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AN ACRIN DOWKER SPACE

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ABSTRACT. Under a set-theoretic assumption, we construct a normal not countably paracompact space X with the property that any continuous regular image of X is normal.

1. INTRODUCTION

A space is ACRIN if All Continuous Regular Images are Normal. A Dowker space is a normal space that is not countably paracompact. The goal of this paper is to construct an ACRIN Dowker space.

Our construction assumes $CH + \diamondsuit(\{\alpha < \omega_2 : cf(\alpha) = \omega_1\})$. We will build a de Caux type Dokwer space on $\omega_2 \times \omega$ (see [1] and [3]). Results from [2] show that lifting to ω_2 is indeed necessary—there is no ACRIN Dowker space of size ω_1.

We begin by defining some of the terminology we will use.

- A *P-space* is a space in which G_δ's are open.
- A space is *locally Lindelöf* if every point has a closed Lindelöf neighborhood.
- A space is *almost Lindelöf* if the non-Lindelöf closed sets form a filterbase. Thus, if X is almost Lindelöf and A and B are disjoint closed subsets of X, then at least one of A and B is Lindelöf.
- π_1 and π_2 denote the projections of $\omega_2 \times \omega$ onto the first and second coordinates, respectively.

Next, we prove some easy but useful lemmas about almost Lindelöf spaces.

Proposition 1.1. *A regular almost Lindelöf space is locally Lindelöf.*

Proof: Let X be a regular almost Lindelöf space. Fix an open cover $\mathcal{U}$ with no countable subcover, and let $x \in X$ be arbitrary. Take a $U \in \mathcal{U}$ with $x \in U$. By regularity, we can find an open V with $x \in V \subseteq \overline{V} \subseteq U$. Now, $\overline{V}$ and $X \setminus U$ are disjoint closed sets and $X \setminus U$ is non-Lindelöf, so $\overline{V}$ is a Lindelöf neighborhood of x. □

Proposition 1.2. *Suppose X is almost Lindelöf, Y is non-Lindelöf, $f: X \to Y$ is a continuous surjection, and $A \subseteq Y$ is closed. Then Y is almost Lindelöf, and A is Lindelöf if and only if $f^{-1}[A]$ is Lindelöf.*

Proof: To see that Y is almost Lindelöf, fix disjoint closed subsets H and K of Y. Then $f^{-1}[H]$ and $f^{-1}[K]$ are disjoint closed subsets of X, so one of them, say $f^{-1}[H]$, is Lindelöf. But then $H = f[f^{-1}[H]]$ is a continuous image of a Lindelöf set, and so is Lindelöf.

To prove the second assertion, first suppose that A is a Lindelöf subset of Y. Let $\mathcal{V} = \{V_i : i \in I\}$ be an open cover of Y that has no countable subcover. Since A is Lindelöf, there is a countable $J \subseteq I$ such that $A \subseteq \bigcup_{i \in J} V_i$.

For each $i \in I$, let $U_i = f^{-1}[V_i]$, then $\mathcal{U} = \{U_i : i \in I\}$ is an open cover of X with no countable subcover. Now, $f^{-1}[A] \subseteq \bigcup_{i \in J} U_i$, so $f^{-1}[A]$ and $X \setminus \bigcup_{i \in J} U_i$ are disjoint closed subsets of X, and $X \setminus \bigcup_{i \in J} U_i$ is non-Lindelöf. Therefore, $f^{-1}[A]$ is Lindelöf.

The reverse implication is trivial, so the proof is complete. □

Proposition 1.3. *A regular almost Lindelöf P-space is normal.*

Proof: Let H and K be disjoint closed subsets of a regular almost Lindelöf P-space X, with H Lindelöf. There is an open cover $\{U_n : n \in \omega\}$ of H with $\overline{U_n} \cap K = \emptyset$ for each $n \in \omega$. Let $U = \bigcup_{n \in \omega} U_n$. Since X is a P-space, $\overline{U} = \bigcup_{n \in \omega} \overline{U_n}$, so $\overline{U} \cap K = \emptyset$. □

2. Construction of X

Our plan is to construct a de Caux type Dowker space X with point set $\omega_2 \times \omega$, using an Ostaszewski type inductive construction. Like the de Caux space, our example will be almost Lindelöf. To help make the space not countably paracompact, each $F_n = \omega_2 \times [n, \omega)$ will closed and non-Lindelöf.

Let $E = \{\alpha \in \omega_2 : cf(\alpha) = \omega_1\}$. We assume $CH + \diamondsuit(E)$. By $2^{\omega_1} = \omega_2$, we can enumerate $\{A \in [\omega_2 \times \omega]^{\omega_1} : |\pi_2(A)| < \omega\}$ as $\{A_\alpha : \alpha \in E\}$, with $A_\alpha \subseteq \alpha \times \omega$.

By $\diamondsuit(E)$, there are sequences $\{B_\alpha : \alpha \in E\}$ and $\{C_\alpha : \alpha \in E\}$ such that for each $\alpha \in E$:

(1) $B_\alpha \cup C_\alpha \subseteq \alpha \times \omega$;
(2) $|\pi_2(B_\alpha \cup C_\alpha)| < \omega$;
(3) $\pi_1(B_\alpha)$ and $\pi_1(C_\alpha)$ are cofinal in α and have order type ω_1;
(4) whenever H and K are elements of $[\omega_2 \times \omega]^{\omega_2}$, there is an $\beta \in E$ such that $B_\beta \subseteq H$ and $C_\beta \subseteq K$.

We construct the topology on X by replacing "cofinite" with "co-countable" in the standard Ostaszewski construction, declaring (α, n) to be isolated if $cf(\alpha) \neq \omega_1$, taking the topology generated by the union of the preceding topologies at limits, and proceeding as follows for points (α, n) with $cf(\alpha) = \omega_1$. Begin by choosing an $n \in \omega$ such that $A_\alpha \cup B_\alpha \cup C_\alpha \subseteq \alpha \times n$. Let (α, m) be isolated if $m \neq n$. Make sure that $(\alpha, n) \in \overline{B_\alpha} \cap \overline{C_\alpha}$ and that if A_α is closed discrete in the topology defined so far, that $(\alpha, n) \in \overline{A_\alpha}$.

3. Properties of X

As constructed, X is a locally Lindelöf P-space and the character of X is ω_1. The open cover $\{[0, \alpha) \times \omega : \alpha < \omega_2\}$ has no countable subcover, so X is not Lindelöf. Because each A_α has a limit point, X is $\aleph_1$-compact (i.e., X has no uncountable closed discrete sets).

We claim that every open cover of X of size ω_1 has a countable subcover. If not, there is an increasing open cover $\mathcal{U} =$

$\{U_\alpha : \alpha < \omega_1\}$ of X that has no countable subcover. Take $x_\alpha \in U_{\alpha+1} \setminus U_\alpha$, then because X is a P-space, $\{x_\alpha : \alpha < \omega_1\}$ is closed discrete, contradicting the fact that X is $\aleph_1$-compact. Let A be a closed subset of X. This claim also shows that any open cover of A of size ω_1 has a countable subcover, hence A is Lindelöf if and only if $|A| \leq \omega_1$.

We made sure that each B_α and C_α have a common limit point, so any pair of closed non-Lindelöf subsets of X intersect. Thus, X is almost Lindelöf and (because X is a regular P-space) normal.

X is a Dowker space because $\{\omega_2 \times [0,n] : n \in \omega\}$ is countable open cover of X with no closed shrinking. To see this, suppose that for each $n \in \omega$, F_n is a closed subset of $\omega_2 \times [0,n]$. Because the complement of $\omega_2 \times [0,n]$ is non-Lindelöf, F_n must be Lindelöf. But then $\bigcup_{n\in\omega} F_n$ is Lindelöf, and hence not all of X.

We need a lemma before we can prove that X is ACRIN.

Lemma 3.1. *Suppose that $f : Z \to Y$ is continuous with Y regular. If $f(Z)$ is dense in Y, then $w(Y) \leq w(Z)^{L(Z)}$.*

Proof: Let $\mathcal{B}$ be a base for Z of size $w(Z)$. We show that $\{\mathrm{int}_Y(\mathrm{cl}_Y(f[\bigcup\mathcal{A}]) : \mathcal{A} \in [\mathcal{B}]^{\leq L(Z)}\}$ is a base for Y. Fix a $y \in Y$ and an open $U \subseteq Y$ with $y \in U$. By regularity, there are open V and W with $y \in V \subseteq \overline{V} \subseteq W \subseteq \overline{W} \subseteq U$. In Z, find $\mathcal{A} \in [\mathcal{B}]^{\leq L(Z)}$ such that $f^{-1}[\overline{V}] \subseteq \bigcup\mathcal{A} \subseteq f^{-1}[W]$. Then $y \in \mathrm{int}_Y(\mathrm{cl}_Y(f[\bigcup\mathcal{A}]) \subseteq \overline{W} \subseteq U$. □

Theorem 3.2. *X is ACRIN.*

Proof: Let $f : X \to Y$ be continuous with Y regular. By Lemma 1.2, Y is either Lindelöf or almost Lindelöf, so we can reduce to considering disjoint H and K with H Lindelöf. Since Y is regular and locally Lindelöf, there is a countable open cover $\{U_n : n \in \omega\}$ of H such that each $\overline{U_n} \cap K = \emptyset$ and $\overline{U_n}$ is Lindelöf. Set $F = \overline{\bigcup_{n\in\omega} U_n}$.

Claim: If F is Lindelöf, then H and K are separated.

To prove the claim, suppose that F is Lindelöf. Then because H and $K \cap F$ are disjoint closed Lindelöf sets, there is an open U containing H such that $\overline{U} \cap (K \cap F) = \emptyset$. But then $H \subseteq U \cap \bigcup_{n \in \omega} U_n$ and

$$\overline{(U \cap \bigcup_{n \in \omega} U_n)} \cap K \subseteq \overline{U} \cap F \cap K = \emptyset,$$

so H and K are separated.

Thus, to complete the proof, we need only show that F is Lindelöf. By Lemma 1.2, each $f^{-1}[\overline{U_n}]$ is a closed Lindelöf subset of X. Since X is a P-space, $A = \bigcup_{n \in \omega} f^{-1}[\overline{U_n}]$ is closed and Lindelöf. As mentioned above, A must have cardinality ω_1. Since the character of X is ω_1, the weight of A is also ω_1.

Now, $\bigcup_{n \in \omega} U_n \subseteq f[A] \subseteq \overline{f[A]} \subseteq F$, so $\overline{f[A]} = F$. By CH and Lemma 3.1, the weight of F is ω_1. The following claim finishes the proof.

Claim: If D is a closed subset of Y that has weight ω_1, then D is Lindelöf.

To see this, fix an open cover $\mathcal{V} = \{V_i : i \in I\}$ of D. Since the weight of D is ω_1, we can assume that $|I| = \omega_1$. Let $U_i = f^{-1}[V_i]$, then $\mathcal{U} = \{U_i : i \in I\}$ is an open cover of the closed subset $f^{-1}[D]$ of X. Since every open cover of a closed subset of X of size ω_1 has a countable subcover, there is a countable $J \subseteq I$ such that $f^{-1}[D]$ is covered by $\{U_i : i \in J\}$. Clearly, $\{V_i : i \in J\}$ covers D. □

We would like to express our thanks to Amer Bešlagić, who greatly simplified our original construction and provided Lemma 3.1. Though more complicated, our original construction gave a space Y with $\omega_1 = hd(Y) < hl(Y) = \omega_2$. We used a computation in $C(Y)$ to show ACRIN.

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