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A NOTE ON GLOBAL AND LOCAL SELECTIONS

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1. Introduction

The purpose of this note is to prove the following theorem, which was stated without proof in [5, Theorem 2.9].

Theorem 1.1. *Let Y be metrizable and $\mathcal{E} \subset 2^Y$ hereditary. Then the following are equivalent.*

(a) *If X is metrizable, then every l.s.c. $\varphi : X \to \mathcal{E}$ has a selection.*

(b) *If X is metrizable, then every l.s.c. $\varphi : X \to \mathcal{E}$ is equi-connected and has selections locally.*

Before defining our terms, we pause to note that it may sometimes be difficult to verify both parts of condition (b) of Theorem 1.1. This is illustrated by two questions in [5]. In [5, Question 3.2], it is clear that φ has selections locally, but it is not known whether φ is equi-connected. By contrast, in [5, Question 3.3] the map φ is clearly equi-connected, but it is unkonwn whether φ has selections locally.

We now turn to our definitions. First, $2^Y = \{E \subset Y : E \neq \emptyset\}$, and a collection $\mathcal{E} \subset 2^Y$ is *hereditary* if $\{y\} \in \mathcal{E}$ whenever $y \in \bigcup \mathcal{E}$. A function $\varphi : X \to 2^Y$ is *l.s.c.* if $\{x \in X : \varphi(x) \cap V \neq \emptyset\}$ is open in X whenever V is open in Y. A *selection* for $\varphi : X \to 2^Y$ is a continuous $f : X \to Y$

such that $f(x) \in \varphi(x)$ for every $x \in X$. A map $\varphi : X \to 2^Y$ *has selections locally* if each $x \in X$ has a neighborhood U in X such that $\varphi \mid U$ has a selection. Equi-connected maps, finally, are defined as follows (where I denotes the closed interval $[0, 1]$).

Definition 1.2. *A map $\varphi : X \to 2^Y$ is* equi-connected *if there exists a continuous $\alpha : M \times I \to Y$, where*

$$M = \{(x, y_0, y_1) \in X \times Y^2 : y_0, y_1 \in \varphi(x)\},$$

such that the following conditions are satisfied for all $x \in X$, $y_0, y_1, y \in \varphi(x)$, and $t \in I$:

(a) $\alpha(x, y_0, y_1, t) \in \varphi(x)$,
(b) $\alpha(x, y_0, y_1, 0) = y_0$ *and* $\alpha(x, y_0, y_1, 1) = y_1$,
(c) $\alpha(x, y, y, t) = y$.

Example 1.3. If $\varphi : X \to 2^Y$, with Y a topological linear space and $\varphi(x)$ convex for every $x \in X$, then φ is equi-connected. (One can define the required $\alpha : M \times I \to Y$ by $\alpha(x, y_0, y_1, t) = ty_1 + (1 - t)y_0$.)

In Section 2 it is shown that $\varphi : X \to 2^Y$ is equi-connected if and only if φ has a formally stronger property which we call equi-convex, and this result is then applied in Section 3 to prove Theorem 1.1.

2. Equi-convex maps

In the following definition, $P_n = \{t \in I^n : \sum_{i=1}^n t_i = 1\}$, and, if Z is any set and $1 \leq i \leq n$, then $\partial_i : Z^n \to Z^{n-1}$ is defined by

$$\partial_i z = (z_1, \ldots , z_{i-1}, z_{i+1}, \ldots , z_n)$$

for all $z = (z_1, \ldots , z_n) \in Z^n$.

Definition 2.1. *A map* $\varphi : X \to 2^Y$ *is* equi-convex *if for all* $n \geq 1$ *there exists a continuous* $k_n : M_n \times P_n \to Y$*, where*

$$M_n = \{(x,y) \in X \times Y^n : y_i \in \varphi(x) \text{ for } i = 1, \dots, n\},$$

such that the following conditions are satisfied for all $(x,y) \in M_n$ *and* $t \in P_n$*:*

(a) $k_n(x,y,t) \in \varphi(x)$,
(b) $k_n(x,y,t) = y_1$ *if* $y_1 = \cdots = y_n$,
(c) $k_n(x,y,t) = k_{n-1}(x, \partial_i y, \partial_i t)$ *if* $t_i = 0$.

Remark. The map $\varphi : X \to 2^Y$ in Example 1.3 is equi-convex. (One can define the required $k_n : M_n \times P_n \to Y$ by $k_n(x,y,t) = \sum_{i=1}^n t_i y_i$.)

Theorem 2.2. *The following properties of a map* $\varphi : X \to 2^Y$ *are equivalent.*

(a) φ *is equi-convex.*
(b) φ *is equi-connected.*

Proof: (a) $\to$ (b). Let $k_n : M_n \times P_n \to Y$ be as in Definition 2.1. Let $M \subset X \times Y^2$ be as in Definition 1.2, and define $\alpha : M \times I \to Y$ by

$$\alpha(x, y_0, y_1, t) = k_2(x, (y_0, y_1), t).$$

This α satisfies the requirements of Definition 1.2.

(b) $\to$ (a). This result is similar to [3, Proposition 5.3] and to [1, Theorem 4]. We present an outline of the proof, referring the reader to [3] and [1] for certain details.

Let M and $\alpha : M \times I \to Y$ be as in Definition 1.2. With M_n as in Definition 2.1, we now choose $k_n : M_n \times P_n \to Y$ for all n by induction.

Define $k_1 : M_1 \times P_1 \to Y$ by $k_1(x,y,1) = y$. Supposing that we have $k_1, \dots, k_n$, define $k_{n+1} : M_{n+1} \times P_{n+1} \to Y$ by

$$k_{n+1}(x,y,t) = y_{n+1} \qquad \text{if } t_{n+1} = 1,$$

$$k_{n+1}(x,y,t) = \alpha(x, k_n(x, \tilde{y}, \tilde{t}), y_{n+1}, t_{n+1}) \qquad \text{if } t_{n+1} \neq 1,$$

where $\widetilde{y} \in Y^n$ and $\widetilde{t} \in P_n$ are defined by $\widetilde{y} = \partial_{n+1}y$ and $\widetilde{t}_i = t_i(1-t_{n+1})^{-1}$ $(i = 1, \dots, n)$. We must check that k_{n+1} satisfies conditions (a)–(c) of Definition 2.1 and that k_{n+1} is continuous.

That k_{n+1} satisfies 2.1(a) and 2.1(b) is clear. The proof of 2.1(c) can be found in [3]; for the case $i = n+1$ see [3, p. 565, lines -6 and -5], and for $i \neq n+1$ see [3, p. 566, lines 2 to 9] (where M_{n+1} should read P_{n+1} in line 3).

For the continuity of k_{n+1}, the reader is referred to [1, p. 48, lines 3 to 6]. There it is shown (with slightly different notation) that there are continuous maps

$$\begin{aligned} \pi_{n+1} &: M_{n+1} \times P_n \times I \to M_{n+1} \times P_{n+1}, \\ h_{n+1} &: M_{n+1} \times P_n \times I \to Y, \end{aligned}$$

with π_{n+1} a quotient map, such that $h_{n+1} = k_{n+1} \circ \pi_{n+1}$. It follows that k_{n+1} is also continuous. □

3. Proof of Theorem 1.1

(a) → (b). Assume (a), and let $\varphi : X \to \mathcal{E}$ be l.s.c. with X metrizable. Clearly φ has selections locally, so let us show that it is equi-connected.

Let $M \subset X \times Y^2$ be as in Definition 1.2. Clearly M is metrizable. Define $\psi : M \times I \to \mathcal{E}$ by $\psi(x, y_1, y_2, t) = \varphi(x)$, so ψ is l.s.c. Define $A, B_0, B_1 \subset M \times I$ by

$$\begin{aligned} A &= \{(x, y_0, y_1, t) \in M \times I : y_0 = y_1\}, \\ B_0 &= \{(x, y_0, y_1, t) \in M \times I : t = 0\}, \\ B_1 &= \{(x, y_0, y_1, t) \in M \times I : t = 1\}. \end{aligned}$$

Since A, B_0 and B_1 are closed in $M \times I$, so is $C = A \cup B_0 \cup B_1$. Define $g : C \to Y$ by

$$\begin{aligned} g(x, y_0, y_1, t) &= y_0 \quad \text{if } y_0 = y_1, \\ g(x, y_0, y_1, t) &= y_0 \quad \text{if } t = 0, \\ g(x, y_0, y_1, t) &= y_1 \quad \text{if } t = 1. \end{aligned}$$

Clearly g is a selection for $\psi|C$. Since $\mathcal{E}$ is hereditary, we can define $\psi_g : M \times I \to \mathcal{E}$ to be g on C and to be ψ on $(M \times I)\backslash C$. Then ψ_g is also l.s.c. [2, Examples 1.3 and 1.3*], and hence it has a selection $\alpha : M \times I \to Y$ by (a). This α satisfies the requirements of Definition 1.2.

(b) $\to$ (a). Assume (b). Let $\varphi : X \to \mathcal{E}$ be l.s.c., with X metrizable. We must show that φ has a selection.

Since X is paracompact and φ has selections locally, there exists a locally finite open cover $(U_\lambda)_{\lambda \in \Lambda}$ of X such that $\varphi|U_\lambda$ has a selection f_λ for each $\lambda \in \Lambda$. Let (p_λ) be a partition of unity on X subordinated to (U_λ). Well-order the index set Λ. Since φ is equi-connected, it is equi-convex by Theorem 2.2. Let $M_n \subset X \times Y^n$ and $k_n : M_n \times P_n \to Y$ be as in Definition 2.1.

For each $x \in X$, let $\Lambda(x) = \{\lambda \in \Lambda : p_\lambda(x) > 0\}$, write $\Lambda(x) = \{\lambda_1(x), \dots, \lambda_{n(x)}(x)\}$ in the order inherited from Λ, and define

$$f(x) = k_{n(x)}(x, (f_{\lambda_1(x)}(x), \dots, f_{\lambda_{n(x)}(x)}), (p_{\lambda_1(x)}(x), \dots, p_{\lambda_{n(x)}(x)})).$$

Then $f(x) \in \varphi(x)$ by 2.1(a). It remains to show that f is continuous at every $x_0 \in X$.

Let V be a neighborhood of x_0 in X which intersects U_λ for only finitely many $\lambda \in \Lambda$, say for $\lambda \in \Lambda_0 \subset \Lambda$. Write $\Lambda_0 = \{\lambda_1, \dots, \lambda_m\}$ in the order inherited from Λ, and define $g : V \to Y$ by

$$g(x) = k_m(x, (f_{\lambda_1}(x), \dots, f_{\lambda_m}(x)), (p_{\lambda_1}(x), \dots, p_{\lambda_m}(x))).$$

Clearly g is continuous. Moreover, if $x \in V$ then $\Lambda(x) \subset \Lambda_0$ and $p_\lambda(x) = 0$ for all $\lambda \in \Lambda_0\backslash\Lambda(x)$, so $f(x) = g(x)$ by condition 2.1(c). Hence $f|V = g$, so f is continuous on V and hence at x_0. □

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