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PERFECTLY NORMAL NON-ARCHIMEDEAN SPACES IN MITCHELL MODELS

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ABSTRACT. We investigate the metrizability of perfectly normal non-archimedean spaces in Lévy and Mitchell-collapse models. By collapsing a supercompact cardinal to $\aleph_2$, we prove that in the extension all perfectly normal non-archimedean spaces of size essentially greater than or equal to $\aleph_2$ must be metrizable. It follows that κ^+-Souslin lines with small subspaces metrizable do not exist in these models.

A *non-archimedean space* is a topological space which has a basis which is a tree under the inclusion relation. We will call a non-metrizable, perfectly normal non-archimedean space an *archvillain*.

A still-open question raised by Nyikos is whether it is consistent that all perfectly normal non-archimedean spaces are metrizable [N_1], [N_2], [R]. For quite a long time, the only such space known was the branch space of a Souslin line or a trivial modification thereof. On the other hand, Todorčević [T_1] proved that under the hypothesis of $MA + \neg wKH$ (the consistency of which can be obtained from an inaccessible cardinal), there is no such space of weight $\aleph_1$. However, the exis-

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tence of such spaces of higher weights remained unsolved. The first author, in another paper [Q], proved that the existence of non-trivial perfectly normal, non-metrizable, non-archimedean spaces essentially of size $\aleph_2 = 2^{\aleph_0}$ is consistent with ZFC by constructing such spaces in MA models. Here we provide a counterpoint to Todorčević's result by providing a model in which all archvillains have size essentially $\aleph_1 < 2^{\aleph_0} = \aleph_2$.

1. Topological Lemmas

Since Souslin trees can always trivially produce archvillains of size as big as one likes, e.g. by taking the sum of a Souslin line and a large metrizable space, we take some measures to regulate the situation.

Definition 1.0. Let T be a tree, let $X = \{x \subset T : x \text{ is a countable branch of } T\}$. Then (X, T), i.e. X with the topology generated by the basic open sets U_t, $t \in T$, defined by $U_t = \{x \in X : t \in x\}$, is called the *branch space* of T.

Note such spaces are first countable and non-archimedean.

A space is called a *Souslin space* if it has a base which is a Souslin tree.

Definition 1.1. Let (X, T) be a non-archimedean space, where T is its topological basis which is a tree. We call (X, T) *stout* if

(1) For any $t \in T$, the *upper cone of* t, $T_t = \{t' \in T : t <_T t'\}$, has the same size as T;
(2) $|X| = |T|$;
(3) $\forall t \in T, d(t) = |T_t| = |X| = |T|$, where $d(t)$ is the density of U_t.

Our focus on stout non-archimedean spaces is justified by the following lemma.

Lemma 1.1. *Every archvillain includes a stout archvillain subspace.*

To prove Lemma 1.1, we need the notion of *σ-density.*

Definition. 1.2. A poset is *σ-dense*, if the intersection of any countable family of dense open sets is dense open.

Definition 1.3. A space X is *developable* if it has a countable family of open coverings $\{W_n : n < \omega\}$, such that for each point $x \in X$, $\{\bigcup\{w \in W_n : x \in w\} : n < \omega\}$ forms a base for x.

Note that in a non-archimedean space any open covering has a refinement which is an antichain of the tree base. (Here and elsewhere we shall systematically confuse t with U_t.) Hence a non-archimedean space is developable if and only if it has a countable family of antichain coverings which forms a base.

Lemma 1.2. *If a non-archimedean space X without isolated points has any σ-dense tree base T, then X is not metrizable.*

Proof: Non-archimedean spaces are collectionwise normal, therefore being developable is equivalent to being metrizable for a non-archimedean space. An antichain covering of (X, T) is a maximal antichain of T. Since X does not have isolated points, the *upper part* of a maximal antichain covering, defined as $\{t \in T : \exists a \in$ the maximal antichain, such that $a <_T t\}$, is not empty. Since T is σ-dense, any countable family of upper parts of maximal antichain coverings will have a dense open intersection in T. Hence there will be some basic open sets which are above all these maximal antichains, i.e., the union of the coverings is not a base. □

Lemma 1.3. *Every archvillain has an open subspace without isolated points whose subspace topological base is a σ-dense tree* $[T_1]$.

Proof of Lemma 1.1: Perfect normality is hereditary. To see there is a stout non-metrizable subspace, we need only find an upper cone which is stout because an upper cone is σ-dense in a

σ-dense tree and a σ-dense tree base induces a non-metrizable topology.

For each $t \in T$, let $T_t = \{t' \in T : t < t'\}$. Let $\kappa = min\{|T_t| : t \in T\}$. Pick a $t_0 \in T$, such that $|T_{t_0}| = \kappa$. Obviously, T_{t_0} is a tree in which for any $t \in T_{t_0}$, T_t has the same size as T_{t_0}.

Let $\lambda = min\{d(t) : t \in T_{t_0}\}$. Pick a $t_1 \in T_{t_0}$, such that $d(t_1) = \lambda$. Obviously, T_{t_1} is a subspace in which for any $t \in T_{t_1}$, $d(t) = d(t_1)$.

Always $d(t_1) \leq |T_{t_1}|$. Take a dense subset $Y \subset t_1$ so that $|Y| = d(t_1)$. Since X is first countable, there is a $T' \subset T_{t_1}$ such that $|T'| = d(t_1) = |Y|$ and T' is a base for the subset Y. For any $t' \in T'$, $Y \cap t'$ is dense in t', therefore $|Y \cap t'| = d(t') = |Y|$; and $|T'_{t'}| \leq |T'| = |Y| = d(t') \leq |T'_{t'}|$, hence $|T'_{t'}| = |Y| = d(t')$.

Y is a non-metrizable subspace since T' is its base and it is dense in T_{t_1}; T' is σ-dense since T_{t_1} is σ-dense. Hence (Y, T') is a stout archvillain. □

Lemma 1.4. *If (X, T) is a perfectly normal non-archimedean space, A is an antichain of T, $Y = \{x \in X : x \notin \bigcup A$ & $(\forall t \in T)(x \in t \to (\exists a \in A)(t <_T a))\}$, then Y is metrizable.*

Proof: Let $\bigcup A = \bigcup\{F_n : n < \omega\}$, where the F_n's are closed, $A_n = \{a \in A : a \cap F_n \neq \emptyset\}$, and $G_n = \bigcup\{a : a \in A_n\}$. Clearly $F_n \subset G_n$. Claim that each G_n is closed. Since for any point $x \in X$, if x is not in G_n there is a basic open set $v \in T$ such that $x \in v$ and v is disjoint from F_n. However, if v meets G_n, then v meets some $a \in A_n$. Either $v \subset a$ or $a \subset v$. If $v \subset a$, then x is in a, hence x is in G_n; if $a \subset v$, then v meets F_n, contradiction.

Let $W_n = \{$ minimal $t \in T :\ t \cap Y \neq \emptyset,\ t \cap G_n = \emptyset\}$. For each n, W_n is an antichain of T because of the minimality. For each $y \in Y$ and each n, there is a $t \in T$ so that $y \in t$ but $t \cap G_n = \emptyset$. For otherwise, y is in G_n, and therefore y is in $\bigcup A$.

Hence each W_n is a cover of Y. Moreover, for each y in Y, since y is a limit point of $\bigcup A$, for any $t \in T$ such that $y \in t$, there is some $n \in w$ and some $a \in A_n$, such that $t < a$. Then there must be some $t' \in W_n$ such that $t < t'$ and $y \in t'$.

Therefore the union of the W_n's forms a basis for Y, so as noted earlier, Y is developable and hence metrizable, since subspaces of non-archimedean spaces are non-archimedean. $\square$

In the proof, the antichain A is *positioned above* Y, or just *above* Y, that is to say, $(\forall y \in Y)(\forall t \in T)(y \in t \to (\exists a \in A)(t <_T a))$.

The following lemma relates the metrizability of subspaces to the perfectness of the space.

Lemma 1.5. *Suppose (X,T) is a perfectly normal non-archimedean space. If $Y \subset X$, and Y is nowhere dense, then Y is metrizable.*

Proof: If Y is nowhere dense, we will find an antichain which is positioned above Y. Then by Lemma 1.4, Y is metrizable.

Let $T(Y) = \{t \in T : Y \cap t \neq \emptyset\}$. Since Y is nowhere dense, for each $t \in T(Y)$, there is an $t' \in T$ so that $t' \cap Y = \emptyset$ and $t < t'$. For each $t \in T(Y)$, let $a(t) = least\{s \in T : t < s$ but $s \cap Y = \emptyset\}$. Then $A = \{a(t) : t \in T(Y)\}$ is an antichain above Y, for if $t, t' \in T(Y)$, $a(t) < a(t')$, then $a(t) < t'$, so $a(t) \cap Y \neq \emptyset$. Hence for $t, t' \in T(Y)$, either $a(t) = a(t')$ or $a(t) \cap a(t') = \emptyset$. By Lemma 1.4, since X is perfectly normal, Y is metrizable. $\square$

Lemma 1.6. *Suppose (X,T) is a stout archvillain. If $Y \subset X, |Y| < |X|$, then Y is metrizable.*

Proof: It suffices to prove Y is nowhere dense in X. This is obvious by stoutness. $\square$

We can prove a decomposition theorem for archvillains:

Theorem 1.1. *If (X,T) is an archvillain, then $X = M \cup (\bigcup_\alpha Y_\alpha) \cup (\bigcup_\beta X_\beta)$, where the Y_α's are basic open sets whose subspace bases are Souslin trees, the X_β's are basic open sets whose subspace bases do not include Souslin trees, and M is a metrizable subspace.*

First, we prove a lemma.

Lemma 1.7. *If an archvillain (X, T) includes a Souslin subspace, it includes an open Souslin space.*

Proof: If Y is a Souslin subspace, then $\exists t \in T(Y)$ so that $T_t = \{s \in T : t < s\}$ is a Souslin tree. For otherwise, $\forall t \in T(Y)$, T_t has an uncountable antichain, therefore there must be some $t' \in T_t$, so that $t' \cap Y = \emptyset$. Let A be an antichain with union equaling $\bigcup\{t \in T : (\exists y \in Y)(\exists t' \in T)(y \in t' \text{ and } t > t' \text{ and } t \cap Y = \emptyset)\}$. Then A is an antichain positioned above Y. Since Y is not metrizable, by Lemma 1.4 X could not be perfectly normal. Therefore, if an archvillain includes a Souslin subspace, one of its basic open sets is a Souslin space. □

Lemma 1.7. shows that a stout archvillain of size greater than or equal to $\aleph_2$ does not include Souslin subspaces. For if it did, it would include an open subspace whose topological base is a Souslin tree, and therefore one of its upper cones would not have size $\aleph_2$ which contradicts the assumption that the space is stout.

Proof of Theorem 1.1: If X includes a Souslin subspace, then by Lemma 1.7, there must be some basic open set t whose subspace base is a Souslin tree. Then there must be a maximal basic open set whose upper cone is a Souslin tree and includes t. Applying Lemma 1.7 repeatedly, we find there must be a maximal family which consists of basic open sets whose upper cones are Souslin trees. Note that this family is an antichain. Extend this antichain to a maximal antichain of basic open sets. The basic open sets in the extended part do not include any Souslin subspace; some of them may be metrizable. The maximal antichain is positioned above a metrizable space. For if the space below is not metrizable, then by Lemma 1.4 X cannot be perfectly normal. □

2. Supercompact cardinals and archvillains

In order to obtain a model which has no stout archvillains of size $> \aleph_1$, we will collapse a supercompact cardinal to $\aleph_2$. The forcing poset used is an iteration of ccc posets of adding Cohen reals and countably closed posets of collapsing ordinals. The resulting model is the Mitchell model [M] which will be more precisely described later on. In the extension, any archvillain of size $\geq \aleph_2$ has a reflection in some intermediate stage, and the reflection would have size $< \kappa$, which is the large cardinal in the ground model. Then preservation lemmas would imply the non-metrizability of the reflection in the extension. As a result we would have a small-sized archvillain included in a stout archvillain, which would be a contradiction to Lemma 1.6.

Here are the preservation lemmas.

Lemma 2.1. *Adding Cohen reals preserves non-developability.*

Proof: See [DTW_2]. □

Lemma 2.2. *Countably closed forcing preserves σ-density.*

Proof: A poset is σ-dense if and only if it does not add any new subset of ω (see e.g. [J]). Let P be a σ-dense poset in V. Claim P is σ-dense in V^Q, where Q is countably closed. Since $P \in V$, the claim is equivalent to $Q \times P$ and hence $P \times Q$ being σ-dense. But since P is σ-dense, Q is still countably closed in V^P, so $P \times Q$ adds no new subset of ω. □

From now on we assume that κ is a supercompact cardinal.

We follow the formulation of the Mitchell collapse in [DJW].

The Mitchell collapse $\mathrm{Mi}(\kappa)$ is defined to be the iteration of $\{\langle P_\alpha, \dot{Q}_\alpha\rangle : \alpha < \kappa\}$ such that if α is even, then $\dot{Q}_\alpha$ is a name for $Fn(\omega, 2)$ (Cohen real forcing); and if α is odd, then $\dot{Q}_\alpha$ is a name for $Fn(\omega_1, 2, \omega_1)$ (countable partial functions from ω_1 into 2); the support is finite on the even ordinals and countable on the odd.

Lemma 2.3. [DJW] *Suppose κ is strongly inaccessible.*

(1) *There is an $\dot{R}_\kappa$ such that $Mi(\kappa) * \dot{R}_\kappa$ is forcing equivalent to $Fn(\kappa, 2) \times Q_\kappa$ where Q_κ is countably closed.*
(2) *If $\lambda < \kappa$, then $Mi(\kappa) = Mi(\lambda) * \dot{Q}$, and if G_λ is $Mi(\lambda)$-generic then $V[G_\lambda] \models \dot{Q} \cong Mi(\kappa)$.*
(3) *$Mi(\kappa)$ has the κ-c.c.*
(4) *If G is $Mi(\kappa)$-generic, then in $V[G], 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 = \kappa$ and $\aleph_1 = \aleph_1^V$.*

Definition. κ is a supercompact cardinal if for each $\lambda \geq \kappa$ there is an elementary embedding j from the universe V into a transitive class M such that
(i) $j(\alpha) = \alpha$ for all $\alpha < \kappa$ but $j(\kappa) > \lambda$;
(ii) $M^\lambda \subset M$.

Definition. A formula with one free variable we call a *property.* A property ϕ of a topological structure X is *preserved* by forcing with P over V if whenever G is a P-generic filter over V and $V \models \phi[X]$ then $V[G] \models \phi[X]$.

Lemma 2.4. *Mitchell forcing preserves the non-developability of archvillains.*

Proof: Since developability is upward absolute, by 2.3 it suffices to show non-developability is preserved by countably closed forcing and by adding Cohen reals. The latter is true by 2.1; as for the countably closed Q_κ, if X is not developable in the ground model V, then by Lemma 1.3 there is an open subspace Y without isolated points which has a base T which is a σ-dense tree. Since Q_κ preserves σ-density (Lemma 2.2), T remains σ-dense and therefore Y remains non-developable in the model V^{Q_κ}. Since X includes a non-metrizable subspace Y, X is not developable in the extension. □

Machinery for supercompact and weak compact reflection is exposited in [DTW$_1$], [DTW$_2$] and [DJW]. In either formulation of supercompact reflection, it is not difficult to check that the Mitchell collapse satisfies the forcing requirements needed to operate the machinery, and that the property of being a perfectly normal, non-archimedean, non-developable space satisfies the linguistic requirements, e.g. is local and structural in the sense of [DTW$_1$]. The one thing that does need to be pointed out is that such spaces have character less than the critical point; one can prove directly that perfectly normal non-archimedean spaces are first countable, or quote [P] to the effect they are linearly ordered, whence it is then easy to see that pseudocharacter equals character. In any event, we can conclude that after Mitchell-collapsing a supercompact, every archvillain of size $\geq \kappa = \aleph_2$ includes one of smaller size. In particular then, we have the following:

Theorem 2.1. *Mitchell-collapse a supercompact. Then every archvillain has a subspace of size $\aleph_1$ which is an archvillain.*

Proof: By 2.4 and reflection, an archvillain will have a non-developable subspace of size $\aleph_1$. Perfect normality and non-archimedeanness are hereditary, so the subspace will be an archvillain. $\square$

Corollary. *Assume κ is supercompact. $Mi(\kappa) \models$ There is no stout archvillain of size $\geq \aleph_2$.*

Proof: By Theorem 2.1 a stout archvillain of size $\aleph_2$ would have a subspace which would be an archvillain of size $\aleph_1$. This contradicts Lemma 1.6. $\square$

Note these results also hold if we Lévy-collapse the supercompact to $\aleph_2$ with countable conditions. In that case, CH will hold.

3. Weakly compact cardinals and archvillains of size $\aleph_2$

In [Q] it is proven that by using an inaccessible not weakly compact cardinal κ in L, one can obtain an MA model in which there is an archvillain of size $\aleph_2$. However, starting with a weakly compact cardinal κ, one can Mitchell-collapse κ to $\aleph_2$ to obtain a model in which there is no stout archvillain of size $\aleph_2$. Recall from e.g. [DJW]:

Definition 3.1. A Π^1_1-formula $\varphi(X_1, ..., X_n)$ is a second order formula of set theory which is of the form

$$\forall X_0 \psi(X_0, X_1, \ldots, X_n),$$

where ψ is a formula of the usual predicate logic in the language $\{\in, X_0, \ldots, X_n\}$, where X_i are unary predicates.

Definition 3.2. If M is a set and $A_1, \ldots, A_n \in \mathcal{P}(M)$,

$$\langle M, \in, A_1, \ldots, A_n \rangle \models \varphi$$

means for all $A_0 \subset M$, we have $\psi^M(A_0, \ldots, A_n)$, where $\psi^M(A_0, \ldots, A_n)$ is the relativization of the formula ψ to M.

Lemma 3.1. *Suppose κ is weakly compact. For any Π^1_1-formula $\varphi(X_1, ..., X_n)$ and any $A_1, ..., A_n$ in $V_{\kappa+1}$ such that $\langle V_\kappa, \in, A_1, ..., A_n \rangle \models \varphi$, there is a strong inaccessible $\theta < \kappa$ such that*

$$\langle V_\theta, \in, A_1 \cap V_\theta, ..., A_n \cap V_\theta \rangle \models \varphi\ .$$

Using standard techniques, this can be used to prove the following theorem:

Theorem 3.1. *Assume κ weakly compact. $Mi(\kappa) \models$ There is no stout archvillain of size $\aleph_2$.*

Proof: The basic idea is to show that the statement Ψ: "X is an archvillain of size κ" is Π^1_1. Then the standard argument as in e.g. [DJW] or [DTW$_2$] yields that since we can take

$Mi(\kappa) \subset V_\kappa$, the assertion that this statement is forced is also Π^1_1. Then the preservation lemmas and the machinery in either of those articles finishes the proof. There are, however, several difficulties to be surmounted. First of all, we don't know if perfect normality is preserved. So, as in section 2, we preserve non-developability and use the fact that perfect normality is hereditary. Second, saying "X is non-archimedean" a priori requires a second-order existential quantifier, so instead we deal with (X, T), where T is a tree base for X. By first countability, the reflection (X', T') is indeed a subspace of (X, T), but we have to check it is non-archimedean. But it is, since that property is hereditary, and in any event, T' is a tree. Third, we need to observe that T can be taken to have size κ, but that follows by first countability. Thus all we need to do is convince the reader that the statement "T is a first countable σ-dense tree base of size κ for X of size κ" is Π^1_1. Saying that T is a tree and a first countable base is routine coding; σ-density may be expressed as "for every countable collection of maximal antichains of T, the set of nodes which are in all the antichains is a dense open subset of T", which is easily seen to be Π^1_1. □

Unfortunately Mi(κ) does have archvillains in it, in fact it contains Souslin trees.

Since $2^{\aleph_0} = \aleph_2$ in Mi(κ), the existence of stout archvillains is excluded from the level of continuum and above. If we Lévy-collapse a supercompact cardinal to $\aleph_2$, we will obtain a model of CH in which there are no stout archvillains of size larger than continuum. Therefore the non-existence of stout archvillains of size larger than or equal to the continuum is independent of the Continuum Hypothesis. The proof of the Lévy-collapse result is similar to but easier than the proof via Mitchell-collapse. We leave it to the reader.

4. κ^+-SOUSLIN TREES

Todorčević [T$_2$] has shown that archvillains can be constructed from what he calls *coherent κ^+-Souslin trees*, which he notes can be constructed from $\square$'s and $\diamondsuit$'s in a fashion similar to that of Jensen's κ^+-Souslin trees. It follows by covering lemma considerations that obtaining the consistency of no perfectly normal non-metrizable non-archimedean space requires large cardinals (and possibly $0 = 1$!).

Todorčević's spaces are first countable linearly ordered spaces with cellularity less than density, in which subspaces of weight not exceeding the cellularity are metrizable. Assuming the consistency of a supercompact cardinal, we shall show it consistent that no such space exists.

It is an open problem whether it is consistent with GCH that there are no κ-Souslin trees, even for $\kappa = \omega_2$. However we can prove, consistent with GCH, that coherent κ^+-Souslin trees do not exist, assuming large cardinals, since, as Todorčević notes, the corresponding spaces have small subspaces metrizable.

Theorem 4.1. *If it is consistent that there is a supercompact (weakly compact) cardinal, it is consistent with GCH that there is no first countable linearly ordered topological space X (of size $\leq \aleph_2$) with $c(X) < d(X)$ such that subspaces of X of size $\aleph_1$ are metrizable.*

Proof: We work in the model obtained by Lévy-collapsing a supercompact (or weakly compact) cardinal to ω_2 with countable conditions. By [QT], as a first countable linearly ordered topological space, X has a dense non-archimedean subspace Y. It suffices to show that Y has a σ-disjoint π-base since then $c(X) = c(Y) = d(Y) \geq d(X)$.

Suppose Y did not have a σ-disjoint π-base. Then by Lemma 1.3, some open subspace Y' of it would have a σ-dense tree base. Note Y' is uncountable. Countably closed forcing preserves σ-density, so by the usual reflection argument, some subspace Y'' of Y' of cardinality $\aleph_1$ would have a σ-dense tree base. But that would contradict the assumed metrizability of Y'', since by Lemma 1.3, a non-archimedean space with a σ-dense tree base is not metrizable (and of course subspaces of metrizable spaces are metrizable).

We can't so quickly wave our hands at the weakly compact case, since as noted earlier the existence of a σ-dense tree base is unlikely to be Π^1_1. Instead, as in section 3, we have to work with a pair, say (Y', T), reflect, and then note the resulting non-archimedean space is a subspace of (Y', T). $\square$

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