x)\} = 0$ for any $x \in X \setminus V$.

[27] As before, we suppose that the upper topology consists of the open *increasing* sets; recall that $G \subseteq X$ is called *increasing* if $f \in G$ and $x \geq f$ imply that $x \in G$.

uniformity are studied. (Note that by [40, Corollary 1], $\mathcal{N}$ necessarily determines X in this case.) It is shown for instance [40, Lemma 4 and Example 4] that a topological ordered space with a countably compact sequential topology satisfies the stated condition if and only if the set of all neighborhoods of its diagonal forms a uniformity. While each GO space has the discussed property [40, Corollary 4], $\boldsymbol{R}^2$ equipped with its usual topology and order does not possess it [40, Example 2].

Question due to J. Lawson [47]*:* Let $(X, \mathcal{T}, \leq)$ be a completely regular ordered topological space, let F be a closed *increasing* set in X and let $a \in X \setminus F$. Is there a continuous increasing function $f : X \to [0,1]$ such that $f(F) = 1$ and $f(a) = 0$?

In [36, Proposition 1] it is observed that a completely regular ordered space X satisfies both Lawson's condition and its order dual if and only if the bispace given by the upper and lower topology of X is completely regular. In general, Lawson's question has a negative answer.

Example 7. [36, Example 6] On the set $X = [0, \omega_1] \times [0, \omega_0]$ equipped with the product topology $\mathcal{T}$ of the interval topologies on the factor sets define a partial order by $(a, b) \leq (c, d)$ if and only if $a \geq c$ and $b \leq d$ whenever $(a, b), (c, d) \in X$. Let $\mathcal{V}$ be the unique quasi-uniformity on the compact space X such that $\mathcal{T}(\mathcal{V}^*) = \mathcal{T}$ and $\cap\mathcal{V} = \leq$, i.e. $\mathcal{V}$ is the coarsest quasi-uniformity compatible with the upper topology of X. For each $V \in \mathcal{V}$ set $H_V = V \setminus \{(x, y) \in X \times X : x = (\omega_1, \omega_0)$ and $y \neq (\omega_1, \omega_0)\}$. Let $\mathcal{U}$ be the quasi-uniformity on X generated by $\{H_V : V \in \mathcal{V}\}$. For the completely regular ordered space $Z = (X, \mathcal{T}(\mathcal{U}^*), \cap\mathcal{U})$ there does not exist a continuous increasing function $f : X \to [0,1]$ such that $f([0, \omega_1[\times \{\omega_0\}) = 1$ and $f((\omega_1, \omega_0)) = 0$, although the set $[0, \omega_1[\times \{w_0\}$ is closed and increasing.

We observe that the subspace $Y = X \setminus ([0, \omega_1[\times \{2n : n \in \omega_0\})$ of the space Z defined above is a non-pseudo-compact completely regular ordered space which is determined by a unique quasi-uniformity [41, Example 3] (compare [33]).

Under additional conditions some positive answers to Lawson's question are known.

Proposition 9. [36] *Each completely regular ordered topological lattice and each locally compact completely regular ordered I-space*[28] *has the separation property formulated above.*

Problem 11. (compare [36, Problem]) *Does there exist a completely regular ordered I-space with a normal topology that does not have Lawson's separation property?*[29]

6. Canonical quasi-uniformities

At the end of this survey we would like to pose a problem about the so-called canonical quasi-uniformities.

Definition: (e.g. [3,37]) Let T denote the forgetful functor from the category $\boldsymbol{Quu}$ of quasi-uniform spaces and quasi-uniformly continuous maps to the category $\boldsymbol{Top}$ of topological spaces and continuous maps. A *functorial admissible quasi-uniformity* on the topological spaces is a functor $F : \boldsymbol{Top} \to \boldsymbol{Quu}$ such that $TF = 1$, i.e. F is a right inverse or section of T, briefly a *T-section.* (Consider e.g. the functor that endows each $X \in \boldsymbol{Top}$ with its semi-continuous quasi-uniformity.) Functorial admissible quasi-uniformities on subcategories of $\boldsymbol{Top}$ are defined similarly. For T-sections F and G we say that F is coarser than G (written $F \leq G$) provided that the quasi-uniformity of FX is coarser than the quasi-uniformity of GX whenever $X \in \boldsymbol{Top}$.

[28] A topologically ordered space $(X, \mathcal{T}, \leq)$ is called an *I-space* provided that $d(E)$ and $i(E)$ are open for any open subset E of X. Here $d(E)$ ($i(E)$, respectively) denotes the intersection of the decreasing sets (increasing sets) containing E.

[29] Remark added during revision: The answer to this question is positive (see [45]).

Examples: [37] (a) Given $X \in \boldsymbol{Top}$ let $\mathcal{U}_X$ be the finest uniformity on X such that $\mathcal{T}(\mathcal{U}_X)$ is coarser than the topology of X. Denote the supremum of $\mathcal{U}_X$ and the Pervin quasi-uniformity $\mathcal{P}_X$ of X by $\mathcal{S}(X)$. The functor $S : \boldsymbol{Top} \to \boldsymbol{Quu}$ that equips each topological space X with the quasi-uniformity $\mathcal{S}(X)$ is a T-section. In [37, Proposition 2.4] it is noted that for a paracompact T_2-space X, the quasi-uniformity $\mathcal{S}(X)$ is transitive if and only if X is boundedly paracompact[30].

(b) Let X be the subset $\{k + \frac{s}{k} : k \in \boldsymbol{N} \text{ and } s \in \{0, \dots, k-1\}\}$ of the set $\boldsymbol{Q}$ of rationals. Define a quasi-pseudo-metric d on X as follows: $d(x,y) = 0$ if $x \leq y$ and $d(x,y) = x - y$ if $x > y$. Then $\mathcal{M} = \{\mathcal{V}$: there is a T-section $S : \boldsymbol{Top} \to \boldsymbol{Quu}$ such that $\mathcal{V}$ is the quasi-uniformity of the space $SX\}$ consists of the Pervin quasi-uniformity and the fine quasi-uniformity of X. In particular $\mathcal{U}_d \notin \mathcal{M}$.

Remarks: It is well-known [3] that the Pervin functor is the coarsest T-section on $\boldsymbol{Top}$ and it is readily checked [37] that the Čech functor $C^* : \boldsymbol{Creg} \to \boldsymbol{Unif}$ is the coarsest functorial admissible quasi-uniformity on the completely regular topological spaces and continuous maps.[31] Furthermore it is shown in [37, Proposition 3.1] that the Pervin quasi-uniformity is the coarsest admissible functorial quasi-uniformity on the category of Hausdorff spaces and continuous maps. The answer to the following question however is unknown.

Problem 12. [37] *Is the Pervin quasi-uniformity the coarsest admissible functorial quasi-uniformity on the category of regular topological spaces and continuous maps?*

Example 8. *Let X be a regular topological space and let $\mathcal{R}_X$ be the quasi-uniformity on X that is generated by the subbase*

[30] A regular topological space X is *boundedly paracompact* if each open cover $\mathcal{C}$ of X has an open refinement that is the union of finitely many discrete families.

[31] For each $X \in \boldsymbol{Creg}$ the uniform space C^*X carries the uniformity initial for the continuous maps from X to $[0,1]$ where $[0,1]$ is equipped with its unique compatible uniformity.

$\{[G \times G] \cup [(X \setminus G) \times X] : G$ is regular open[32] *in $X\}$. Clearly $\mathcal{R}_X$ is a compatible quasi-uniformity on X. However this construction is not functorial on the category of regular spaces and continuous maps.*

7. APPENDIX

In this final section we would like to answer a question put to the author by G.C.L. Brümmer at the International Conference in Memory of F. Hausdorff, Berlin, 1992. Subsequently the result was used in the proof of [4, Proposition 4.2]. Proposition 10 nicely illustrates some of the theory presented in this paper. The terminology is the same as in the last two sections. We work in the category of T_0-spaces and $M : \boldsymbol{Top} \to \boldsymbol{Quu}$ denotes the functor which assigns the well-monotone open covering quasi-uniformity. By $K : \boldsymbol{Quu} \to \boldsymbol{Quu}$ we shall denote the bicompletion functor.

Proposition 10. *For any T-section F, the inequality $KF \geq FTKF$ implies that $F \geq M$.*

Proof. Let F be a T-section satisfying $KF \geq FTKF$. For any (infinite) regular cardinal α define a topological space X_α by endowing the set α with the lower topology $\{\alpha\} \cup \{[0, \beta[: \beta \in \alpha\}$. We wish to show first that $FX_\alpha \geq MX_\alpha$ for all α. Note that MX_α is the fine quasi-uniformity of X_α.

Assume that there is a regular cardinal γ such that $FX_\gamma < MX_\gamma$. The quasi-uniformity of FX_γ will be denoted by $\mathcal{U}$ in the following. We want to prove that the property of FX_γ implies that for each entourage $W \in \mathcal{U}$ there exists $\alpha \in \gamma$ such that $([\alpha, \to[) \times \gamma \subseteq W$.

Let $W \in \mathcal{U}$ and let $V \in \mathcal{U}$ be such that $V^2 \subseteq W$ where we assume without loss of generality that for each $\delta \in \gamma$, $V(\delta)$ is open in X_γ ([17, p. 3]). Note that it suffices to show that there is $\beta \in \gamma$ such that $V(\beta) = \gamma$, since then for any β' with $\beta \leq \beta' < \gamma$ we have that $\beta \in [0, \beta'] \subseteq V(\beta')$ and, thus, $\gamma = V(\beta) \subseteq W(\beta')$ and $([\beta, \to[) \times \gamma \subseteq W$.

Therefore, in order to reach a contradiction we assume that $V(\delta) \neq \gamma$ whenever $\delta \in \gamma$. Inductively define a strictly increasing transfinite sequence $(x_\beta)_{\beta<\gamma}$ of points in γ such that $x_\beta \notin V(x_\alpha)$ whenever $\alpha < \beta$. Note that the construction is possible, because for any $\gamma' < \gamma$ there is $\gamma'' < \gamma$ such that $\bigcup_{\beta<\gamma'} V(x_\beta) \subseteq [0, \gamma'']$, since each $V(x_\beta)$ is open and distinct from γ and γ is regular.

[32] An open set G of a topological space is called *regular open* if G is equal to the interior of its closure.

The map $j : X_\gamma \to X_\gamma$ defined by $j(\beta) = x_\beta$ whenever $\beta < \gamma$ is continuous and we have $(j \times j)^{-1}(V) = \bigcup_{\alpha\in\gamma}(\{\alpha\} \times [0, \alpha])$. Hence FX_γ carries the fine quasi-uniformity of X_γ — a contradiction. We have verified that each entourage W of $\mathcal{U}$ has the property stated above.

It follows that the filter $\mathcal{F}$ generated by $\{]\beta, \to [: \beta \in \gamma\}$ on γ is a $\mathcal{U}^*$-Cauchy filter. Denote the minimal $\mathcal{U}^*$-Cauchy filter contained in $\mathcal{F}$ by $\mathcal{G}$ [17, Proposition 3.30]. We now observe that the singleton $\{\mathcal{G}\}$ is closed in $TKFX_\gamma$: By the property of the entourages of $\mathcal{U}$ established above, for any $W \in \mathcal{U}$ there is $\beta < \gamma$ such that $W^{-1}(]\beta, \to [) \times \gamma \subseteq W^2$. Hence $(\mathcal{G}, \mathcal{H}) \in \cap\widetilde{\mathcal{U}}$ whenever $\mathcal{H} \in \widetilde{\gamma}$. The assertion follows, since $TKFX_\gamma$ is a T_0-space. Because F is finer than the Pervin functor, $U = [\{\mathcal{G}\} \times \widetilde{\gamma}] \cup [(\widetilde{\gamma} \setminus \{\mathcal{G}\}) \times (\widetilde{\gamma} \setminus \{\mathcal{G}\})]$ belongs to the quasi-uniformity of $FTKFX_\gamma$. However by the obvious density argument it does not belong to $\widetilde{\mathcal{U}}$, the quasi-uniformity of the space KFX_γ, since $U^{-1}(\mathcal{G}) = \{\mathcal{G}\}$ and $\mathcal{G}$ belongs to the remainder of KFX_γ, for $\mathcal{F}$ does not converge in X_γ. On the other hand by our original assumption, $FTKFX_\gamma \leq KFX_\gamma$. We have reached a contradiction and finally conclude that $FX_\alpha \geq MX_\alpha$ for any regular cardinal α.

Let X be an arbitrary topological space and let $\{G_\beta : \beta < \delta\}$ be any well-monotone open cover of X. Define a continuous map $f : X \to X_\gamma$ where γ is some regular cardinal larger than δ, by setting for each $x \in X$, $f(x)$ equal to the minimal $\beta < \delta$ such that $x \in G_\beta$. Since $FX_\gamma \geq MX_\gamma$, we see that $\bigcup_{\alpha\in\gamma}(\{\alpha\} \times [0, \alpha])$ is a member of the quasi-uniformity of FX_γ. Therefore $(f \times f)^{-1}(\bigcup_{\alpha\in\gamma}\{\alpha\} \times [0, \alpha]) = \bigcup_{x\in X}(\{x\} \times G_{f(x)})$ belongs to the quasi-uniformity of FX. It follows that $FX \geq MX$. We have shown that $F \geq M$.

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