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THE PRODUCTS OF METALINDELOF SPACES

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In 1991, T. Hui and K. Chiba investigated the various covering properties of σ-products. They obtained the following results:

A. ([1]) Let $X = \sigma\{X_\alpha : \alpha \in A\}$. If every finite subproduct of X is metacompact, then X is metacompact.

B. ([1]) Let $X = \{X_\alpha : \alpha \in A\}$. If every finite subproduct of X is subparacompact and X is subnormal, then X is subparacompact.

C. ([2]) Let $X = \sigma\{X_\alpha : \alpha \in A\}$ and X is normal. If every finite subproduct of X is submetacompact, then X is submetacompact.

In this paper, we first prove that σ-product of metalindelöf spaces has the result which is similar to (A). Secondly, we discuss Tychonoff product of two metalindelöf spaces on the basis of ([3], Theorem 6.25). The following two results are obtained:

(i) Suppose X is a P-space and Y is a strong Σ-space. If X and Y are both metalindelöf space then $X \times Y$ is also metalindelöf.

(ii) Let X be metalindelöf P-space, Y has a point countablebase, then $X \times Y$ is metalindelöf.

1. Definition and Preliminaries

In this paper, $N(K)$ denotes neighbourhood system of a set K; $(\mathcal{U})_x$, $\mathcal{U}|_A$ and $N(x)$ denote respectively $\{U \in \mathcal{U} : x \in U\}, \{A \cap U : U \in \mathcal{U}\}$ and $N(\{x\})$; $\mathbf{N}$ and $\mid A \mid$ denote respectively the set of all natural numbers and the cardinal numbers of A. A^n denotes $\{a : a \subset A \text{ and } \mid a \mid = n\}$. $A^{<\omega} = \cup\{A^n : n \in \omega\}$. And all the spaces do not add the axioms of separation if without special statement.

Definition 1.1. ([1]) Let $s = (s_\alpha)_{\alpha \in A}$ be fixed point in Tychonoff product $\prod\{X_\alpha : \alpha \in A\}$. For each $x = (x_\alpha) \in \prod\{X_\alpha : \alpha \in A\}$, put $Q(x) = \{\alpha \in A : x_\alpha \neq s_\alpha\}$ and define $\sigma\{X_\alpha : \alpha \in A\} = \{x = (x_\alpha)_{\alpha \in A} \in \prod\{X_\alpha : \alpha \in A\} : |Q(x)| < \omega\}$. We call $\sigma\{X_\alpha : \alpha \in A\}$ the σ-product of $\{X_\alpha : \alpha \in A\}$ and s the base point of it. And for every $a \in A^{<\omega}, \prod\{X_\alpha : \alpha \in A\}$ is called a finite subproduct of $\sigma\{X_\alpha : \alpha \in A\}$.

Definition 1.2. A space X is metalindelöf if its every open cover has a point countable open refinement.

Definition 1.3. ([4]) A space X is a P-space if for any index set Ω and for any collection $\{U(\alpha_1, \dots, \alpha_n) : (\alpha_1, \dots, \alpha_n) \in \Omega^n\}$ of open sets in X such that $U(\alpha_1, \dots, \alpha_n) \subset U(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$ for each $(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \in \Omega^{n+1}$, there exists a collection $\{F(\alpha_1, \dots, \alpha_n) : (\alpha_1, \dots, \alpha_n) \in \Omega^n\}$ of closed sets in X such that

(i) $F(\alpha_1, \dots, \alpha_n) \subset U(\alpha_1, \dots, \alpha_n)$ for each $(\alpha_1, \dots, \alpha_n) \in \Omega^n$.

(ii) $\bigcup\{F(\alpha_1, \dots, \alpha_n) : n \in N\} = X$ for any sequence $\{\alpha_n\}$ such that $X = \bigcup\{\cup(\alpha_1, \dots, \alpha_n) : n \in N\}$.

Definition 1.4. ([5]) Let $\{\mathcal{F}_i\}_{i \in N}$ be a sequence of locally finite closed coverings satisfying the following condition:

If $K_1 \supset K_2 \supset \dots$ is a sequence of non-empty closed sets of X such that

$$K_i \subset \bigcap\{F : x \in F \in \mathcal{F}_i\}$$

for some point x in X and for each $i \in \boldsymbol{N}$, then

$$\bigcap\{K_i : i \in N\} \neq \emptyset$$

We set

$$C(x) = \bigcap\{\bigcap\{F : x \in F \in \mathcal{F}_i\} : i \in N\}$$

then it is to be noted that every $C(x)$ is closed and countable compact. Particularly, if $C(x)$ is compact for each $x \in X$, then X is called a strong Σ-space.

Lemma 1.1. ([5]) *If X is a strong Σ-space, then there exists a sequence $\{F_i\}_{i \in N}$ of locally finite closed covers of X and an index set Ω^i, satisfying*

(a) $\mathcal{F}_i = \{F(\alpha_1, \ldots, \alpha_i) : (\alpha_1, \ldots, \alpha_i) \in \Omega_i\}$
(b) $F(\alpha_1, \ldots, \alpha_i) = \bigcup\{F(\alpha_1, \ldots, \alpha_i, \alpha_{n+1}) : \alpha_{i+1} \in \Omega\}$ *for each* $(\alpha_1, \ldots, \alpha_i) \in \Omega^i$
(c) *for each $x \in X$ there is $(\alpha_1, \ldots, \alpha_i, \ldots) \in \Omega^\omega$ such that*
  (i) $x \in \bigcap\{F(\alpha_1, \ldots, \alpha_i) : i \in N\}$
  (ii) $C(x) = \bigcap\{F \in \mathcal{F}_i : x \in F$ *and* $i \in N\}$ *is compact and if U is open in X, $C(x) \subset U$, then there is $i \in N$ such that $C(x) \subset F(\alpha_1, \ldots, \alpha_i,) \subset U$.*

We say that the sequence $< \mathcal{F}_i = \{F(\alpha_1, \ldots, \alpha_i) : (\alpha_1, \ldots, \alpha_i) \in \Omega^i\} >_{i \in N}$ of closed covers of X is a strong Σ-net of X.

Definition 1.5. Let $\mathcal{A}^F = \{\bigcup B : B \in \mathcal{A}^{<\omega}\}$, the collection $\mathcal{A}$ is said to be directed if $\mathcal{A}^F$ refined $\mathcal{A}$.

The following is easily proved by Definition 1.2 and Definition 1.5:

Lemma 1.2. *X is metalindelöf iff every directed open cover of X has a point countable open refinement.*

2. Main results and Proofs

Theorem 2.1. *Let $\{X_\alpha : \alpha \in A\}$ be a family of T_1 spaces and $X = \sigma\{X_\alpha : \alpha \in A\}$. If every finite subproduct of X is metalindelöf, then X is metalindelöf.*

Proof: For each $a \in A^{<\omega}$ and $n \in \omega$ denote $Y_a = \Pi\{X_\alpha : \alpha \in a\} \times \{s_\alpha : \alpha \in A - a\}$ and $Z_n = \{x \in X : |Q(x)| < n+1\}$. Define the mapping $p_a : X \to Y_a$ such that $p_a(x) = (x^*_\alpha)_{\alpha \in a}$ for each $x = (x_\alpha)_{a \in A} \in X$, where

$$x^*_\alpha = \begin{cases} x_\alpha & \alpha \in A \\ s_\alpha & \alpha \in A - a. \end{cases}$$

Let $\mathcal{U}$ be an open cover of X. By induction we construct a sequence $< \mathcal{V}_n >_{n \in \omega}$ of the collections of open subsets of X such that

(1) For each $n \in \omega, \mathcal{V}_n$ is a point countable partial refinement of $\mathcal{U}$.
(2) $\bigcup\{\mathcal{V}_i : i < n+1\}$ covers Z_n for each $n \in \omega$.
(3) $\bigcup \mathcal{V}_n \subset X - Z_{n-1}$.

When $n = 0$, put $U_0 \in \mathcal{U}$ such that $s \in U_0$. Let $\mathcal{V}_0 = \{U_0\}$.

Assume that $\mathcal{V}_1$ has been constructed for $i < n+1$ such that it satisfies (1)-(3).

We set $L_a = Y_a - \bigcup\{\bigcup \mathcal{V}_i : i < n+1\}$ for each $a \in A^{n+1}$. Then L_a is a closed subspace of Y_a. $\mathcal{U}|_{Y_a}$ has a point countable open refinement $\mathcal{W}^*_a$ since it is an open cover of Y_a.

Let $\mathcal{W}_a = \{w^* - Z_n : w* \in \mathcal{W}^*_a\}$, then $\mathcal{W}_a$ is an open cover of L_a and refines partly $\mathcal{U}$, i.e., for each $w \in \mathcal{W}_a$ there is $U(w) \in \mathcal{U}$ such that $w \subset U(w) \cap Y_a$ and $P_a^{-1}(w)$ is open in X. Define $\mathcal{V}_a = \{P_a^{-1}(w) \bigcap U(w) : w \in \mathcal{W}_a\}$ and $\mathcal{V}_{n+1} = \bigcup\{\mathcal{V}_a : a \in A^{n+1}\}$.

(i) $\mathcal{V}_{n+1}$ is a point countable collection of open subset of X.

For $x \in \bigcup \mathcal{V}_{n+1}$, let $\Delta = \{a \in A^{n+1} : x \in \bigcup \mathcal{V}_a\}$, then $|\Delta| \leq \omega$. Otherwise, $\bigcup \Delta$ is noncountable and for each $a \in \Delta$

there is $w_a \in \mathcal{W}_a$ such that $x \in P_a^{-1}(w_a) \bigcap U(w_a)$, $P_a(x) \in w_a \subset Y_a - Z_n$, then $x_\alpha \neq s_\alpha$ for each $\alpha \in a$. i.e., $x_\alpha \neq s_\alpha$ for each $\alpha \in \bigcup \Delta$, $|Q(x)| \geq |\bigcup \Delta| > \omega$. This is contrary to $x \in X$.

For each $a \in \Delta$, let $\mathcal{W}_a(x) = \{w \in \mathcal{W}_a : x \in P_a^{-1}(w) \bigcap U(w)\}$, then $\mathcal{W}_a(x)$ is countable.

In fact, for each $w \in \mathcal{W}_a$ and $x \in P_a^{-1}(w) \bigcap U(w)$, then $P_a(x) \in w \in \mathcal{W}_a$. Therefore $|(\mathcal{V}_{n+1})_x| \leq \omega$.

(ii) $Z_{n+1} - \bigcup\{\bigcup \mathcal{V}_i : i < n+1\} \subset \bigcup \mathcal{V}_{n+1}$

For each $x \in Z_{n+1} - \bigcup\{\bigcup \mathcal{V}_i : i < n+1\} \subset Z_{n+1} - Z_n$, then $|Q(x)| = n+1$. There is $a \in A^{n+1}$ such that $x_\alpha \neq s_\alpha$ for each $\alpha \in a$, then $x \in Y_a - Z_n \subset \bigcup \mathcal{W}_a$. There is $w \in \mathcal{W}_a$, such that $x \in w \subset p_\alpha^{-1}(w) \bigcap U(w) \in \mathcal{V}_a \subset \mathcal{V}_{n+1}$.

(iii) $\bigcup \mathcal{V}_{n+1} \subset X - Z_n$

For each $x \in \bigcup \mathcal{V}_{n+1}$, there is $a \in A^{n+1}$ such that $x \in \bigcup \mathcal{V}_a$. And there is $w \in \mathcal{W}_a$ such that $x \in P_a^{-1}(w) \bigcap U(w)$, then $P_a(x) \in w = w^* - Z_n$. Hence $x \notin Z_n$. The induction is completed.

By (2), $\bigcup\{\bigcup \mathcal{V}_n : n \in \omega\}$ is an open cover of X and refines $\mathcal{U}$.

Now we discuss Tychonoff products of two metalindelóf spaces.

Lemma 2.2. *If X is metalindelöf, then every locally finite family of closed sets of X has a point countable open expansion.*

Proof: Let $\{F_\alpha : \alpha \in A\}$ is a locally finite family of closed sets of metalindelöf space X. For each $s \in A^{<\omega}$, put $G(s) = X - \bigcup\{F_\alpha : \alpha \in A - s\}$, then $\mathcal{G} = \{G(s) : s \in A^{<\omega}\}$ is an open cover of X it has a point countable open refinement. Let $U_\alpha = \bigcup\{V \in \mathcal{V} : V \bigcap F_\alpha \neq \emptyset\}$ for each $\alpha \in A$. It is easy to check that $\{U_\alpha : \alpha \in A\}$ is a point countable open expansion of $\{F_\alpha : \alpha \in A\}$.

Theorem 2.3. *Let X be a P-space and Y a strong Σ-space. If X and Y are both metalindelöf then $X \times Y$ is metalindelöf.*

Proof: Let $\mathcal{U}$ be a directed open cover of $X \times Y$ and $< \mathcal{F}_i = \{F(\alpha_1, \dots, \alpha_i) : (\alpha_1, \dots \alpha_i) \in \Omega^i\} >_{i \in n}$ is a strong Σ-net of Y. It has a point countable open expansion $\mathcal{H}_i = \{H(\alpha_1, \dots, \alpha_i) : (\alpha_1, \dots \alpha_i) \in \Omega^i\}$. Since $\mathcal{F}_i$ is a locally finite closed cover of Y then for $(\alpha_1, \dots, \alpha_i) \in \Omega^i$, let $\mathcal{G}(\alpha_1, \dots, \alpha_i) = \{V_\lambda \times W_\lambda : \lambda \in \Lambda(\alpha_1, \dots, \alpha_i)\}$ be a maximal collection satisfying the following (1)-(3)

(1) V_λ is open in X

(2) W_λ is open in Y and $F(\alpha_1, \dots, \alpha_i) \subset W_\lambda \subset H(\alpha_1, \dots, \alpha_i)$

(3) $\mathcal{G}(\alpha_1, \dots, \alpha_i)$ is a partial refinement of $\mathcal{U}$.

For each $i \in N$ and $(\alpha_1, \dots, \alpha_i) \in \Omega^i$, let $V(\alpha_1, \dots, \alpha_i), = \bigcup\{V_\lambda : \lambda \in \Lambda(\alpha_1, \dots, \alpha_i)\}$

(4) $V(\alpha_1, \dots, \alpha_i) \subset V(\alpha_1, \dots, \alpha_i, \alpha_{i+1})$ for each $(\alpha_1, \dots, \alpha_i, \alpha_{i+1}) \in \Omega^{i+1}$.

In fact , for each $t \in V(\alpha_1, \dots, \alpha_i)$ there is $\lambda_t \in \Lambda(\alpha_1, \dots, \alpha_i)$ such that $t \in V_{\lambda t}$ and $F(\alpha_1, \dots, \alpha_i) \subset W_{\lambda t} \subset H_{\lambda t}(\alpha_1, \dots, \alpha_i)$. Then $F(\alpha_1, \dots, \alpha_i) \subset W_{\lambda t} \subset H(\alpha_1, \dots, \alpha_i)$. $F(\alpha_1, \dots, \alpha_i, \alpha_{i+1}) \subset W_\lambda \bigcap H(\alpha_1, \dots, \alpha_i, \alpha_{i+1}) \subset H(\alpha_1, \dots, \alpha_i, \alpha_{i+1})$ since $F(\alpha_1, \dots, \alpha_i, \alpha_{i+1}) \subset F(\alpha_1, \dots, \alpha_i)$. Let $W = W_{\lambda t} \bigcap H(\alpha_1, \dots, \alpha_i, \alpha_{i+1})$, then $V_{\lambda t} \times W$ satisfies (1)-(3). There is $\lambda \in \Lambda(\alpha_1, \dots, \alpha_i)$ such that $V_\lambda = V_{\lambda t}$ and $W_\lambda = W$. Then $t \in V_{\lambda t} \subset V(\alpha_1, \dots, \alpha_i, \alpha_{i+1})$, i.e., $V(\alpha_1, \dots \alpha_1) \subset V(\alpha_1, \dots, \alpha_i, \alpha_{i+1})$.

Since X is a P-space, it has a collection $\{C(\alpha_1, \dots, \alpha_i) : (\alpha_1, \dots, \alpha_i) \in \Omega^i$ and $i \in \omega\}$ of closed sets of X such that

(5) $C(\alpha_1, \dots, \alpha_i) \subset V(\alpha_1, \dots, \alpha_i)$

(6) $\bigcup\{C(\alpha_1, \dots, \alpha_i) : i \in N\} = X$ if $\bigcup\{V\alpha_1, \dots, \alpha_i) : i \in N\} = X$.

Put $\mathcal{V}(\alpha_1, \dots, \alpha_i) = \{V_\lambda : \lambda \in \Lambda(\alpha_1, \dots, \alpha_i)\}$, then $\mathcal{V}(\alpha_1, \dots, \alpha_i) \bigcup\{X - C(\alpha_1, \dots, \alpha_i)\}$ is an open cover of X and it has a point countable open refinement $\{O_\lambda : \lambda \in \Lambda(\alpha_1, \ldots, \alpha_i)\} \bigcup \{O'\}$ such that $O' \subset X - C(\alpha_1, \ldots, \alpha_i)$ and $O_\lambda \subset V_\lambda$ for each $\lambda \in \Lambda(\alpha_1, \ldots, \alpha_i)$.

By the above, the following is obvious

(7) $C(\alpha_1, \ldots, \alpha_i) \subset \{O_\lambda : \lambda \in \Lambda(\alpha_1, \ldots, \alpha_i)\}$

Let $\zeta(\alpha_1, \ldots, \alpha_i) = \{O_\lambda \times W_\lambda : \lambda \in \Lambda(\alpha_1, \ldots, \alpha_i)\}, \zeta_i = \bigcup\{\zeta(\alpha_1, \ldots, \alpha_i) : (\alpha_1, \ldots, \alpha_i) \in \Omega^i\}$, then

(8) ζ_i is point countable collection of $X \times Y$ and refines partly $\mathcal{U}$.

In fact, for each $(x, y) \in \bigcup \zeta_i$, put $\Delta = \{(\alpha_1, \ldots, \alpha_i) \in \Omega^i : y \in H(\alpha_1, \ldots, \alpha_i)\}$, then $|\Delta| \leq \omega$ since $\{H(\alpha_1, \ldots, \alpha_i) : (\alpha_1, \ldots, \alpha_i) \in \Omega^i\}$ is a point countable open cover of Y. For each $(\alpha_1, \ldots, \alpha_i) \in \Delta$, let $\Lambda^0(\alpha_1, \ldots, \alpha_i) = \{\lambda \in \Lambda(\alpha_1, \ldots, \alpha_i) : x \in O_\lambda\}$, then $|\Lambda^0(\alpha_1, \ldots, \alpha_i)| \leq \omega$. Since $\{O_\lambda : \lambda \in \Lambda(\alpha_1, \ldots, \alpha_i)\}$ is point countable. $(\zeta(\alpha_1, \ldots, \alpha_i))_{(x,y)} \subset \{O_\lambda \times W_\lambda : \lambda \in \Lambda^0(\alpha_1, \ldots, \alpha_i)\}, (\zeta_i)_{(x,y)} \subset \bigcup\{(\zeta(\alpha_1, \ldots, \alpha_i)) : (\alpha_1, \ldots, \alpha_i) \in \Delta\}$, then $|(\zeta_i)_{(x,y)}| \leq \omega$. And it is easy to check that ζ_i refines partly $\mathcal{U}$ by (3).

(9) $\bigcup\{\zeta_i : i \in N\}$ is a cover of $X \times Y$

For $(x, y) \in X \times Y$, there is $(\alpha_1, \ldots, \alpha_i, \ldots) \in \Omega^\omega$ such that for each $W \in N(C(y))$, then there is $i \in N$ such that $C(y) \subset F(\alpha_1, \ldots, \alpha_i) \subset W$.

Now we assert $\bigcup\{V(\alpha_1, \ldots, \alpha_i) : i \in \boldsymbol{N}\} = X$.

For each $x' \in X$ there is $U \in \mathcal{U}$ such that $\{x'\} \times C(y) \subset U$ since $\{x'\} \times C(y)$ is compact. There are $V \in N(x')$ and $W' \in N(C(y))$ such that $\{x'\} \times C(y) \subset V \times W' \subset U$. Then there is $i \in N$ such that $C(y) \subset F(\alpha_1, \ldots, \alpha_i) \subset W'$ since $C(y) \subset W'$. Put $W = W' \bigcap H(\alpha_1, \ldots, \alpha_i)$, $V \times W$ satisfies (1)-(3) there is $\lambda \in \Lambda(\alpha_1, \ldots, \alpha_i)$ such that $V_\lambda = V$ and $W_\lambda = W$ by the maximun of $\mathcal{G}(\alpha_1, \ldots, \alpha_i)$. Therefore $x' \in V = V_\lambda \subset V(\alpha_1, \ldots, \alpha_i)$, i.e., $\bigcup\{V(\alpha_1, \ldots, \alpha_i) : i \in N\} = X$.

By (6), $\bigcup\{C(\alpha_1, \ldots, \alpha_n) : n \in N\} = X$. There is $n \in \boldsymbol{N}$ such that $x \in C(\alpha_1, \ldots, \alpha_n) \subset \bigcup\{O_\lambda : \lambda \in \Lambda(\alpha_1, \ldots, \alpha_n)\}$. And there is $\lambda \in \Lambda(\alpha_1, \ldots, \alpha_n)$ such that $x \in O_\lambda$. $(x, y) \in$

$O_\lambda \times W \in \zeta(\alpha_1, \ldots, \alpha_n)$ since $y \in C(\alpha_1, \ldots, \alpha_n) \subset W_\lambda \subset H(\alpha_1, \ldots, \alpha_n)$.

ζ is a point countable open refinement of $\mathcal{U}$ by (8) and (9). □

Theorem 2.4. *If X is a metalindelöf P-space and Y has a point countable base then $X \times Y$ is metalindelöf.*

Proof: Let $\mathcal{B} = \{B_\alpha : \alpha \in \Omega\}$ a point countable base of Y and $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ an open cover of $X \times Y$.

For each $n \in N, (\alpha_1, \ldots, \alpha_n) \in \Omega^n$ and $\lambda \in \Lambda$, put $H(\alpha_1, \ldots, \alpha_n : \lambda) = \bigcup\{U : U \times \bigcap_{i=1}^n B_{\alpha i} \subset U_\lambda$ and U is open in $X\}, G(\alpha_1, \ldots, \alpha_n) = \bigcup\{H(\alpha_1, \ldots, \alpha_n; \lambda) : \lambda \in \Lambda\}$.

It is easy to check that for each $(\alpha_1, \ldots, \alpha_n, \ldots) \in \Omega^\omega$, $\{G(\alpha_1, \ldots, \alpha_n) : n \in N\}$ is a monotone increasing collection of open sets of X. Since X is a P-space there is a collection $\{F(\alpha_1, \ldots, \alpha_n) : n \in N\}$ of closed sets of X such that

(i) $F(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n)$ for each $n \in N$.

(ii) $\bigcup\{F(\alpha_1, \ldots, \alpha_n) : n \in N\} = X$ if $\bigcup\{G(\alpha_1, \ldots, \alpha_n) : n \in N\} = X$.

$\mathcal{G}(\alpha_1, \ldots, \alpha_n) = \{H(\alpha_1, \ldots, \alpha_n; \lambda) : \lambda \in \Lambda\}$ is an open cover of $F(\alpha_1, \ldots, \alpha_n)$. There is a point countable collection $\mathcal{V}(\alpha_1, \ldots, \alpha_n) = \{V(\alpha_1, \ldots, \alpha_n; \lambda) : \lambda \in \Lambda\}$ of open sets satisfying.

(iii) $F(\alpha_1, \ldots, \alpha_n) \subset \bigcup \mathcal{V}(\alpha_1, \ldots, \alpha_n)$

(iv) $V(\alpha_1, \ldots, \alpha_n) \subset H(\alpha_1, \ldots, \alpha_n; \lambda)$ for each $\lambda \in \Lambda$.

Define $\mathcal{W}(\alpha_1, \ldots, \alpha_n) = \{V(\alpha_1, \ldots, \alpha_n; \lambda) \times \bigcap_{i=1}^n B_{\alpha i} : \lambda \in \Lambda\}, \mathcal{W} = \bigcup\{\mathcal{W}(\alpha_1, \ldots, \alpha_n) : (\alpha_1, \ldots, \alpha_n) \in \Omega^n$ and $n \in N\}$.

(v) $\mathcal{W}$ is point countable.

In fact, for each $(x, y) \in \bigcup \mathcal{W}$, $\Delta = \{\alpha \in \Omega : y \in B_\alpha\}$ is countable and $(\mathcal{W})_{(x,y)} = \bigcup\{(\mathcal{W}(\alpha_1, \ldots, \alpha_n))_{(x,y)} : (\alpha_1, \ldots, \alpha_n) \in \Delta^n$ and $n \in N\}$. Since $(\mathcal{W}(\alpha_1, \ldots, \alpha_n))_{(x,y)} \subset \{V(\alpha_1, \ldots, \alpha_n; \lambda) \times \bigcap_{i=1}^n B_{\alpha i} : x \in V(\alpha_1, \ldots, \alpha_n; \lambda)$ and $\lambda \in \Lambda\}$ and $\mathcal{V}(\alpha_1, \ldots, \alpha_n)$ is point countable then $|(\mathcal{W})_{(x,y)}| \leq \omega$.

(vi) $\mathcal{W}$ is an open cover of $X \times Y$ and refines $\mathcal{U}$.

For each $y \in Y$, we prove that $\mathcal{W}$ covers $X \times \{y\}$. Put $\Delta(y) = \{\alpha_i \in \Omega : y \in B_\alpha\}$. Then $|\Delta(y)| \leq \omega$. Without loss of generality, we assume $\Delta(y) = \{\alpha_i : i \in N\}$.

First, we prove $\bigcup\{G(\alpha_1, \dots, \alpha_n) : n \in N\} = X$.

For each $x \in X$ there is $\lambda \in \Lambda$ such that $(x, y) \in U_\lambda$. Then there are $n \in N$ and $U \in N(x)$ such that $(x, y) \in U \times \bigcap_{i=1}^n B_{\alpha i} \subset U_\lambda$. Hence $x \in U \subset H(\alpha_1, \dots, \alpha_n; \lambda) \subset G(\alpha_1, \dots, \alpha_n)$, i.e., $\bigcup\{G(\alpha_1, \dots, \alpha_n) : n \in N\} = X$

By (ii), $\bigcup\{F(\alpha_1, \dots, \alpha_n) : n \in N\} = X$, there is $n \in N$ such that $x \in F(\alpha_1, \dots, \alpha_n)$. And there is $\lambda \in \Lambda$ such that $x \in V(\alpha_1, \dots, \alpha_n; \lambda)$. Hence $(x, y) \in V(\alpha_1, \dots, \alpha_n; \lambda) \times \bigcap_{i=1}^n B_{\alpha i} \subset U_i$, i.e., $\mathcal{W}$ is a point countable open refinement of $\mathcal{U}$.

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